Math. 4347, Homework assignment, due on November 20, 2017.

1. (6 pts) Solve the following initial value problem. Find the shock curves and check that the entropy condition is satisfied. Draw a picture depicting what happens for all times $t \geq 0$ that include characteristics and the shock curves.

$$u_t + (u^2)_x = 0$$
$$u(x,0) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| > 1. \end{cases}$$

2. (5 pts) Find the general solution of

$$u_{xx} + 2u_{xy} - 3u_{yy} - u_x + u_y = 0.$$

3. (5 pts) Find Green's function for $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < r^2, y > 0\}$.

4. (6 pts) Let u be a harmonic function in $\mathbb{R}^3$ and let $a > 0$. Show that the function v defined in spherical coordinates by

$$v(r,\theta,\psi) = \frac{a}{r} u(\frac{a^2}{r}, \theta, \psi)$$

is harmonic in $\mathbb{R}^3 \setminus \{0\}$.

5. (6 pts) Solve the initial boundary value problem

$$u_{tt} = 4u_{xx} \quad x > 0, t > 0$$
$$u(x,0) = \frac{x^2}{8}, \; u_t(x,0) = x \quad x \geq 0$$
$$u(0,t) = t^2 \quad t \geq 0.$$

6. (6 pts) Use separation of variables to solve the wave equation

$$\begin{cases} u_{tt} = a^2 u_{xx} & \text{for } 0 < x < 1, t > 0, \\ u(x,0) = 0, \; u_t(x,0) = \cos \pi x & \text{for } 0 < x < 1, \\ u_x(0,t) = u_x(1,t) = 0 & \text{for } 0 < t. \end{cases}$$

7. (6 pts) Let $u \in C^2(\mathbb{R} \times [0,+\infty))$ be the solution of

$$u_{tt} - u_{xx} = 0 \quad x \in \mathbb{R}, t > 0$$
$$u(x,0) = f(x), \; u_t(x,0) = g(x) \quad x \in \mathbb{R}.$$

Suppose that the functions f, g are equal to 0 for $|x| > K$ for some $K > 0$. Show that the "kinetic energy" of the solution

$$k(t) = \frac{1}{2} \int_{-\infty}^{+\infty} u_t^2(x,t)\,dx$$

is equal to the "potential energy" of the solution

$$p(t) = \frac{1}{2} \int_{-\infty}^{+\infty} u_x^2(x,t)\,dx$$

for all large enough times t. (Hint: Use D'Alembert's solution formula.)