SUGGESTED HOMEWORK PROBLEMS

①
$$\begin{cases} u_{xx} + u_{yy} = 0, & 0<x<1,\ 0<x<1 \\ u(0,y) = \sin \pi y,\ u(1,y) = 0, & 0<y<1 \\ u(x,0) = u(x,1) = 0, & 0<x<1 \end{cases}$$

②
$$\begin{cases} \nabla^2 u = 0 & 0<x<1,\ 0<y<1 \\ u(0,y) = \sin \pi(\tfrac{1}{2}-y),\ u(1,y) = 0, & 0<y<1 \\ u_y(x,0) = x(1-x),\ u_y(x,1) = 0, & 0<x<1 \end{cases}$$

③
$$\begin{cases} \nabla^2 u = 0, & 0<x<1,\ y>0 \\ u(x,0) = x - \frac{x^2}{2}, & 0<x<1 \\ u(0,y) = u_x(1,y) = 0, & y>0 \\ u \text{ is bounded} \end{cases}$$

④
$$\begin{cases} u_{xx} + u_{yy} = 0, & x>0,\ 0<y<1 \\ u(0,y) = f(y), & 0<y<1 \\ u_y(x,0) = u_y(x,1) = 0, & x>0 \\ u \text{ is bounded} \end{cases}$$

1 $u(x,y) = X(x)\,Y(y)$

Then we have

$$-Y''(y) = \mu Y(y), \quad Y(0) = Y(1) = 0$$

$$X''(x) = \mu X(x), \quad X(1) = 0$$

Solving the first we get

$$\boxed{\mu_n = (n\pi)^2, \quad Y_n(y) = \sin n\pi y} \quad n = 1, 2, \ldots$$

and then

$$X_n(x) = A_n e^{n\pi(x-1)} + B_n e^{-n\pi(x-1)}$$, which satisfies $X_n(1) = 0$ if $A_n = -B_n$

$$\therefore \boxed{X_n(x) = A_n \sinh n\pi(x-1)} \quad n = 1, 2, \ldots$$

$$\therefore u(x,y) = \sum_{n=1}^{\infty} A_n \sinh n\pi(x-1) \sin n\pi y$$

Now $u(0,y) = \sum_{n=1}^{\infty} A_n \sinh(-n\pi) \sin n\pi y = \sin \pi y$

this is already Fourier series expansion of $\sin \pi y$

$\therefore A_n = 0$ for $n \neq 1$

and $A_1 = \dfrac{1}{\sinh(-\pi)}$

$$\boxed{u(x,y) = \frac{1}{\sinh(-\pi)} \sinh \pi(x-1) \sin \pi y}$$ is the solution

2. Solve 2 subproblems:

① $\begin{cases} \nabla^2 u = 0 \\ u(0,y) = \sin\pi(\frac{1}{2}-y),\ u(1,y)=0 \\ u_y(x,0) = u_y(x,1) = 0 \end{cases}$

② $\begin{cases} \nabla^2 u = 0 \\ u(0,y) = u(1,y) = 0 \\ u_y(x,0) = x(1-x),\ u_y(x,1) = 0 \end{cases}$

First one is similar to the one in #3.

$u(x,y) = X(x)Y(y)$ and then

$-Y'' = \mu Y \qquad Y'(0) = Y'(1) = 0$

$X'' = \mu X \qquad X(1) = 0$

First equation is SL problem and we have:

$$\boxed{\begin{aligned} &\mu_0 = 0,\ Y_0 = 1 \\ &\mu_n = (n\pi)^2,\ Y_n(y) = \cos n\pi y \end{aligned}}$$

Then $\boxed{\begin{aligned} &X_0(x) = A_0 x - A_0 \\ &X_n(x) = A_n \sinh n\pi(x-1) \end{aligned}}$

$$\therefore u_1(x,y) = A_0 x - A_0 + \sum_{n=1}^{\infty} A_n \sinh n\pi(x-1)\cos n\pi y$$

$$u_1(0,y) = -A_0 + \sum_{n=1}^{\infty} A_n \sinh(-n\pi)\cos n\pi y = \sin\pi(\tfrac{1}{2}-y) = \cos(-\pi y)$$
$$= \cos\pi y$$

$$\therefore A_n = 0 \text{ if } n \neq 1 \text{ and } A_1 = \frac{1}{\sinh(-\pi)}$$

$$\therefore \boxed{u_1(x,y) = \frac{1}{\sinh(-\pi)} \sinh\pi(x-1)\cos\pi y}$$

Second problem: Separation of variables gives

$$-X''(x) = \mu X(x), \quad X(0) = X(1) = 0$$

$$Y''(y) = \mu Y(y), \quad Y'(1) = 0$$

$$\boxed{\begin{aligned} &\mu_n = (n\pi)^2, \quad X_n(x) = \sin n\pi x \\ &Y_n(y) = A_n \cosh n\pi (y-1) \end{aligned}}$$

$$\therefore \; u_2(x,y) = \sum_{n=1}^{\infty} A_n \cosh n\pi(y-1) \sin n\pi x$$

$$u_2(x,0) = \sum_{n=1}^{\infty} A_n n\pi \sinh(-n\pi) \sin n\pi x = x(1-x)$$

$$x(1-x) = 2\sum_{n=1}^{\infty} \left(x(1-x), \sin n\pi x\right) \sin n\pi x$$

$$\left(x(1-x), \sin n\pi x\right) = \int_0^1 x(1-x)\sin n\pi x \, dx = \underbrace{\frac{2}{(n\pi)^3}\left(1-(-1)^n\right)}_{\text{Integrate by parts}}$$

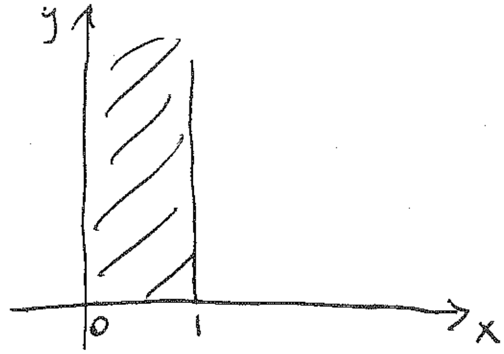
$$\therefore \; A_n = \frac{4\left(1-(-1)^n\right)}{(n\pi)^4 \sinh(-n\pi)}$$

$$\therefore \boxed{\begin{aligned} u(x,y) = u_1(x,y) + u_2(x,y) = &\frac{1}{\sinh(-\pi)} \sinh \pi(x-1) \cos \pi y \\ &+ \sum_{n=1}^{\infty} \frac{4\left(1-(-1)^n\right)}{(n\pi)^4 \sinh(-n\pi)} \cosh n\pi(y-1) \sin n\pi x \end{aligned}}$$

$\boxed{3}$

Solve first

$$\begin{cases} \nabla^2 u = 0 \\ u(0,y) = u_x(1,y) = 0 \\ u \text{ bounded} \end{cases}$$



Separate variables:

$$-X''(x) = \mu X(x), \quad X(0) = X'(1) = 0$$

$$Y''(y) = \mu Y(y), \quad Y(y) \text{ is bounded}$$

Then $\boxed{\mu_n = \frac{(2n+1)^2\pi^2}{4}, \quad X_n(x) = \sin \frac{(2n+1)\pi}{2} x, \quad n = 0,1,2,\ldots}$

$$Y_n(y) = A_n e^{\frac{2n+1}{2}\pi y} + B_n e^{-\frac{2n+1}{2}\pi y}$$

$Y(y)$ bounded $\Longrightarrow A_n = 0 \quad \therefore \boxed{Y_n(y) = B_n e^{-\frac{2n+1}{2}\pi y}}$

$$\therefore u(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{2n+1}{2}\pi x = x - \frac{x^2}{2}$$

$$x - \frac{x^2}{2} = 2\sum_{n=0}^{\infty} \left(x - \frac{x^2}{2}, \sin \frac{2n+1}{2}\pi x\right) \sin \frac{2n+1}{2}\pi x$$

$$\left(x - \frac{x^2}{2}, \sin \frac{2n+1}{2}\pi\right) = \left(x - \frac{x^2}{2}\right) \frac{-\cos \frac{2n+1}{2}\pi x}{\frac{2n+1}{2}\pi}\Bigg|_0^1 + \int_0^1 (1-x) \frac{\cos \frac{2n+1}{2}\pi x}{\frac{2n+1}{2}\pi} dx$$

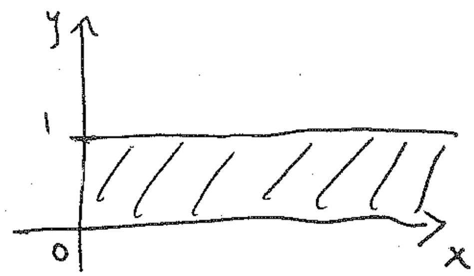
$$= 0 + (1-x)\frac{\sin \frac{2n+1}{2}\pi x}{\left(\frac{2n+1}{2}\pi\right)^2}\Bigg|_0^1 + \frac{1}{\left(\frac{2n+1}{2}\pi\right)^2} \int_0^1 \sin \frac{2n+1}{2}\pi x \, dx$$

$$= \frac{-1}{\left(\frac{2n+1}{2}\pi\right)^3} \cos \frac{2n+1}{2}\pi x \Bigg|_0^1 = \frac{8}{(2n+1)^3\pi^3}$$

$$\therefore B_n = \frac{16}{(2n+1)^3\pi^3}$$

$$\therefore \boxed{u(x,y) = 16\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3\pi^3} e^{-\frac{2n+1}{2}\pi y} \sin\frac{2n+1}{2}\pi x}$$

$\boxed{4}$ Solve first $\begin{cases} \nabla^2 u = 0 \\ u_y(x,0) = u_y(x,1) = 0 \\ u \text{ bounded} \end{cases}$



Separation of variables

$$-Y''(y) = \mu Y, \quad Y'(0) = Y'(1) = 0$$

$$X''(x) = \mu X, \quad X(x) \text{ is bounded}$$

$$\therefore \boxed{\begin{array}{l} \mu_0 = 0, \quad Y_0 = 1 \\ \mu_n = (n\pi)^2, \quad Y_n = \cos n\pi y \end{array}}$$

Then $X_0(x) = A_0 x + B_0$ is bounded if $A_0 = 0$

$$\therefore \boxed{X_0 = B_0}$$

$X_n(x) = A_n e^{n\pi x} + B_n e^{-n\pi x}$ is bounded if $A_n = 0$

$$\therefore \boxed{X_n(x) = B_n e^{-n\pi x}} \quad \text{for } n = 1, 2, \ldots$$

$$\therefore u(x,y) = \sum_{n=1}^{\infty} B_n e^{-n\pi x} \cos n\pi y$$

$$u(0,y) = f(y) = \sum_{n=0}^{\infty} B_n \cos n\pi y$$

But $f(y) = (f,1) + \sum_{n=1}^{\infty} \frac{(f, \cos n\pi y)}{\|\cos n\pi y\|^2} \cos n\pi y$

$$= \int_0^1 f(y)\,dy + 2\sum_{n=1}^{\infty} \int_0^1 f(y)\cos n\pi y\,dy \,\cos n\pi y$$

$$\therefore \quad B_0 = \int_0^1 f(y)\,dy$$

$$B_n = 2\int_0^1 f(y)\cos n\pi y\,dy \;, \quad n = 1, 2 \ldots$$

and thus

$$\boxed{u(x,y) = \int_0^1 f(y)\,dy + 2\sum_{n=1}^{\infty} \int_0^1 f(y)\cos n\pi y\,dy\; e^{-n\pi x} \cos n\pi y}$$