Math. 4581, Test 2

Name: Solutions

1. Solve the following boundary value problem:

$$\begin{cases} \nabla^2 u(x,y) = x & \text{for } 0 < x, y < 1, \\ u_x(0,y) = u(1,y) = 0 & \text{for } 0 < y < 1, \\ u(x,0) = u(x,1) = 0 & \text{for } 0 < x < 1. \end{cases}$$

We first find an orthogonal approximating basis of $L^2((0,1) \times (0,1))$ composed of the eigenfunctions of the Laplacian with the required boundary conditions. This is done by doing separation of variables, i.e. by looking for eigenfunctions of the form $X(x)Y(y)$. We then have two Sturm-Liouville problems

$$X'' + \mu X = 0, \quad X'(0) = X(1) = 0$$

and

$$Y'' + \lambda Y = 0, \quad Y(0) = Y(1) = 0.$$

The first has solutions

$$\mu_m = (\frac{2m+1}{2}\pi)^2, \ m = 0, 1, 2, ..., \quad X_m = \cos(\frac{2m+1}{2}\pi x)$$

and the second

$$\lambda_n = (n\pi)^2, \ n = 1, 2, ..., \quad Y_n = \sin(n\pi y).$$

The family $\{\cos(\frac{2m+1}{2}\pi x)\}_{m=0}^{\infty}$ is an orthogonal approximating basis of $L^2(0,1)$ and the family $\{\sin(n\pi y)\}_{n=1}^{\infty}$ is also an orthogonal approximating basis of $L^2(0,1)$. Thus the products of functions of these families

$$\cos(\frac{2m+1}{2}\pi x)\sin(n\pi y), \ m = 0, 1, 2, ..., n = 1, 2, ...$$

create an orthogonal approximating basis of $L^2((0,1) \times (0,1))$, which is composed of the eigenfunctions of the Laplacian with the required boundary conditions.

We now expand the solution $u$ in terms of the elements of this basis to get

$$u(x,y) = \sum_{m=0,n=1}^{\infty} A_{nm} \cos(\frac{2m+1}{2}\pi x)\sin(n\pi y).$$

It remains to find the $A_{nm}$. To do this we expand

$$x = \sum_{m=0,n=1}^{\infty} B_{nm} \cos(\frac{2m+1}{2}\pi x)\sin(n\pi y),$$

where

$$B_{nm} = \frac{\int_0^1 \int_0^1 x \cos(\frac{2m+1}{2}\pi x) \sin(n\pi y) dx dy}{\int_0^1 \int_0^1 \cos^2(\frac{2m+1}{2}\pi x) \sin^2(n\pi y) dx dy}$$

$$= \frac{\int_0^1 x \cos(\frac{2m+1}{2}\pi x) dx \int_0^1 \sin(n\pi y) dy}{\frac{1}{2} \cdot \frac{1}{2}}$$

$$= \frac{4}{n\pi}(1 - (-1)^n) \int_0^1 x \cos(\frac{2m+1}{2}\pi x) dx$$

$$= \frac{4}{n\pi}(1 - (-1)^n) \left( \frac{(-1)^m}{(m + \frac{1}{2})\pi} - \frac{1}{(m + \frac{1}{2})^2 \pi^2} \right).$$

Now

$$\nabla^2 u = \sum_{m=0,n=1}^{\infty} -A_{nm}(n^2 + (m + \frac{1}{2})^2)\pi^2 \cos(\frac{2m+1}{2}\pi x) \sin(n\pi y)$$

and so plugging this into the equation we obtain

$$\sum_{m=0,n=1}^{\infty} -A_{nm}(n^2 + (m + \frac{1}{2})^2)\pi^2 \cos(\frac{2m+1}{2}\pi x) \sin(n\pi y) = \sum_{m=0,n=1}^{\infty} B_{nm} \cos(\frac{2m+1}{2}\pi x) \sin(n\pi y).$$

This gives

$$A_{nm} = \frac{-\frac{4}{n\pi}(1 - (-1)^n) \left( \frac{(-1)^m}{(m+\frac{1}{2})\pi} - \frac{1}{(m+\frac{1}{2})^2 \pi^2} \right)}{(n^2 + (m + \frac{1}{2})^2)\pi^2}$$

and we are done.

2. Solve the initial boundary value problem

$$\begin{cases} u_t = D u_{xx} & \text{for } 0 < x < L, t > 0, \\ u(t,0) = T_0,\ u(t,L) = T_1 & \text{for } t > 0, \\ u(0,x) = 0 & \text{for } 0 < x < L, \end{cases}$$

where $T_0$ and $T_1$ are constants.

We first find the steady state solution of our heat equation, i.e. the function $v$ satisfying

$$\begin{cases} v'' = 0 & \text{for } 0 < x < L, \\ v(0) = T_0,\ v(L) = T_1. \end{cases}$$

We get

$$v(x) = T_0 + (T_1 - T_0)\frac{x}{L}.$$

Therefore $u$ satisfies the original equation if and only if $w = u - v$ satisfies

$$\begin{cases} w_t = Dw_{xx} & \text{for } 0 < x < L, t > 0, \\ w(t,0) = 0,\ w(t,L) = 0 & \text{for } \ t > 0, \\ w(0,x) = -T_0 - (T_1 - T_0)\dfrac{x}{L} & \text{for } 0 < x < L. \end{cases}$$

Separation of variables for this problem gives

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad T_n(t) = e^{-(\frac{n\pi}{L})^2 Dt}, \quad n = 1, 2, ...$$

Thus we are looking for the solution $w$ of the form

$$w(t,x) = \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 Dt} \sin\frac{n\pi x}{L}.$$

We must have

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = -T_0 - (T_1 - T_0)\frac{x}{L},$$

and thus

$$A_n = \frac{\int_0^L \left[-T_0 - (T_1 - T_0)\frac{x}{L}\right] \sin\frac{n\pi x}{L} dx}{\int_0^L \sin^2\frac{n\pi x}{L} dx}$$

$$= \frac{2}{L}\int_0^L \left[-T_0 - (T_1 - T_0)\frac{x}{L}\right] \sin\frac{n\pi x}{L} dx = \frac{2}{n\pi}(T_1(-1)^n - T_0).$$

Therefore,

$$u(t,x) = T_0 + (T_1 - T_0)\frac{x}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(T_1(-1)^n - T_0)}{n} e^{-(\frac{n\pi}{L})^2 Dt} \sin\frac{n\pi x}{L}.$$