SUGGESTED HOMEWORK PROBLEMS

(1) Find u satisfying

$$\begin{cases} u_t = D u_{xx}, & 0 < x < 1,\ t > 0 \\ u(x,0) = \begin{cases} 1 & \frac{1}{3} < x < \frac{2}{3} \\ 0 & 0 < x < \frac{1}{3} \text{ or } \frac{2}{3} < x < 1 \end{cases} \\ u(0,t) = 0,\ u_x(1,t) = 0, & t > 0 \end{cases}$$

(2) Solve the problem

$$\begin{cases} u_t = D u_{xx} + \alpha u, & 0 < x < 1,\ t > 0 \\ u(x,0) = x(1-x), & 0 < x < 1 \\ u(0,t) = u(1,t) = 0, & t > 0 \end{cases}$$

(3) Show that the solution v of the problem

$$\begin{cases} v_t = v_{xx}, & 0 < x < 1,\ t > 0 \\ v(x,0) = 0, & 0 < x < 1 \\ v(0,t) = 1,\ v(1,t) = 0, & t > 0 \end{cases}$$

tends to a time independent steady-state solution $v_e(x)$ as $t \to +\infty$. Show (by direct computation) that $v_e(x)$ satisfies

$$\begin{cases} v_e'' = 0, & 0 < x < 1 \\ v_e(0) = 1,\ v_e(1) = 0 \end{cases}$$

(1)

$u(x,y) = X(x)\,T(t)$ yields

$$-X''(x) = \mu X(x) \qquad X(0) = X'(1) = 0$$
$$-T'(t) = \mu D\, T(t)$$

Eigenvalues and eigenfunctions for the SL Problem are

$$\mu_n = \left(n-\tfrac{1}{2}\right)^2 \pi^2 \quad , \quad X_n(x) = \sin\left(n-\tfrac{1}{2}\right)\pi x \qquad n = 1, 2, \ldots$$

Then $T_n(t) = C_n e^{-\left(n-\frac{1}{2}\right)^2 \pi^2 D t}$

$$\therefore \quad u(x,y) = \sum_{n=1}^{\infty} C_n e^{-\left(n-\frac{1}{2}\right)^2 \pi^2 D t} \sin\left(n-\tfrac{1}{2}\right)\pi x$$

$$(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(n-\tfrac{1}{2}\right)\pi x = \begin{cases} 1 & \frac{1}{3} < x < \frac{2}{3} \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad C_n = 2\int_{\frac{1}{3}}^{\frac{2}{3}} \sin\left(n-\tfrac{1}{2}\right)\pi x \, dx = \frac{2\left(\cos\left(n-\frac{1}{2}\right)\frac{\pi}{3} - \cos\left(n-\frac{1}{2}\right)\frac{2\pi}{3}\right)}{\left(n-\frac{1}{2}\right)\pi}$$

(2) Change of variables:

$$u(x,t) = e^{\alpha t}\, v(x,t)$$

Then $v(x,t)$ satisfies

$$\begin{cases} v_t(x,t) = D\, v_{xx}(x,t) & 0 < x < 1, \quad t > 0 \\ v(x,0) = x(1-x) & 0 < x < 1 \\ v(0,t) = v(1,t) = 0 & t > 0 \end{cases}$$

Eigenvalues and eigenfunctions are:

$$\mu_n = (n\pi)^2, \quad X_n(x) = \sin n\pi x,$$

$$\therefore \quad v(x,t) = \sum_{n=1}^{\infty} C_n e^{-(n\pi)^2 Dt} \sin n\pi x,$$

where $C_n = 2\int_0^1 x(1-x)\sin n\pi x\,dx = -\dfrac{2x(1-x)\cos n\pi x}{n\pi}\Big|_0^1$

$$+ 2\int_0^1 \frac{(1-2x)\cos n\pi x}{n\pi}\,dx = \frac{2}{(n\pi)^2}(1-2x)\sin n\pi x\Big|_0^1 + \frac{4}{(n\pi)^2}\int_0^1 \sin n\pi x$$

$$= -\frac{4}{(n\pi)^3}\cos n\pi x\Big|_0^1 = \frac{4}{(n\pi)^3}\left((-1)^{n+1}+1\right).$$

$$\therefore \boxed{u(x,t) = e^{\alpha t}\sum_{n=1}^{\infty} \frac{4}{(n\pi)^3}\left((-1)^{n+1}+1\right) e^{-(n\pi)^2 Dt} \sin n\pi x}$$

(3) The function v satisfies the original problem if and only if $u(x,t) = v(x,t) - (1-x)$ satisfies

$$\begin{cases} u_t = u_{xx}, & 0<x<1,\ t>0 \\ u(x,0) = x-1, & 0<x<1 \\ u(0,t) = u(1,t) = 0, & t>0 \end{cases}$$

By separation of variables we obtain

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi)^2 t} \sin n\pi x,$$

where

$$c_n = 2\int_0^1 (x-1)\sin n\pi x \, dx = -\frac{2}{n\pi}$$

Therefore

$$u(x,t) = (1-x) + \sum_{n=1}^{\infty} \left(-\frac{2}{n\pi}\right) e^{-(n\pi)^2 t} \sin n\pi x$$

We use $e^y \geq 1+y$. Thus $e^{-y} \leq \frac{1}{1+y}$ and hence

$$e^{-(n\pi)^2 t} = e^{-t} e^{-[(n\pi)^2-1]t} \leq e^{-t}\frac{1}{1+[(n\pi)^2-1]t} < \frac{e^{-t}}{(n\pi)^2} \quad \text{if } t \geq 2$$

$$\therefore \left| \sum_{n=1}^{\infty} \left(-\frac{2}{n\pi}\right) e^{-(n\pi)^2 t} \sin n\pi x \right| \leq e^{-t} \underbrace{\sum_{n=1}^{\infty} \frac{2}{(n\pi)^3}}_{\text{convergent series}} \Longrightarrow 0 \quad \text{as } t \to +\infty$$

$$\therefore \lim_{t\to+\infty} u(x,t) = 1-x =: V_e(x),$$ which obviously satisfies

$$\begin{cases} V_e'' = 0, & 0<x<1 \\ V_e(0) = 1, \ V_e(1) = 0 \end{cases}$$