Math. 4581, Practice Test 2

1. Solve the following boundary value problem:

$$\begin{cases} \nabla^2 u(x,y) = x\cos \pi y & \text{for } 0 < x, y < 1, \\ u(0,y) = u(1,y) = 0 & \text{for } 0 < y < 1, \\ u_y(x,0) = \sin \pi x,\ u_y(x,1) = 0 & \text{for } 0 < x < 1. \end{cases}$$

2 subproblems:

(1) $\begin{cases} \nabla^2 u = 0 \\ u(0,y) = u(1,y) = 0 \\ u_y(x,0) = \sin \pi x, u_y(x,1) = 0 \end{cases}$

(2) $\begin{cases} \nabla^2 u = x\cos \pi y \\ u(0,y) = u(1,y) = 0 \\ u_y(x,0) = u_y(x,1) = 0 \end{cases}$

(1) $u(x,y) = X(x)Y(y)$. Then

$-X''(x) = \mu X(x)$, $X(0) = X(1) = 0$

$Y''(y) = \mu Y(y)$, $Y'(1) = 0$

Solving the SL problem we get $\mu_n = (n\pi)^2$, $X_n(x) = \sin n\pi x$ for $n = 1, 2, 3, \ldots$

Then $Y_n(y) = A_n \cosh (y-1)n\pi$

$\therefore$ $u(x,y) = \sum_{n=1}^{\infty} A_n \cosh n\pi(y-1) \sin n\pi x$ solves the homogeneous part of (1).

$u_y(x,0) = \sum_{n=1}^{\infty} A_n n\pi \sinh(-n\pi) \sin n\pi x = \sin \pi x$

Since $\sin \pi x$ is equal to its expansion in terms of $\{\sin n\pi x\}_{n=1}^{\infty}$ we obtain $A_n = 0$ for $n = 2, 3, \ldots$, and

$$A_1 = \frac{-1}{\pi \sinh \pi}$$

$$\therefore\ u_1(x,y) = \frac{-1}{\pi \sinh \pi} \cosh \pi(y-1) \sin \pi x \text{ solves (1)}$$

(2)

Orthogonal bases of eigenfunctions associated with homogeneous boundary conditions are $\{\sin n\pi x\}$, $n=1,2,\ldots$, and $\{\cos m\pi y\}$, $m=0,1,2,\ldots$

Therefore we will look for a solution in the form

$$u(x,y)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} u_{nm}\sin n\pi x\cos m\pi y$$

$$\nabla^2 u(x,y)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} u_{nm}(-n^2-m^2)\pi^2\sin n\pi x\cos m\pi y$$

Now expand:

$$x\cos\pi y=4\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left(x\cos\pi y,\sin n\pi x\cos m\pi y\right)\sin n\pi x\cos m\pi y$$

$$\left(x\cos\pi y,\sin n\pi x\cos m\pi y\right)=\int_0^1\int_0^1 x\cos\pi y\sin n\pi x\cos m\pi y\,dy\,dx$$

$$=\begin{cases}0 & \text{if } m\neq 1\\ \dfrac{(-1)^{n+1}}{2n\pi} & \text{if } m=1\end{cases}$$

Therefore we get from the equation

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} u_{nm}(-n^2-m^2)\pi^2\sin n\pi x\cos m\pi y=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n\pi}\sin n\pi x\cos\pi y$$

$$\therefore\quad u_{nm}=\begin{cases}0 & \text{if } m\neq 1\\ \dfrac{2(-1)^n}{n\pi^3(n^2+1)} & \text{if } m=1\end{cases}$$

$$\therefore\quad u_2(x,y)=\sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi^3(n^2+1)}\sin n\pi x\cos\pi y \quad \text{solves (2)}$$

$$\therefore\quad \boxed{u(x,y)=u_1(x,y)+u_2(x,y)=\frac{-1}{\pi\sinh\pi}\cosh\pi(y-1)\sin\pi x+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi^3(n^2+1)}\sin n\pi x\cos\pi y}$$