Math. 4318, Practice Test 1

1. Let $f : \mathbb{R} \to \mathbb{R}$ and let $f'(c)$ exist. Show that

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h}.$$

Give an example to show that the existence of the limit does not imply the existence of the derivative.

2. Prove that $e(\ln x) \leq x$ for all $x > 0$. Show that the equality is true only if $x = e$.

3. Let $f : [0,1] \to \mathbb{R}$ be bounded. Suppose that f is continuous on $[0,1) \setminus \{p_1, p_2, ...\}$, where $0 < p_1 < p_2 < ... < 1, \lim_{n \to +\infty} p_n = 1$. Show that f is Riemann integrable on $[0,1]$.

4. Evaluate the Riemann-Stieltjes integral

$$\int_{-\pi}^{\pi} \cos x \, d\left(|x| + \cos x\right).$$

5. Determine whether the infinite integral

$$\int_0^{+\infty} \sin(e^x)\, dx$$

exists.

Solutions:

1. Let $\epsilon > 0$ and let $\delta > 0$ be such that if $|h| < \delta$ then

$$\left|\frac{f(c+h)-f(c)}{h} - f'(c)\right| < \epsilon.$$

Then if $|h| < \delta$

$$\left|\frac{f(c+h)-f(c-h)}{2h} - f'(c)\right|$$

$$\leq \left|\frac{1}{2}\left(\frac{f(c+h)-f(c)}{h} - f'(c)\right) + \frac{1}{2}\left(\frac{f(c-h)-f(c)}{-h} - f'(c)\right)\right|$$

$$\leq \frac{1}{2}\left|\frac{f(c+h)-f(c)}{h} - f'(c)\right| + \frac{1}{2}\left|\frac{f(c-h)-f(c)}{-h} - f'(c)\right| < \epsilon.$$

If $f(x) = |x|, c = 0$ then the limit is 0 but $f'(0)$ does not exist.

2. Define $f(x) = e(\ln x) - x$. We have $f'(x) = e/x - 1$. Thus $f'(x) > 0$ for $0 < x < e$, and $f'(x) < 0$ for $x > e$. If $0 < x < e$, by Mean Value Theorem $f(e) - f(x) = f'(c)(e-x) > 0$ since c is a point such that $x < c < e$. Similarly we obtain $f(e) > f(x)$ for $e < x$. Thus $f(x) \leq f(e) = 0$ for all $x > 0$ with equality only if $x = e$.

3. We will use the Riemann Criterion for Integrability. Suppose that $M > 0$ is such that $|f(x)| \leq M$ for all $x \in [0,1]$. Let $\epsilon > 0$. Let $p_1, ..., p_k$ be the points of $\{p_1, p_2, ...\}$ which are in $[0, 1 - \frac{\epsilon}{8M}]$. The function f is uniformly continuous on

$$A = [0, 1 - \frac{\epsilon}{8M}] \setminus \bigcup_{i=1}^{k} (p_i - \frac{\epsilon}{16Mk}, p_i + \frac{\epsilon}{16Mk})$$

so there exists n_0 such that if $|x - y| \leq \frac{1}{n_0}, x, y \in A$, then $|f(x) - f(y)| \leq \frac{\epsilon}{2}$. Denote $B = [0,1] \setminus A$. We now define a partition $P_\epsilon = \{\frac{i}{n_0} : i = 0, ..., n_0\} \cup \{p_i - \frac{\epsilon}{16Mk}, p_i + \frac{\epsilon}{16Mk} : i = 1, ..., k\} \cup \{1 - \frac{\epsilon}{8M}\}$. Let $P = \{x_0, x_1, ..., x_m\}$ be a refinement of P_ϵ. We notice that for every j either $[x_{j-1}, x_j] \subset A$ or $(x_{j-1}, x_j) \subset B$. Also if $M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\}, m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$ then $M_j - m_j \leq 2M$. Therefore

$$\sum_{j=1}^{m} (M_j - m_j)(x_j - x_{j-1}) = \sum_{\{j:(x_{j-1},x_j)\subset B\}} + \sum_{\{j:[x_{j-1},x_j]\subset A\}}$$

$$< 2M(\text{total length of intervals in } B) + \frac{\epsilon}{2}(\text{total length of intervals in } A)$$

$$\leq 2M\left(\frac{\epsilon}{8M} + k\frac{\epsilon}{8Mk}\right) + \frac{\epsilon}{2} \cdot 1 = \epsilon.$$

4. Using additivity with respect to the domain of integration, linearity with respect to the integrators, the theorem about the modification of the integral, and the Fundamental Theorem of Calculus, we have

$$\int_{-\pi}^{\pi} \cos x \, d\left(|x| + 2\cos x\right) = -\int_{-\pi}^{0} \cos x \, dx + \int_{0}^{\pi} \cos x \, dx + 2\int_{-\pi}^{\pi} \cos x (\cos x)' \, dx$$

$$= -\sin x|_{-\pi}^{0} + \sin x|_{0}^{\pi} - \int_{-\pi}^{\pi} 2\cos x \sin x \, dx = 0 + 0 - \int_{-\pi}^{\pi} \sin 2x \, dx$$

$$= \frac{1}{2} \cos 2x|_{-\pi}^{\pi} = 0.$$

5. We make the substitution $y = e^x$, i.e. $x = \ln y$. Then for $c > 0$

$$\int_{0}^{c} \sin(e^x) \, dx = \int_{1}^{e^c} \frac{\sin(y)}{y} \, dy.$$

Since the integral on the right converges when $c \to +\infty$ (by Dirichlet test, discussed in class), it thus follows that

$$\int_{0}^{+\infty} \sin(e^x) \, dx$$

exists.