HOMEWORK ASSIGNMENT #2

(30.J) Let $\varepsilon > 0$ and let P_ε be a partition of $[a,b]$ such that if $P_\varepsilon \subseteq P$ then

$$\left| S(P;f) - \int_a^b f \right| < \varepsilon .$$

Let now $P = (x_0, x_1, \dots, x_n)$ be a refinement of P_ε. The function F is continuous on $[a,b]$ and differentiable, so using Mean Value Theorem we get

$$F(b) - F(a) = \sum_{i=1}^{n} \left(F(x_i) - F(x_{i-1}) \right) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

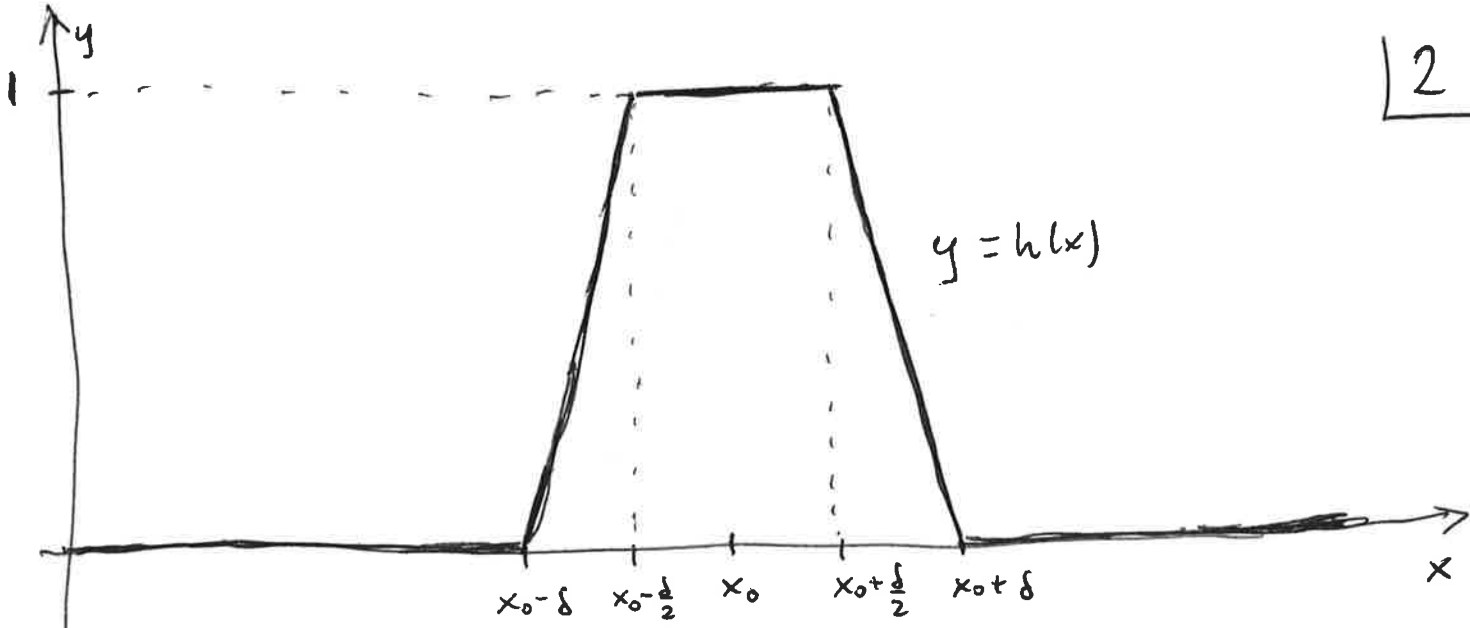
$$= S(P;f)$$

where c_i, $i = 1, \dots, n$, are some numbers such that $x_{i-1} < c_i < x_i$.

Therefore we obtained that for every $\varepsilon > 0$

$$\left| F(b) - F(a) - \int_a^b f \right| < \varepsilon \implies F(b) - F(a) = \int_a^b f .$$

(30.R) Let x_0 be a point of continuity of f. Suppose that $x_0 \in (a,b)$. If $x_0 = a$ or $x_0 = b$ the proof is similar.
If $f(x_0) \neq 0$ then we can assume without loss of generality that $f(x_0) > 0$ (otherwise we can consider $-f$).
Since f is continuous at x_0, there is $\delta > 0$ such that $f(x) \geq \frac{f(x_0)}{2}$ for $x \in [x_0 - \delta, x_0 + \delta]$. Let h be a continuous function such that $h \geq 0$, $h(x) = 1$ if $x \in [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$, $h(x) = 0$ if $x \notin [x_0 - \delta, x_0 + \delta]$.



Then $f(x)h(x) \geq 0$ on $[a,b]$ and thus

$$\int_a^b fh \geq \int_{x_0-\frac{\delta}{2}}^{x_0+\frac{\delta}{2}} fh \geq \frac{f(x_0)}{2}\delta > 0$$

which is a contradiction. Thus we must have $f(x_0)=0$.

30. V(+)

$$\int_{-\pi}^{\pi} \cos x \, d(|\sin x|) = \int_{-\pi}^{0} \cos x \, d(|\sin x|) + \int_{0}^{\pi} \cos x \, d(|\sin x|)$$

$$= \int_{-\pi}^{0} \cos x \, d(-\sin x) + \int_{0}^{\pi} \cos x \, d(\sin x) = -\int_{-\pi}^{0} \cos x \, d(\sin x) + \int_{0}^{\pi} \cos x \, d(\sin x)$$

$$= -\int_{-\pi}^{0} \cos^2 x \, dx + \int_{0}^{\pi} \cos^2 x \, dx = 0.$$

31.D If $x \geq a$ then

$$\left| \frac{\sin nx}{nx} \right| \leq \frac{1}{na} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Therefore the functions $f_n(x) = \dfrac{\sin nx}{nx}$ converge to 0 uniformly on $[a, \pi]$. Therefore

$$\lim_{n\to\infty} \int_a^\pi \frac{\sin nx}{nx}\,dx = \lim_{n\to\infty} \int_a^\pi f_n(x)\,dx = \int_a^\pi \lim_{n\to\infty} f_n(x)\,dx = \int_a^\pi 0\,dx = 0.$$

If $a = 0$ we have

$$\int_0^\pi \frac{\sin nx}{nx}\,dx = \frac{1}{n}\int_0^{n\pi} \frac{\sin y}{y}\,dy.$$

$y = nx$.

$$\int_0^{n\pi} \frac{\sin y}{y}\,dy = \int_0^1 \frac{\sin y}{y}\,dy + \int_1^{n\pi} \frac{\sin y}{y}\,dy$$

Since $\left|\dfrac{\sin y}{y}\right| \leq \dfrac{1}{y}$, we have

$$\left| \int_1^{n\pi} \frac{\sin y}{y}\,dy \right| \leq \int_1^{n\pi} \frac{1}{y}\,dy = \ln y \Big|_1^{n\pi} = \ln n\pi.$$

Therefore

$$\int_0^\pi \frac{\sin nx}{nx}\,dx = \frac{1}{n}\left[\int_0^1 \frac{\sin y}{y}\,dy + \ln n\pi \right] \longrightarrow 0$$

as $n \to \infty$.

(We used the fact that $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$.)

31.G Since f is integrable with respect to g_n, we have

$$\int_a^b f\,dg_n = f(b)g_n(b) - f(a)g_n(a) - \int_a^b g_n\,df$$

Since (g_n) converges uniformly to g on $[a,b]$,

$$f(b)g_n(b) \to f(b)g(b), \quad f(a)g_n(a) \to f(a)g(a) \quad \text{as } n\to\infty.$$

Moreover, we obtain that g is f-integrable and

$$\lim_{n\to\infty} \int_a^b g_n\,df = \int_a^b g\,df.$$

Therefore it follows from integration by parts theorem that f is g-integrable and

$$\int_a^b f\,dg = f(b)g(b) - f(a)g(a) - \int_a^b g\,df = \lim_{n\to\infty} \int_a^b f\,dg_n.$$

31.U By the uniform continuity of f_t, if $\varepsilon > 0$, let $\delta(\varepsilon) > 0$ be such that if $|t - t_0| < \delta(\varepsilon)$, then

$$|f_t(x,t) - f_t(x,t_0)| < \varepsilon \quad \text{for all } x \in [a,b].$$

It then follows from the Mean Value Theorem that if $0 < |t - t_0| < \delta(\varepsilon)$, then

$$\left| \frac{f(x,t) - f(x,t_0)}{t - t_0} - f_t(x,t_0) \right| < \varepsilon, \quad \text{for all } x \in [a,b].$$

Thus (by Theorem 30.5)

$$\left| \frac{F(t)-F(t_0)}{t-t_0} - \int_a^b f_t(x,t_0)\,dg(x) \right| \le \int_a^b \left| \frac{f(x,t)-f(x,t_0)}{t-t_0} - f_t(x,t_0) \right| dg(x)$$

$$\le \varepsilon \left(g(b) - g(a) \right).$$

This shows that F' exists for $t \in [c,d]$ and

$$F'(t) = \int_a^b f_t(x,t)\,dg(x).$$

32.D(a)

$$x + x^2 \ge x \implies 0 \le \frac{1}{(x+x^2)^{\frac{1}{2}}} \le \frac{1}{\sqrt{x}}$$

~~Therefore~~ Since $\int_0^1 \frac{dx}{\sqrt{x}}$ exists we thus have

$$\int_0^1 \frac{dx}{(x+x^2)^{\frac{1}{2}}} \text{ is convergent.}$$

32.F(c) Since for $x \ge 1$, $0 \le \frac{1}{x} \le 1$, we get

$$0 \le \frac{\sin\left(\frac{1}{x}\right)}{x} = \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \frac{1}{x^2} \le \frac{1}{x^2}.$$

(We used $\sin y \le y$ for $y \ge 0$)

Since $\int_1^{+\infty} \frac{1}{x^2}\,dx$ converges, $\int_1^{+\infty} \frac{\sin\left(\frac{1}{x}\right)}{x}\,dx$ converges absolutely.

32.G(c) $\boxed{p<0}$ Similarly to 32.F(c), for $x \ge 1$ we get

$$0 \le \frac{\sin x^p}{x} \le \frac{\sin x^p}{x^p} \frac{x^p}{x} \le x^{p-1}.$$

$\int_1^{+\infty} x^{p-1}\,dx$ converges since $p-1 < -1 \implies \int_1^{+\infty} \frac{\sin x^p}{x}\,dx$ converges absolutely.

$\boxed{p=0}$ Then

$$\frac{\sin(x^p)}{x} = \frac{1}{x} \Longrightarrow \int_1^{+\infty} \frac{\sin x^p}{x}\,dx \text{ does not exist for } p=0.$$

$\boxed{p>0}$ Set $y = x^p$. Then

$$\int_1^c \frac{\sin x^p}{x}\,dx = \frac{1}{p}\int_1^{c^p} \frac{\sin y}{y}\,dy$$

Since the latter integrals converge (but not absolutely) we obtain that $\int_1^{+\infty} \frac{\sin x^p}{x}\,dx$ converges but not absolutely.

33.C(e)

$$\left| e^{-x^2-\frac{t^2}{x^2}} \right| = e^{-x^2} e^{-\frac{t^2}{x^2}} \le e^{-x^2} \quad \text{for all } t \in \mathbb{R}.$$

Since $\int_0^{+\infty} e^{-x^2}\,dx$ exists, $\int_0^{+\infty} e^{-x^2-\frac{t^2}{x^2}}\,dx$

converges uniformly for $t \in \mathbb{R}$ by the Weierstrass M-test.

30.S Define $F(x) = \int_a^x p(t)\,dt$. Then F is continuously differentiable and $F'(x) = p(x)$. Therefore we have

$$F'(x) \le c\,F(x) \quad \text{for } x \in [a,b].$$

Therefore

$$\left(e^{-cx} F(x)\right)' \le 0 \Longrightarrow e^{-cx} F(x) \le e^{-ca} F(a) = 0.$$

Thus we obtained $F(x) \le 0$ for $x \in [a,b]$ which then implies

$$0 \le p(x) \le c\,F(x) \le 0 \Longrightarrow p(x) = 0 \text{ on } [a,b].$$