Math. 4581, Test 1

Name: SOLUTIONS

1. Let

$$f(x) = \begin{cases} 2x & \text{for } 0 < x < \dfrac{1}{2} \\ 2(1-x) & \text{for } \dfrac{1}{2} < x < 1. \end{cases}$$

(a)(6 pts) Find the (half-range) Fourier cosine series for $f(x)$.

For a function defined on $(0, L)$ the coefficients of the Fourier cosine series are

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \quad \text{for } n = 0, 1, 2, ...$$

Here $L = 1$ and so

$$a_0 = 2\Big(\int_0^{\frac{1}{2}} 2xdx + \int_{\frac{1}{2}}^{1} 2(1-x)dx\Big) = 1.$$

For $n > 0$

$$a_n = 2\Big(\int_0^{\frac{1}{2}} 2x\cos n\pi x dx + \int_{\frac{1}{2}}^{1} 2(1-x)\cos n\pi x dx\Big)$$

$$= 4\Big(\frac{x}{n\pi}\sin n\pi x\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}}\frac{\sin n\pi x}{n\pi}dx\Big) + 4\Big(\frac{1-x}{n\pi}\sin n\pi x\Big|_{\frac{1}{2}}^{1} + \int_{\frac{1}{2}}^{1}\frac{\sin n\pi x}{n\pi}dx\Big)$$

$$= 2\frac{\sin\frac{n\pi}{2}}{n\pi} + \frac{4\cos n\pi x}{(n\pi)^2}\Big|_0^{\frac{1}{2}} - 2\frac{\sin\frac{n\pi}{2}}{n\pi} - \frac{4\cos n\pi x}{(n\pi)^2}\Big|_{\frac{1}{2}}^{1}$$

$$= \frac{4}{(n\pi)^2}\Big(2\cos\frac{n\pi}{2} - 1 - \cos n\pi\Big) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\dfrac{8}{(n\pi)^2}[1-(-1)^{\frac{n}{2}}] & \text{if } n \text{ is even.} \end{cases}$$

Therefore the Fourier cosine series is

$$S(x) = \frac{1}{2} + \sum_{n=1}^{\infty} a_n\cos n\pi x.$$

(b)(3 pts) Sketch the graph on the interval $(-2, 2)$ of the function $\tilde{f}(x)$ to which the series converges and the graph of its derivative.

$S(x)$ converges to the 2-periodic even extension $\tilde{f}$ of f.

(c)(3 pts) Does the differentiated Fourier series for $f(x)$ converge to $[\tilde{f}(x)]'$?

Since $\tilde{f}$ is continuous and $\tilde{f}'$ and $\tilde{f}''$ are piecewise continuous, the differentiated Fourier series $S'(x)$ converges to

$$\frac{\tilde{f}'(x^+) + \tilde{f}'(x^-)}{2} = \begin{cases} 0 & \text{if } x = \pm\frac{n}{2}, n = 0, 1, 2, ... \\ \tilde{f}'(x) & \text{otherwise.} \end{cases}$$

2.(8 pts) Is $\{x, \cos x, \cos 2x, \cos 3x, \cos 4x, \cos 5x\}$ an orthonormal family in $L^2(-\pi, \pi)$? Does $f(x) = \sin 2x$ belong to the space spanned by the family?

The family is not orthonormal since for $f(x) = x$ we have

$$\|f\|^2 = \int_{-\pi}^{\pi} x^2 dx = \frac{2}{3}\pi^3 \neq 1.$$

However the family is orthogonal.

Suppose that there are constants $c_1, ..., c_6$ such that

$$\sin 2x = c_1 x + c_2 \cos x + ... + c_6 \cos 5x.$$

Then

$$\sin 2x - c_1 x = c_2 \cos x + ... + c_6 \cos 5x.$$

The left-hand side is an odd function while the right-hand side is an even one. Therefore they both have to be 0 and then we get that

$$\sin 2x = c_1 x.$$

This is impossible and therefore $\sin 2x$ does not belong to the space spanned by the family.

3.(a)(6 pts) Compute all the eigenvalues and the corresponding eigenfunctions for the Sturm-Liouville problem

$$u''(x) + \lambda u(x) = 0, \quad u'(-\frac{\pi}{2}) = u(0) = 0.$$

It is easy to check that if $\lambda \leq 0$ then λ is not an eigenvalue. If $\lambda > 0$ the general solution has the form

$$u(x) = c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x.$$

Then $u(0) = 0$ implies that $c_2 = 0$ and therefore $u'(x) = c_1 \sqrt{\lambda} \cos \sqrt{\lambda} x$. Therefore

$$0 = u'(-\frac{\pi}{2}) = c_1 \sqrt{\lambda} \cos(\sqrt{\lambda}\frac{\pi}{2}).$$

This implies that

$$\sqrt{\lambda}\frac{\pi}{2} = (n + \frac{1}{2})\pi$$

and so the eigenvalues are

$$\lambda_n = (2n+1)^2 \qquad n = 0, 1, 2, ...$$

and the corresponding eigenfunctions

$$u_n(x) = \sin(2n+1)x.$$

(b)(4 pts) Write the first two terms of the generalized Fourier series expansion in $L^2(-\frac{\pi}{2}, 0)$ for

$$f(x) = \begin{cases} 0 & \text{for } -\dfrac{\pi}{2} < x < -\dfrac{\pi}{3} \\ 1 & \text{for } -\dfrac{\pi}{3} < x < 0. \end{cases}$$

in terms of the orthonormal approximating basis of eigenfunctions of the Sturm-Liouville problem (i.e the two terms involving the eigenfunctions corresponding to the two smallest eigenvalues).

The first two eigenfunctions are $u_0(x) = \sin x$ and $u_1(x) = \sin 3x$. Therefore the first two terms of the generalized Fourier series expansion in terms of the eigenfunctions of the Sturm-Liouville problem for f are

$$\frac{(f, u_0)}{\|u_0\|^2} u_0 + \frac{(f, u_1)}{\|u_1\|^2} u_1$$

$$= \frac{\int_{-\frac{\pi}{3}}^{0} \sin x dx}{\int_{-\frac{\pi}{2}}^{0} \sin^2 x dx} \sin x + \frac{\int_{-\frac{\pi}{3}}^{0} \sin 3x dx}{\int_{-\frac{\pi}{2}}^{0} \sin^2 3x dx} \sin 3x = -\frac{2}{\pi} \sin x - \frac{8}{3\pi} \sin 3x.$$