PRACTICE TEST: SOLUTIONS

(1) The function

$$v(t,x) = e^{\frac{\sigma}{2}t}\,u(t,x)$$

satisfies

$$\begin{cases} v_{tt} = a^2 v_{xx} + \frac{\sigma^2}{4} v \\ v(0,x) = f(x), \quad v_t(0,x) = \frac{\sigma}{2} f(x) \\ v(t,0) = v(t,1) = 0 \end{cases}$$

Separation of variables now gives

$$\frac{T''}{a^2 T} - \frac{\sigma^2}{4a^2} = \frac{X''}{X} = -\mu = \text{const.}$$

Therefore

$$X'' + \mu X = 0, \quad X(0) = X(1) = 0$$

$$T'' + \left(a^2\mu - \frac{\sigma^2}{4}\right) T = 0$$

The first problem gives us $\mu_n = (n\pi)^2$, $X_n(x) = \sin n\pi x$, $n = 1, 2, \ldots$

Since σ is small we can assume that

$$a^2 (n\pi)^2 - \frac{\sigma^2}{4} > 0 \quad \text{for all } n = 1, 2, \ldots$$

and denoting $\theta_n^2 = a^2 - \frac{\sigma^2}{(2n\pi)^2}$ we obtain

$$T_n(t) = A_n \cos(n\pi\theta_n t) + B_n \sin(n\pi\theta_n t)$$

The solution v thus has the form

$$v(t,x) = \sum_{n=1}^{\infty} \left[A_n \cos(n\pi\theta_n t) + B_n \sin(n\pi\theta_n t) \right] \sin n\pi x .$$

Initial conditions imply

$$f(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x \implies A_n = 2 \int_0^1 f(x) \sin n\pi x \, dx$$

$$\frac{6}{2} f(x) = \sum_{n=1}^{\infty} n\pi\theta_n B_n \sin n\pi x \implies B_n = \frac{6}{n\pi\theta_n} \int_0^1 f(x) \sin n\pi x .$$

$$\therefore \quad u(t,x) = e^{-\frac{6}{2}t} \sum_{n=1}^{\infty} \left[A_n \cos(n\pi\theta_n t) + B_n \sin(n\pi\theta_n t) \right] \sin n\pi x .$$

(2) We can either use parallelogram rule or D'Alembert's solution formula. Using parallelogram rule we get

$$u(2,1) = -u(0,1) = -1$$

$$u(3.5,.5) = -u(1.5,1.5) = -\left(u(.5,.5) - u(0,1) \right)$$

Now $u(.5,.5) = \frac{1}{2}\left[u(0,0) + u(0,1) \right] + \frac{1}{2} \int_0^1 u_t(0,x)\,dx$

$$= \frac{1}{2} + \frac{1}{2} \int_0^1 x(2-x)\,dx = \frac{1}{2} + \frac{1}{2} \left(x^2 - \frac{x^3}{3} \right) \Big|_0^1$$

$$= \frac{1}{2} + \frac{1}{3}$$

$$\therefore \quad u(3.5,.5) = -\left(\frac{1}{2} + \frac{1}{3} - u(0,1) \right) = -\left(\frac{1}{2} + \frac{1}{3} - 1 \right) = \frac{1}{6}$$