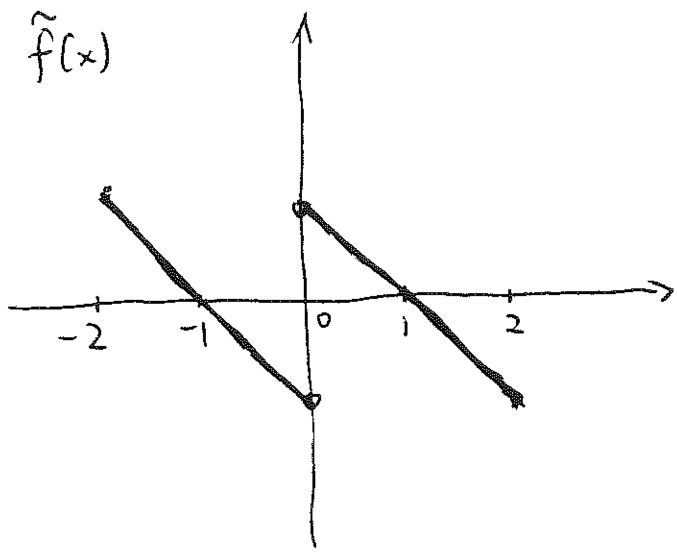
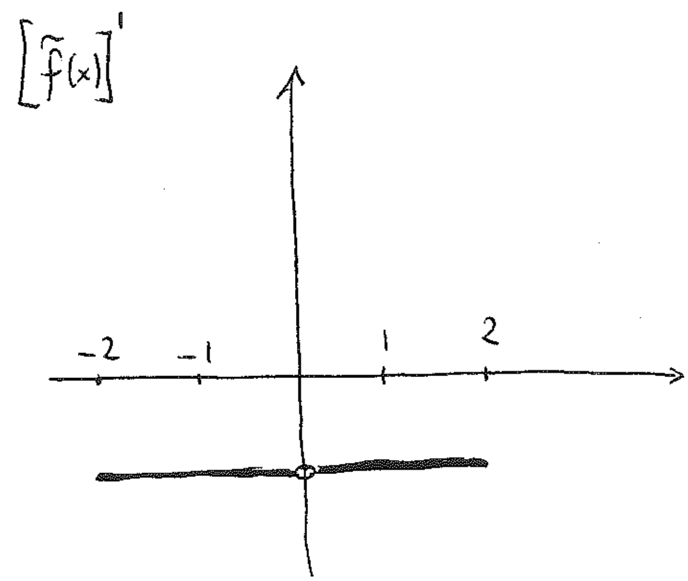
MATH-4581 PRACTICE TEST 1, SOLUTIONS

① (a)





(b) f is odd $\Longrightarrow a_n = 0$ for $n = 0, 1, 2, \ldots$

$$b_n = 2\int_0^1 (1-x)\sin n\pi x \, dx = \frac{2}{n\pi}(1-x)\left(-\cos n\pi x\right)\Big|_0^1 - \frac{2}{n\pi}\int_0^1 \cos n\pi x \, dx$$

$$= \frac{2}{n\pi} - \frac{2}{(n\pi)^2}\sin n\pi x \Big|_0^1 = \boxed{\frac{2}{n\pi}} \qquad \text{for } n = 1, 2, 3, \ldots$$

(c) The differentiated Fourier series for f is

$$\sum_{n=1}^{\infty} 2\cos n\pi x$$

which is divergent and so it does not converge to $[\tilde{f}(x)]'$.
(Notice that $\tilde{f}$ is not continuous so no convergence theorem applies.)

② The family is not orthonormal since it is not orthogonal, for instance $(1, \sqrt{2}\sin\pi x) \neq 0$.

$(1+\cos\pi x)\sin x = \sin\pi x + \frac{1}{2}\sin 2\pi x$ and therefore it belongs to the space spanned by the family.

3 (a) $\boxed{\lambda<0}$, $\lambda=-\beta^2$, $\beta>0$. Then $u(x)=Ae^{\beta x}+Be^{-\beta x}$

$u'(x)=\beta Ae^{\beta x}-\beta Be^{-\beta x}$.

$$\therefore \begin{cases} A-B=0 \\ Ae^{\beta\frac{\pi}{2}}-Be^{-\beta\frac{\pi}{2}}=0 \end{cases} \implies A=B=0$$ No negative eigenvalues

$\boxed{\lambda=0}$ $u(x)=Ax+B$ and $\boxed{u_0(x)=1}$ is an eigenvector

$\boxed{\lambda>0}$, $\lambda=\beta^2$; $\beta>0$

Then $u(x)=A\cos\beta x+B\sin\beta x$, $u'(x)=-A\beta\sin\beta x+B\beta\cos\beta x$

$$\therefore \begin{cases} B=0 \\ -A\beta\sin\beta\frac{\pi}{2}+B\beta\cos\beta\frac{\pi}{2}=0 \end{cases} \implies \frac{\beta\pi}{2}=n\pi \iff \beta=2n,\ n=1,2,\ldots$$

$$\boxed{u_n(x)=\cos 2nx}$$ are eigenvectors for eigenvalues $\lambda_n=4n^2$, $n=1,2,\ldots$

(b) The two smallest eigenvalues are 0 and 4 and so (in the current notation) $u_1(x)=1$, $u_2(x)=\cos 2x$.

$$\mathrm{Proj}(f;\operatorname{span}\{u_1,u_2\})=\frac{(\sin 2x,u_1)}{\|u_1\|^2}u_1+\frac{(\sin 2x,u_2)}{\|u_2\|^2}u_2$$

$$=\frac{\int_0^{\frac{\pi}{2}}\sin 2x\,dx}{\frac{\pi}{2}}+\frac{\int_0^{\frac{\pi}{2}}\sin 2x\cos 2x\,dx}{\int_0^{\frac{\pi}{2}}\cos^2 2x\,dx}\cos 2x$$

$$=-\frac{\cos 2x}{\pi}\Big|_0^{\frac{\pi}{2}}+\frac{\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin 4x}{\int_0^{\frac{\pi}{2}}\cos^2 2x\,dx}\cos 2x=\frac{2}{\pi}+0$$

$=\frac{2}{\pi}$ so the projection is the constant function $g(x)\equiv\frac{2}{\pi}$.