Math. 4581, Suggested homework problems 1.

1. Show that $(1, i, -1, -i), (1, -1, 1, -1), (1, -i, -1, i)$ are orthogonal.
2. Find (f, g) in $L^2(-\pi, \pi)$ if $f(x) = e^{ix}, g(x) = e^{-ix}$.
3. Find $\|f\|$ in $L^2(-1, 1)$ if $f(x) = -1$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$.
4. Let $L > 0$. Calculate

(a)

$$(\cos \frac{n\pi x}{L}, \cos \frac{m\pi x}{L}) \quad \text{in } L^2(-L, L),$$

(b)

$$(\sin \frac{n\pi x}{L}, \cos \frac{m\pi x}{L}) \quad \text{in } L^2(-L, L),$$

where n, m are nonnegative integers.

5. Find an orthonormal basis in $L^2(-1, 1)$ for $span\{1, x, x^2, x^3\}$.
6. Define Legendre polynomials $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

(a) Are these functions mutually orthogonal in $L^2(-1, 1)$?
(b) Are they orthonormal?
(c) Is $span\{P_0, P_1, P_2, P_3\} = span\{1, x, x^2, x^3\}$?

7. Find $f(x) = ax^2 + bx + c$ such that in $L^2(-1, 1)$

$$\|f - e^x\| = \min_{g \in span\{1, x, x^2\}} \|g - e^x\|,$$

i.e. find a quadratic polynomial that best approximates e^x on $[-1, 1]$ in the least squares sense.

Answers:
2. 0.
3. $\sqrt{2}$.
4. (a) 0 if $n \neq m$, L if $n = m > 0$, and $2L$ if $n = m = 0$.
(b) 0 for all n, m.
5.

$$\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \frac{1}{2}\sqrt{\frac{5}{2}}(3x^2 - 1), \frac{1}{2}\sqrt{\frac{7}{2}}(5x^3 - 3x).$$

6. (a) Yes. (b) No. (c) Yes.
7. We compute

$$\|P_0\| = \sqrt{2}, \quad \|P_1\| = \sqrt{\frac{2}{3}}, \quad \|P_2\| = \sqrt{\frac{2}{5}}$$

and we set

$$\tilde{P}_0 = \frac{P_0}{\|P_0\|} = \frac{1}{\sqrt{2}}, \quad \tilde{P}_1 = \frac{P_1}{\|P_1\|} = \sqrt{\frac{3}{2}}x, \quad \tilde{P}_2 = \frac{P_2}{\|P_2\|} = \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1).$$

Then $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2$ are orthonormal and $span\{1, x, x^2\} = span\{P_0, P_1, P_2\} = span\{\tilde{P}_0, \tilde{P}_1, \tilde{P}_2\}$. The f that minimizes the distance is equal to the orthogonal projection of e^x onto $span\{1, x, x^2\}$ Therefore we have

$$f = proj(e^x; span\{1, x, x^2\}) = proj(e^x; span\{\tilde{P}_0, \tilde{P}_1, \tilde{P}_2\}) = \sum_{i=0}^{2}(e^x, \tilde{P}_i)\tilde{P}_i.$$

Thus

$$f(x) = \left(\int_{-1}^{1} \frac{1}{\sqrt{2}}e^x dx\right)\frac{1}{\sqrt{2}} + \left(\int_{-1}^{1} \sqrt{\frac{3}{2}}xe^x dx\right)\sqrt{\frac{3}{2}}x + \left(\int_{-1}^{1} \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)e^x dx\right)\frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)$$

$$= \left(\frac{33}{4e} - \frac{3e}{4}\right) + \frac{3}{e}x + \left(\frac{15e}{4} - \frac{105}{4e}\right)x^2.$$