Practice Test 2

1. Let $f : \mathbb{R} \to \mathbb{R}, f \in C^2(\mathbb{R}), f'(0) = 0$. Define $g : \mathbb{R}^n \to \mathbb{R}$ by $g(x) = f(\|x\|)$. Show that $g \in C^1(\mathbb{R}^n)$.

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, $f(1) = g(1) = 0$, and $|f'(1)| \neq |g'(1)|$. Show that the system

$$f(xy) + g(yz) = 0, \quad g(xy) + f(yz) = 0$$

can be solved for y and z as functions of x near the point $(x, y, z) = (1, 1, 1)$. Compute the derivatives of y and z with respect to x at 1 if $f'(1) = 1, g'(1) = 0$.

3. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (e^x \cos y, e^x \sin y)$. What is the image of T? Show that for every $(x, y) \in \mathbb{R}^2$ there is an open neighborhood U of (x, y) such that $T_{|U}$ is invertible and its inverse is of class $C^1(T(U))$. Is it true that if $f : \Omega \to \mathbb{R}^p$, where Ω is an open subset of $\mathbb{R}^p$, $f \in C^1(\Omega)$ and $J_f(z) \neq 0$ for every $z \in \Omega$ then f is invertible as a map from Ω onto $f(\Omega)$?

4. Let A be a symmetric $p \times p$ matrix. Show that the maximum value of $f(x) = (Ax) \cdot x$ on the unit sphere $\{x \in \mathbb{R}^p : \|x\| = 1\}$ is equal to the largest eigenvalue of A.

5. Let g be the function from problem 1. Does $D^2 g(0)$ exist? Find $D^2 g(0)$ if it exists.

Solutions:

1. Away from the origin $g(x) = f(\|x\|) = f(\sqrt{x_1^2 + ... + x_n^2})$ is the composition of two differentiable functions and therefore using chain rule we have

$$D_i g(x) = f'(\|x\|) D_i(\|x\|) = f'(\|x\|) \frac{x_i}{\|x\|} \quad \text{for } i = 1, ..., n,$$

and these functions are continuous. Moreover $\lim_{\|x\| \to 0} D_i g(x) = 0$ for every i, since

$$\left| f'(\|x\|) \frac{x_i}{\|x\|} \right| \leq |f'(\|x\|)| \to |f'(0)| = 0$$

as $x \to 0$, and

$$D_i g(0) = \lim_{t \to 0} \frac{g(te_i) - g(0)}{t} = \lim_{t \to 0} \frac{f(t) - f(0)}{t} = f'(0) = 0.$$

Therefore all partial derivatives of g exist and are continuous on $\mathbb{R}^n$ and thus $g \in C^1(\mathbb{R}^n)$.

2. Define the map $F : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$F(x, y, z) = (f(xy) + g(yz), g(xy) + f(yz)).$$

We have $F(1, 1, 1) = 0$ and

$$\frac{\partial(F_1, F_2)}{\partial(y, z)}(x, y, z) = \det \begin{pmatrix} f'(xy)x + g'(yz)z & g'(xy)y \\ g'(xy)x + f'(yz)z & f'(yz)y \end{pmatrix}.$$

Therefore

$$\frac{\partial(F_1, F_2)}{\partial(y, z)}(1, 1, 1) = \det \begin{pmatrix} f'(1) + g'(1) & g'(1) \\ g'(1) + f'(1) & f'(1) \end{pmatrix} = (f'(1))^2 - (g'(1))^2 \neq 0.$$

Therefore, by the Implicit Function theorem there exists a continuously differentiable function $\varphi(x) = (y(x), z(x))$ in a neighborhood W of $x = 1$ and a neighborhood U of $(1, 1, 1)$ such that $(x, y, z) \in U$ satisfies $F(x, y, z) = 0$ if and only if $F(x, \varphi(x)) = F(x, y(x), z(x)) = 0$ for $x \in W$, i.e. the equations can be solved for y and z as functions of x. Moreover we know that

$$D\varphi(1) = \begin{pmatrix} y'(1) \\ z'(1) \end{pmatrix} = - \begin{pmatrix} f'(1) + g'(1) & g'(1) \\ g'(1) + f'(1) & f'(1) \end{pmatrix}^{-1} \begin{pmatrix} f'(1) \\ g'(1) \end{pmatrix}$$

$$= - \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

3. Let $(u,v) \neq (0,0) \in \mathbb{R}^2$. Then if $x = \ln\sqrt{u^2+v^2}$ and y is such that $\cos y = u/\sqrt{u^2+v^2}, \sin y = v/\sqrt{u^2+v^2}$ we get

$$T(x,y) = (e^{\ln\sqrt{u^2+v^2}}\frac{u}{\sqrt{u^2+v^2}}, e^{\ln\sqrt{u^2+v^2}}\frac{v}{\sqrt{u^2+v^2}}) = (u,v).$$

Since $\|T(x,y)\| = e^x > 0$, $(0,0)$ does not belong to the image of T. Therefore the image of T is $\mathbb{R}^2 \setminus \{(0,0)\}$.

Since the partial derivatives of the coordinate functions of T are continuous, $T \in C^1(\mathbb{R}^2)$. Now for every $(x,y) \in \mathbb{R}^2$

$$J_T(x,y) = \det\begin{pmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{pmatrix} = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} \neq 0.$$

Thus, by the Inverse Mapping theorem, there exists an open neighborhood U of (x,y) such that $T_{|U}$ is invertible and its inverse is of class $C^1(T(U))$.

The map T is an example of a map such that $J_T(x,y) \neq 0$ for every $(x,y) \in \mathbb{R}^2$ but T is not invertible as a map from $\mathbb{R}^2$ onto $\mathbb{R}^2 \setminus \{(0,0)\}$.

4. We want to maximize f subject to the constraint $g(x) = \|x\|^2 - 1 = 0$. Since $f(x+h) - f(x) - 2Ax\cdot h = Ah\cdot h$, we have

$$\lim_{\|h\|\to 0}\frac{\|f(x+h)-f(x)-2Ax\cdot h\|}{\|h\|} = \lim_{\|h\|\to 0}\frac{\|Ah\cdot h\|\cdot h}{\|h\|} \leq \lim_{\|h\|\to 0}\frac{\|A\|\|h\|^2}{\|h\|} = 0,$$

$Df(x)(h) = 2Ax\cdot h$. Also $Dg(x)(h) = 2x\cdot h$. Since the unit sphere is compact, the maximum of f is attained at some point c. Thus by Lagrange's theorem there must exist λ such that

$$2Ac = \lambda 2c.$$

Therefore λ is an eigenvalue of A and c is an eigenvector. Moreover

$$f(c) = \lambda c\cdot c = \lambda\|c\|^2 = \lambda,$$

i.e. the maximum value of f is equal to an eigenvalue of A. Obviously the maximum value must be equal to the largest eigenvalue of A.

5. Identifying $Dg(x)$ with the gradient of g at x we need to show that the map $x \to f'(\|x\|)\frac{x}{\|x\|}$ is differentiable at 0. Recall that $Dg(0) = 0$. But

$$\lim_{\|x\|\to 0}\frac{\|f'(\|x\|)\frac{x}{\|x\|} - 0 - f''(0)x\|}{\|x\|} \leq \lim_{\|x\|\to 0}\left|\frac{\|f'(\|x\|) - f'(0)}{\|x\|} - f''(0)\right| = 0.$$

Therefore $D^2g(0)$ exists, the Hessian matrix (of second partial derivatives) at 0 is equal to $f''(0)I$, and

$$D^2g(0)(u,v) = f''(0)\sum_{i=1}^n u_i v_i.$$