Math. 4581, Practice Final: Answers

1.

$$u(r,\theta) = A_0 + 4r\cos\theta - \frac{1}{2}r^2\cos 2\theta + \frac{1}{3}r^3\cos 3\theta$$

2.

$$\mu_{nm} = -\left[\left(n+\frac{1}{2}\right)^2 + \left(m+\frac{1}{2}\right)^2\right]\pi^2, \quad n,m = 0,1,2,...$$

$$X_{nm}(x,y) = \cos\left((n+\frac{1}{2})\pi x\right)\sin\left((m+\frac{1}{2})\pi y\right)$$

3.

$$u(x,t) = T_0 + 2(T_1 - T_0)\sum_{n=0}^{\infty}\frac{1}{\pi(n+\frac{1}{2})}e^{-D(n+\frac{1}{2})^2\pi^2 t}\sin((n+\frac{1}{2})\pi x)$$

4.

$$u(x,t) = \frac{1}{a\pi}\sin(a\pi t)\cos(\pi x) + \sum_{n=0}^{\infty}u_n(t)\cos(n\pi x),$$

where

$$u_0(t) = \frac{t^2}{\pi}, u_n(t) = \frac{A_n}{(n\pi a)^2}(1-\cos(n\pi a t))$$

$$A_n = 0 \text{ if } n \text{ is odd}, \quad A_n = \frac{-4}{\pi(n^2-1)} \text{ if } n \text{ is even.}$$

5. $u(0.5, 3) = -0.5$

6. $Y(t) = e^{-t} + ct^3e^{-t}$.

7. Denote $U(x,s) = \mathcal{L}\{u(x,\cdot)\}(t)$. After taking the Laplace transform we have

$$U_{xx} - (s+\gamma^2)U = -T_0 - \frac{\gamma^2 T}{s}$$

$$U_x(0,s) = U_x(1,s) = 0.$$

Solving the above differential equation we obtain

$$U(x,s) = c_1 e^{\sqrt{s+\gamma^2}x} + c_2 e^{-\sqrt{s+\gamma^2}x} + \frac{T_0}{s+\gamma^2} + \frac{\gamma^2 T}{s(s+\gamma^2)}.$$

The boundary conditions imply

$$0 = U_x(0,s) = c_1 - c_2, \qquad 0 = U_x(1,s) = c_1\sqrt{s+\gamma^2}e^{\sqrt{s+\gamma^2}} - c_2\sqrt{s+\gamma^2}e^{\sqrt{s+\gamma^2}}$$

which give $c_1 = c_2 = 0$. Thus

$$U(x,s) = \frac{T_0}{s+\gamma^2} + \frac{\gamma^2 T}{s(s+\gamma^2)}.$$

Taking the inverse Laplace transform we get

$$u(x,t) = T + (T_0 - T)e^{-\gamma^2 t}$$

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