Math. 4581, Practice Final

1. Find the bounded solution of

$$\nabla^2 u(x,y) = 0 \quad \text{in } D = \{(x,y) : x^2 + y^2 \le 1\}$$

assuming that on the boundary of D, $\frac{\partial u}{\partial n}$ (when written in polar coordinates) is equal to $f(\theta) = 4\cos\theta - \cos 2\theta + \cos 3\theta$. (Hint: Write the problem in polar coordinates (r,θ) and use the fact that $\frac{\partial u}{\partial n} = u_r$.)

2. Find the eigenfunctions and eigenvalues for the problem

$$\begin{cases} \nabla^2 u(x,y) = \mu u(x,y) & \text{for } 0 < x, y < 1, \\ u_x(0,y) = u(1,y) = 0 & \text{for } 0 < y < 1, \\ u(x,0) = u_y(x,1) = 0 & \text{for } 0 < x < 1. \end{cases}$$

3. Solve the heat equation

$$\begin{cases} u_t(x,t) = D u_{xx}(x,t) & \text{for } 0 < x < 1, t > 0, \\ u(x,0) = T_1 & \text{for } 0 < x < 1, \\ u(0,t) = T_0,\ u_x(1,t) = 0 & \text{for } 0 < t. \end{cases}$$

4. Solve the wave equation

$$\begin{cases} u_{tt}(x,t) = a^2 u_{xx}(x,t) + \sin \pi x & \text{for } 0 < x < 1, t > 0, \\ u(x,0) = 0,\ u_t(x,0) = \cos \pi x & \text{for } 0 < x < 1, \\ u_x(0,t) = u_x(1,t) = 0 & \text{for } 0 < t. \end{cases}$$

5. Let u solve

$$\begin{cases} u_{tt}(x,t) = 4u_{xx}(x,t) & 0 < x < 1, t > 0 \\ u(0,t) = u_x(1,t) = 0, & t > 0 \\ u(x,0) = x, u_t(x,0) = 1 & 0 < x < 1. \end{cases}$$

Find $u(0.5, 3)$.

6. Solve

$$\begin{cases} tY''(t) + 2(t-1)Y'(t) + (t-2)Y(t) = 0 \\ Y(0) = 1. \end{cases}$$

7. Use Laplace transform to solve the heat conduction problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) - \gamma^2(u(x,t) - T), & 0 < x < 1, t > 0 \\ u_x(0,t) = u_x(1,t) = 0, & t > 0 \\ u(x,0) = T_0 & 0 < x < 1. \end{cases}$$