Math. 4318, Test 1, 02/17/2020

Name: SOLUTIONS

1.(8 pts) Suppose f is continuous on $[0,+\infty)$, differentiable on $(0,+\infty)$, $f(0)=0$, and f' is an increasing function. Define $g(x)=\frac{f(x)}{x}$ for $x>0$. Prove that g is increasing on $(0,+\infty)$.

It is enough to show that $g'\geq 0$. Using the product rule we have

$$g'(x)=f'(x)\frac{1}{x}-f(x)\frac{1}{x^2}=\frac{xf'(x)-f(x)}{x^2}.$$

Thus we need to show that $xf'(x)\geq f(x)$ for $x>0$. This is however clear since by the Mean Value Theorem $f(x)=f(x)-f(0)=f'(c)(x-0)=f'(c)x$ for some $c\in[0,x]$. Therefore, since f' is increasing, $f(x)=f'(c)x\leq f'(x)x$.

2.(8 pts) Let f,f' be continuous on $[a,b]$. Let $c\in(a,b)$ and let f' satisfy

$$|f'(x)-f'(c)|\leq L|x-c|$$

for all $x\in(a,b)$ for some $L>0$. Show that

$$|f(x)-f(c)-f'(c)(x-c)|\leq\frac{L}{2}|x-c|^2$$

for all $x\in[a,b]$. (You will get 5 points if you can show that the inequality holds with constant L instead of $L/2$.)

By Fundamental Theorem of Calculus

$$f(x)-f(c)=\int_c^x f'(y)dy\leq\int_c^x(f'(c)+L(y-c))dy$$

$$=f'(c)(x-c)+\frac{L}{2}(y-c)^2|_c^x=f'(c)(x-c)+\frac{L}{2}|x-c|^2.$$

Similarly

$$f(x)-f(c)=\int_c^x f'(y)dy\geq\int_c^x(f'(c)-L(y-c))dy$$

$$=f'(c)(x-c)-\frac{L}{2}(y-c)^2|_c^x=f'(c)(x-c)-\frac{L}{2}|x-c|^2.$$

Therefore

$$-\frac{L}{2}|x-c|^2\leq f(x)-f(c)-f'(c)(x-c)\leq\frac{L}{2}|x-c|^2$$

which proves the claim.

3.(8 pts) Let f be Riemann integrable on $[a,b]$. Define for $x \in [a,b]$

$$F(x) = \int_a^x f(t)dt.$$

Show that F is continuous on $[a,b]$ and $F'(x)$ exists and equals $f(x)$ at every x at which f is continuous.

Let $|f(t)| \leq C$ for all $t \in [a,b]$. Then if $x \leq y$ we have

$$|F(y) - F(x)| = \left| \int_x^y f(t)dt \right| \leq \int_x^y C dt = C|y-x|.$$

Thus F is uniformly continuous on $[a,b]$ (in fact it is Lipschitz continuous with Lipschitz constant C).

Let now f be continuous at x. We notice that

$$f(x) = f(x)\frac{1}{y-x}\int_x^y dt = \frac{1}{y-x}\int_x^y f(x)dt.$$

For $\epsilon > 0$ there is $\delta > 0$ such that $|f(t) - f(x)| < \epsilon$ whenever $|t-x| < \delta$. Hence, if $|y-x| < \delta$ we have

$$\left| \frac{F(y)-F(x)}{y-x} - f(x) \right| = \left| \frac{1}{y-x} \int_x^y [f(t) - f(x)]dt \right|$$

$$\leq \frac{1}{|y-x|}\left| \int_x^y |f(t)-f(x)|dt \right| \leq \frac{1}{|y-x|}\left| \int_x^y \epsilon\, dt \right| = \epsilon.$$

Therefore

$$\lim_{y \to x} \frac{F(y)-F(x)}{y-x} = f(x),$$

i.e. $F'(x)$ exists and equals $f(x)$.

4.(8 pts) Explain why the Riemann-Stieltjes integral

$$\int_0^2 x^2 d\left([x^3] + e^{x^2}\right)$$

exists and evaluate it.

The integral exists since $f(x) = x^2$ is continuous and $g(x) = [x^3] + e^{x^2}$ is increasing and bounded on $[0,2]$. We have

$$\int_0^2 x^2 d\left([x^3] + e^{x^2}\right) = \int_0^2 x^2 d([x^3]) + \int_0^2 x^2 d(e^{x^2})$$

$$= \sum_{i=1}^{8} i^{\frac{2}{3}} + x^2 e^{x^2}|_0^2 - \int_0^2 e^{x^2} d(x^2) = \sum_{i=1}^{8} i^{\frac{2}{3}} + 4e^4 - \int_0^2 2xe^{x^2} dx$$

$$= \sum_{i=1}^{8} i^{\frac{2}{3}} + 4e^4 - e^{x^2}|_0^2 = \sum_{i=1}^{8} i^{\frac{2}{3}} + 3e^4 + 1.$$

5.(10 pts) Show that

$$\int_0^{+\infty} \frac{\sin(x+x^2)}{x^\lambda}\, dx$$

exists for $0 < \lambda < 2$.

Since

$$\lim_{x\to 0} \frac{\sin(x+x^2)}{x} = 1$$

we have for $0 < x \leq 1$

$$0 < \frac{\sin(x+x^2)}{x^\lambda} = \frac{\sin(x+x^2)}{x}\, \frac{x}{x^\lambda} \leq C\frac{1}{x^{\lambda-1}}$$

for some constant $C > 0$ so

$$\int_0^1 \frac{\sin(x+x^2)}{x^\lambda} dx$$

exists by the Comparison Test.

As regards the infinite integral on $[1, +\infty)$ we use Dirichlet's Test. The function $1/x^\lambda$ is monotonically decreasing to 0 as $x \to +\infty$. Therefore we need to show that the set $\{\int_1^c \sin(x+x^2)dx : c \geq 1\}$ is bounded. However, making the change of variables $z = x+x^2$, we have

$$\int_1^c \sin(x+x^2)dx = \int_2^{c+c^2} \frac{\sin z}{\sqrt{1+4z}}\, dz.$$

The latter integrals are bounded since the infinite integral

$$\int_2^{+\infty} \frac{\sin z}{\sqrt{1+4z}}\, dz$$

exists, which can be seen after another application of Dirichlet's Test.