33.I(c): We use the identity

$$\sin x \sin ax = \frac{1}{2}(\cos((1-a)x) - \cos((1+a)x)).$$

For $t > 0$ define the function

$$F(a) = \frac{1}{\pi}\int_0^{+\infty} e^{-tx}\frac{\cos((1-a)x) - \cos((1+a)x)}{x}dx.$$

The integrand and its partial derivative with respect to a are continuous on $[0, +\infty) \times \mathbb{R}$ and thus

$$F'(a) = \frac{1}{\pi}\int_0^{+\infty} e^{-tx}(\sin((1-a)x) + \sin((1+a)x))dx$$

since the latter integral is uniformly convergent for $a \in \mathbb{R}$. Integrating by parts we obtain

$$F'(a) = \frac{1}{\pi}\left(\frac{1-a}{t^2 + (1-a)^2} + \frac{1+a}{t^2 + (1+a)^2}\right).$$

Integrating with respect to a and using the fact that $F(0) = 0$ thus gives us

$$F(a) = -\frac{1}{2\pi}\ln[t^2 + (1-a)^2] + \frac{1}{2\pi}\ln[t^2 + (1+a)^2] = \frac{1}{2\pi}\ln\frac{t^2 + (1+a)^2}{t^2 + (1-a)^2}.$$

Now for a fixed a define the function

$$G(t) = F(a) = \frac{1}{\pi}\int_0^{+\infty} e^{-tx}\frac{\cos((1-a)x) - \cos((1+a)x)}{x}dx.$$

The integral defining G is uniformly convergent for $t \geq 0$ by Dirichlet's Test with

$$f(x,t) = \cos((1-a)x) - \cos((1+a)x), \quad \varphi(x,t) = \frac{e^{-tx}}{x}.$$

Therefore F is continuous on $[0, +\infty)$ and this implies

$$G(0) = \frac{1}{\pi}\int_0^{+\infty} \frac{\cos((1-a)x) - \cos((1+a)x)}{x}dx$$

$$= \frac{2}{\pi}\int_0^{+\infty} \frac{\sin x \sin ax}{x}dx = \frac{1}{2\pi}\ln\left(\frac{1+a}{1-a}\right)^2$$

which gives the required identity.

33.M: We notice that for $a > 0$

$$\int_0^{+\infty} \int_a^{+\infty} 2xe^{-x^2(1+y^2)} \cos(ty)dxdy = \int_0^{+\infty} -\frac{e^{-x^2(1+y^2)}\cos(ty)}{1+y^2}\Bigg|_a^{+\infty} dy$$

$$= \int_0^{+\infty} e^{-a^2(1+y^2)} \frac{\cos ty}{1+y^2} dy \to \int_0^{+\infty} \frac{\cos ty}{1+y^2} dy \quad \text{as } a \to 0$$

by the Dominated Convergence Theorem. We need to compute the iterated integral. We notice that

$$|2xe^{-x^2(1+y^2)} \cos(ty)| \leq 2xe^{-x^2} e^{-a^2 y^2} =: M(x)N(y).$$

Therefore we can change the order of integration to get

$$\int_0^{+\infty} \int_a^{+\infty} 2xe^{-x^2(1+y^2)} \cos(ty)dxdy = \int_a^{+\infty} \int_0^{+\infty} 2xe^{-x^2(1+y^2)} \cos(ty)dydx$$

$$= \int_a^{+\infty} \int_0^{+\infty} 2xe^{-x^2} e^{-x^2 y^2} \cos(ty)dydx$$

using 33.G with $c = x^2$

$$= \int_a^{+\infty} 2xe^{-x^2} \frac{1}{2}\sqrt{\frac{\pi}{x^2}} e^{-\frac{t^2}{4x^2}} dx = \sqrt{\pi} \int_a^{+\infty} e^{-(x^2+\frac{t^2}{4x^2})} dx \to \sqrt{\pi} \int_0^{+\infty} e^{-(x^2+\frac{t^2}{4x^2})} dx$$

as $a \to 0$. Therefore, using 33.H with $t := t/2$, we finally get

$$\int_0^{+\infty} \frac{\cos ty}{1+y^2} dy = \sqrt{\pi} \int_a^{+\infty} e^{-(x^2+\frac{t^2}{4x^2})} dx = \sqrt{\pi}\frac{1}{2}\sqrt{\pi} e^{-|t|} = \frac{\pi}{2} e^{-|t|}.$$