Weak Harnack inequality for fully nonlinear uniformly parabolic equations with unbounded ingredients and applications

Shigeaki Koike∗  Andrzej Święch†
Mathematical Institute  School of Mathematics
Tohoku University  Georgia Institute of Technology
Aoba, Sendai 980-8578  Atlanta, GA 30332
JAPAN  USA

and

Shota Tateyama‡
Mathematical Institute
Tohoku University
Aoba, Sendai 980-8578
JAPAN

Abstract

The weak Harnack inequality for \( L^p \)-viscosity supersolutions of fully nonlinear second-order uniformly parabolic partial differential equations with unbounded coefficients and inhomogeneous terms is proved. It is shown that Hölder continuity of \( L^p \)-viscosity solutions is derived from the weak Harnack inequality for \( L^p \)-viscosity supersolutions. The local maximum principle for \( L^p \)-viscosity subsolutions and the Harnack inequality for \( L^p \)-viscosity solutions are also obtained. Several further remarks are presented when equations have superlinear growth in the first space derivatives.

∗Supported in part by Grant-in-Aid for Scientific Research (No. 16H06339, 16H03948, 16H03946) of JSPS, e-mail: koike@m.tohoku.ac.jp
†e-mail: swiech@math.gatech.edu
‡Supported by Grant-in-Aid for JSPS Research Fellow 16J02399, e-mail: nrxcf428@yahoo.co.jp
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1 Introduction

The seminal paper [3] of L. A. Caffarelli was the most influential in the development of modern regularity theory for viscosity solutions of fully nonlinear uniformly elliptic partial differential equations (PDE for short). Various results were proved there, including Harnack inequality, $C^\alpha$, $C^{1,\alpha}$, $C^{2,\alpha}$ and $W^{2,p}$ estimates, and the reader can find a more detailed and complete account of them in [4]. Around the same time similar results like Harnack inequality, $C^\alpha$ and $C^{1,\alpha}$ estimates for viscosity solutions were also proved by different methods in [19, 20, 21]. In order to treat PDE with measurable terms, the notion of $L^p$-viscosity solution of fully nonlinear uniformly elliptic PDE was introduced in [5] and similar idea was also considered in [22]. L. Wang in [22, 23] extended regularity results of [3] to viscosity solutions of fully nonlinear uniformly parabolic PDE. Later, $L^p$-viscosity solutions of parabolic PDE were studied in [6, 7].

The main ingredient in the theory is the Aleksandrov-Bakelman-Pucci (ABP for short) maximum principle, which gives the $L^\infty$-estimates in terms of the $L^p$-norms of the inhomogeneous terms. The ABP maximum principle for viscosity solutions of fully nonlinear uniformly parabolic PDE was proved in [22]. In [5], the ABP maximum principle was proved for $L^p$-viscosity solutions of uniformly elliptic PDE which are uniformly Lipschitz continuous in the first derivatives. It was later extended for elliptic and parabolic PDE to equations which are not uniformly Lipschitz continuous in the first derivative terms in [11], where the Lipschitz coefficient functions (as functions of $x$ and $t$) belong to some $L^q$ spaces. The second ingredient of the regularity theory of [3] is the Harnack inequality for viscosity solutions as well as the weak Harnack inequality and the local maximum principle. Such results for non-divergence form equations started with the work of Krylov and Safonov [15] and the results for strong solutions can be found in classical books [9, 17]. Results for viscosity solutions first appeared in [3, 19] (see [4]). General form of the weak Harnack inequality for $L^p$-viscosity supersolutions of fully nonlinear elliptic PDE (which implies the Hölder continuity of $L^p$-viscosity solutions) was proved in [12], using the ABP estimates of [11], while a general local maximum principle for $L^p$-viscosity solutions can be found in [14]. The corresponding results for viscosity solutions of uniformly parabolic PDE were proved in [22], however only for equations which are uniformly Lipschitz continuous in the first derivatives. In this paper we want to extend them to $L^p$-viscosity solutions of more general equations. The relevant equations are the parabolic extremal equations

$$u_t + P^\pm(D^2u) \pm \mu|Du| + f = 0 \quad \text{in } Q,$$
where \( f \in L^p(Q) \) and \( \mu \in L^q(Q) \).

In this manuscript, combining the argument from [10] with the ABP maximum principle of [11], we first show the weak Harnack inequality when the \( L^q \)-norm of the coefficient function \( \mu \) is small. We then avoid this smallness assumption by the introduction of a new “heat kernel” like barrier function in our proof of the weak Harnack inequality. We will use global estimates on strong solutions of fully nonlinear parabolic equations from a recent paper by Dong, Krylov and Xu [8]. We remark that the weak Harnack inequality yields the (local) Hölder estimate. In order to establish the Harnack inequality, following the argument of [4] (see also [14]), we also obtain the corresponding local maximum principle. We refer to [22] and [10] for the other approach. We also present some results when the PDE contains first space derivative terms which may grow superlinearly.

This paper is organized as follows. In section 2, we recall the definition of \( L^p \)-viscosity solution for parabolic PDE, its properties and known results. Section 3 is devoted to a proof of the weak Harnack inequality for \( L^p \)-viscosity supersolutions. In section 4 we first establish the local Hölder continuity estimate using the weak Harnack inequality. For the completeness of the theory, we show the local maximum principle for \( L^p \)-viscosity subsolutions by a parabolic version of the argument of [4] and then obtain the Harnack inequality. In section 5, we present some results for PDE which may contain superlinearly growing gradient terms.

## 2 Preliminaries

We fix \( n \in \mathbb{N} \), a bounded domain \( \Omega \subset \mathbb{R}^n \), and \( T > 0 \). We denote by \( \mathbb{S}^n \) the set of all \( n \times n \) symmetric matrices with the standard order.

Given \( F : \Omega \times (0, T] \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R} \), we are concerned with the following fully nonlinear parabolic PDE:

\[
 u_t + F(x, t, Du, D^2 u) = 0 \quad \text{in } \Omega \times (0, T],
\]

where \( Du \) and \( D^2 u \), respectively, denote the first and second derivatives with respect to \( x \in \mathbb{R}^n \), \( u_t \) is the time derivative, and \( F \) is at least measurable with respect to all the variables. We will write \( u_{x_k}, u_{x_kx_l} \) for \( \frac{\partial u}{\partial x_k}, \frac{\partial^2 u}{\partial x_k \partial x_l} \), respectively.

In what follows, we assume that \( F \) is uniformly parabolic, i.e. that there exist \( 0 < \lambda \leq \Lambda < \infty \) such that

\[
 \mathcal{P}_{\lambda, \Lambda}^-(X - Y) \leq F(x, t, \xi, X) - F(x, t, \xi, Y) \leq \mathcal{P}_{\lambda, \Lambda}^+(X - Y) \quad \text{(2.2)}
\]

for all \( (x, t, \xi, X, Y) \in \Omega \times (0, T] \times \mathbb{R}^n \times \mathbb{S}^n \times \mathbb{S}^n \), where \( \mathcal{P}_{\lambda, \Lambda}^\pm : \mathbb{S}^n \to \mathbb{R} \) are defined by

\[
 \mathcal{P}_{\lambda, \Lambda}^+(X) := \max\{-\text{Tr}(AX) \mid A \in \mathbb{S}^n, \lambda I \leq A \leq \Lambda I\},
\]

\[
 \mathcal{P}_{\lambda, \Lambda}^-(X) := \min\{-\text{Tr}(AX) \mid A \in \mathbb{S}^n, \lambda I \leq A \leq \Lambda I\}
\]
for $X \in S^n$, where $I$ denotes the $n \times n$ identity matrix. Since we fix $0 < \lambda \leq \Lambda$ in this paper, we simply write $\mathcal{P}^\pm$ for $\mathcal{P}_\lambda^\pm$. For properties of $\mathcal{P}^\pm$, we refer for instance to [5].

Setting $Q := \Omega \times (0, T)$, we denote the parabolic boundary of $Q$ by

$$\partial_p Q := \Omega \times \{0\} \cup \partial \Omega \times [0, T).$$

The parabolic distance is defined by

$$d((x, t), (y, s)) := \sqrt{|x - y|^2 + |t - s|}.$$  

For $U, V \subset \mathbb{R}^{n+1}$, we define the distance between $U$ and $V$

$$\text{dist}(U, V) := \inf \{d((x, t), (y, s)) \mid (x, t) \in U, (y, s) \in V\}.$$  

We will write $\text{diam}(Q)$ for the diameter of $Q$ (measured with respect to the parabolic distance) and $\text{diam}(\Omega)$ for the diameter of $\Omega$.

We will use the anisotropic Sobolev spaces. For $1 \leq p \leq \infty$,

$$W_{p}^{2,1}(Q) := \{f \in L^p(Q) \mid f_{x_k}, f_{x_k x_t}, f_t \in L^p(Q) \ (1 \leq k, \ell \leq n)\},$$

and

$$W_{p,\text{loc}}^{2,1}(Q) := \{f \in W_{p}^{2,1}(Q') \mid \forall Q' \subset Q\}.$$  

Here and later, $Q' \subset Q$ means $\text{dist}(Q', \partial_p Q) > 0$. We define the norm for $f \in W_{p}^{2,1}(Q)$ by

$$\|f\|_{W_{p}^{2,1}(Q)} := \|f\|_{L^p(Q)} + \|f_t\|_{L^p(Q)} + \sum_{k=1}^{n} \|f_{x_k}\|_{L^p(Q)} + \sum_{k, \ell=1}^{n} \|f_{x_k x_\ell}\|_{L^p(Q)}.$$  

We will also use the anisotropic Sobolev spaces

$$W_{p}^{1,0}(Q) := \{f \in L^p(Q) \mid f_{x_k} \in L^p(Q) \ (1 \leq k \leq n)\}$$

for $1 \leq p \leq \infty$, equipped with the norm

$$\|f\|_{W_{p}^{1,0}(Q)} := \|f\|_{L^p(Q)} + \sum_{k=1}^{n} \|f_{x_k}\|_{L^p(Q)}.$$  

We denote by $C^{2,1}(Q)$ be the space of functions $u \in C(Q)$ such that $u_t, u_{x_k}, u_{x_k x_\ell} \in C(Q)$ for $1 \leq k, \ell \leq n$. For $0 < \alpha \leq 1$ we denote by $C^{\alpha}(Q)$ the space of functions which are $\alpha$-Hölder continuous in $Q$ with respect to the parabolic distance. We denote by $W_{p}^{k}(\Omega)$, $k = 1, 2, \ldots$, the standard Sobolev spaces.

We recall the notion of $L^p$-viscosity solutions of parabolic PDE (2.1). To this end, we denote by $B_r(x)$ the open ball in $\mathbb{R}^n$ with the radius $r > 0$ and the center $x$, and define the parabolic cylinders

$$Q_r(x, t) := (x, t) + (-r, r)^n \times (-r^2, 0].$$
**Definition 2.1.** Let $Q'$ be a relatively open subset of $Q$. A function $u \in C(Q')$ is said to be an $L^p$-viscosity subsolution (resp., supersolution) of (2.1) if for $\phi \in W_{p,\text{loc}}^{2,1}(Q')$, we have
\[
\lim_{\varepsilon \to 0} \text{ess inf}_{(y,s) \in Q_\varepsilon(x,t)} \{ \phi_t(y,s) + F(y,s,D\phi(y,s),D^2\phi(y,s)) \} \leq 0
\]
provided that $u - \phi$ attains a maximum (resp., minimum) at $(x,t) \in Q'$ over some parabolic cylinder $Q_r(x,t) \subset Q'$. A function $u \in C(Q')$ is said to be an $L^p$-viscosity solution of (2.1) if $u$ is an $L^p$-viscosity subsolution and supersolution of (2.1).

**Remark 2.2.** We note that $W_{p,\text{loc}}^{2,1}(Q) \subset C(Q)$ for $p > \frac{n+2}{2}$ and if $\Omega$ is regular enough (e.g. if $\partial\Omega$ is $C^{1,1}$) then $W_p^{2,1}(Q) \subset C^\alpha(Q)$ for $\alpha = 2 - (n + 2)/p$ is a bounded imbedding for $\frac{n+2}{2} < p < n + 2$. If $u \in W_{p,\text{loc}}^{2,1}(Q)$ for $p > n + 2$ then $u_{x_i} \in C^\alpha$ for $\alpha = 1 - (n + 2)/p$ (see e.g. [16]). Also, it is known that if $p > \frac{n+2}{2}$ and $u \in W_p^{2,1}(Q)$, then $u_t$, $u_{x_i}$ and $u_{x_ix_j}$ $(1 \leq i, j \leq n)$ exist a.e. in $Q$ (see [6]).

In this section, we recall the ABP maximum principle for $L^p$-viscosity subsolutions of the following extremal uniformly parabolic equations;
\[
u_t + \mathcal{P}^-(D^2u) - \mu|Du| - f = 0 \quad \text{in } Q,
\]
where
\[f \in L^p(Q), \quad \text{and} \quad \mu \in L^q(Q).
\]
We will suppose that the powers $p$ and $q$ satisfy the condition
\[q > n + 2, \quad p_1 < p \leq q,
\]
where $p_1 = p_1(n, \frac{\Lambda}{2}) \in [\frac{n+2}{2}, n + 1]$ is the constant, which gives a range where the ABP maximum principle holds, see e.g. [11].

**Proposition 2.3.** (cf. Theorem 2.8 in [7], Proposition 3.3 in [11]) For $p > p_1$, there exists a constant $C = C(n, \Lambda, \lambda, p) > 0$ such that for $f \in L^p(Q)$, there exists $u \in C(\overline{Q}) \cap W_p^{2,1}(Q)$ such that
\[
\begin{cases}
  u_t + \mathcal{P}^+(D^2u) - f(x,t) = 0 & \text{a.e. in } Q, \\
  u = 0 & \text{on } \partial_p Q,
\end{cases}
\]
and
\[-C\|f^-\|_{L^p(Q)} \leq u \leq C\|f^+\|_{L^p(Q)} \quad \text{in } Q.
\]
Moreover, for $Q' \Subset Q$, there is $C' = C'(n, \Lambda, \lambda, p, \text{diam}(\Omega), \text{dist}(Q', \partial_p Q)) > 0$ such that
\[
\|u\|_{W_p^{2,1}(Q')} \leq C'\|f\|_{L^p(Q)}.
\]
We emphasize that the dependence of constants on various parameters sometimes may mean that a constant may blow up as a parameter converges to 0, for instance the constant \( C' \) in Proposition 2.3 may blow up as \( \lambda \to 0 \). The precise dependence of constants on \( T, \text{diam}(\Omega), \text{diam}(Q) \) can often be found by scaling.

We state below in Proposition 2.5 a scaled version of the ABP maximum principle for \( L^p \)-strong and \( L^p \)-viscosity solutions of (2.3) based on the results of [11]. For \( u \in C(\overline{Q}) \), we introduce the set

\[
Q_+[u] := \left\{ (x,t) \in Q \mid u(x,t) > \sup_{\partial_p Q} u^+ \right\}.
\]

We denote by \( L^p_+(Q) \) for the set of all nonnegative functions in \( L^p(Q) \).

**Remark 2.4.** The non-scaled statement of the classical ABP maximum principle for strong solutions in Proposition 3.2 of [11] was slightly incorrect and might be confusing. The exact ABP inequality from [18] is

\[
\|u\|_{L^\infty(Q)} \leq \|\psi\|_{L^\infty(\partial_p Q)} + C_1 d_\Omega^{\frac{2n}{n+1}} \exp\left( C_2 d_\Omega^{-1} \|\mu\|_{L^{n+1}(Q)}^{n+1} \right) \|f\|_{L^{n+1}(Q)}, \tag{2.5}
\]

where \( d_\Omega = \text{diam}(\Omega) \), which behaves differently from the one in [11] when \( \text{diam}(\Omega) \) is small. This however does not affect the results of [11] since the proofs there only used (2.5) in parabolic cylinders of fixed size which contained \( Q \) and did not depend on \( \text{diam}(\Omega) \).

**Proposition 2.5.** (see Proposition 3.6, Theorem 3.10 of [11]) Let (2.4) hold and let \( f \in L^p_+(Q), \mu \in L^q_+(Q) \). There exists a constant \( C_1 = C_1(n, \Lambda, \lambda, p, q, d_Q^{1-(n+2)/q}\|\mu\|_{L^2(\Omega)}) > 0 \) such that if \( u \in C(\overline{Q}) \) is either an \( L^p \)-strong or an \( L^p \)-viscosity subsolution of (2.3), then

\[
\sup_Q u \leq \sup_{\partial_p Q} u + C_1 d_Q^{2 \frac{n+2}{n}} \|f\|_{L^p(Q)}, \tag{2.6}
\]

where \( d_Q = \text{diam}(Q) \). Moreover, if \( v \in C(\overline{Q}) \) is an \( L^p \)-viscosity subsolution of (2.3) in \( Q_+[v] \) then

\[
\sup_Q v \leq \sup_{\partial_p Q} v^+ + C_1 d_Q^{2 \frac{n+2}{n}} \|f\|_{L^p(Q_+[v])}. \tag{2.7}
\]

We remark that when \( p \geq n+2 \) then (2.6) can be made more precise based on (2.5) or on a scaled version of (2.5) in a unit cylinder. We also remark that it can be proved that if (2.4) holds then an \( L^p \)-strong subsolution of \( u_\ast + \mathcal{P}^\pm(D^2 u_\ast) \pm \mu |Du| = f \) is an \( L^p \)-viscosity subsolution of those. We refer to [12], Section 3, for such a proof in the elliptic case. Similar statement holds for \( L^p \)-viscosity supersolutions.
Proof of Proposition 2.5. To see why (2.6) is true we notice that the function \(w(x,t) = u(d_Qx,d_Q^2t)\) is an \(L^p\)-strong or an \(L^p\)-viscosity subsolution of

\[
w_t + \mathcal{P}^-(D^2w) - \bar{\mu}|Dw| - \bar{f} = 0
\]

in a unit cylinder \(Q_1\), where \(\bar{\mu}(x,t) = d_Q\mu(d_Qx,d_Q^2t)\) and \(\bar{f}(x,t) = d_Q^2 f(d_Qx,d_Q^2t)\). Then, by the estimates of [11]

\[
\sup_{Q_1} w \leq \sup_{\partial_p Q_1} w + C_1(n, \Lambda, \lambda, p, q, \|\bar{\mu}\|_{L^q(Q_1)}) \|\bar{f}\|_{L^p(Q_1)}.
\]

It remains to notice that \(\|\bar{\mu}\|_{L^q(Q_1)} = d_Q^{1-(n+2)/q}\|\mu\|_{L^q(Q)}\) and \(\|\bar{f}\|_{L^p(Q_1)} = d_Q^{-2n/p}\|f\|_{L^p(Q)}\).

Estimate (2.7) is proved similarly by rescaling and adding to \(w\) a subsolution of an extremal equation in a bigger cylinder to eliminate \(\bar{f}\), which can be found using Proposition 3.5 of [11] (or using Proposition 2.6 below if \(q \geq p > n + 2\)). The reader can find a similar argument in the proof of Proposition 2.8 of [11]. \(\square\)

A result similar to Proposition 2.6 can be found in [11] (see Proposition 3.5 there). Using global \(W^{2,1}_p\) estimates by Dong, Krylov and Xu in [8], we present a slightly different existence result.

Proposition 2.6. Assume that \(\partial \Omega\) is \(C^{1,1}\) and \(q \geq p > n + 2\). Let \(\mu \in L^q_+(Q), \psi \in W^{2,1}_p(Q) \cap C(\overline{Q})\) and \(f \in L^p(Q)\). The equation

\[
\begin{cases}
  u_t + \mathcal{P}^+(D^2 u) + \mu|Du| - f = 0 & \text{a.e. in } Q, \\
  u = \psi & \text{on } \partial_p Q
\end{cases}
\]

has an \(L^p\)-strong solution \(u \in C(\overline{Q}) \cap W^{2,1}_p(Q)\). The solution \(u\) satisfies

\[
\|u\|_{L^\infty(Q)} \leq \|\psi\|_{L^\infty(\partial_p Q)} + C_1 d_Q^{-2n/p} \|f\|_{L^p(Q)}
\]

(where \(C_1\) is the constant from (2.6)) and

\[
\|u\|_{W^{2,1}_p(Q)} \leq C_2 \left(\|\psi\|_{W^{2,1}_p(Q)} + \|f\|_{L^p(Q)}\right),
\]

for some constant \(C_2 = C_2(n, \Lambda, \lambda, p, q, \|\mu\|_{L^q(Q)}, T, \text{diam}(\Omega), \partial \Omega) > 0\).

Remark 2.7. The function \(u\) in Proposition 2.6 is also an \(L^p\)-viscosity solution of (2.8). We also note that \(p > n + 1\) is assumed in [8] while we assume \(p > n + 2\).

Proof. Let \(f^j, \mu^j \in C(\overline{Q})\) be such that \(\|f^j - f\|_{L^p(Q)} + \|\mu^j - \mu\|_{L^q(Q)} \to 0\), and \((f^j, \mu^j) \to (f, \mu)\) almost everywhere in \(Q\) as \(j \to \infty\). Let \(u^j \in C(\overline{Q}) \cap C^{2,1}(Q)\) be the classical solution of

\[
\begin{cases}
  u^j_t + \mathcal{P}^+(D^2 u^j) + \mu^j|Du^j| - f^j = 0 & \text{in } Q, \\
  u^j = \psi & \text{on } \partial_p Q.
\end{cases}
\]
Here and later, $C > 0$ stands for various constants depending only on known quantities. We know from [8] that

$$\|u^j\|_{W^{2,1}_p(Q)} \leq C \left( \|\mu^j D u^j\|_{L^p(Q)} + \|\psi\|_{W^{2,1}_p(Q)} \right). \tag{2.12}$$

It is also known (e.g. Lemma 3.3 in [16]) that for sufficiently small $\varepsilon > 0$,

$$\|Du^j\|_{L^\infty(Q)} \leq \varepsilon^{\alpha_1} \left( \|D^2 u^j\|_{L^p(Q)} + \|u^j\|_{L^p(Q)} \right) + \varepsilon^{-\alpha_2} C\|u^j\|_{L^p(Q)}, \tag{2.13}$$

where $\alpha_1 = 1 - \frac{n+2}{p} > 0$ and $\alpha_2 = 1 + \frac{n+2}{p} > 0$. Combining (2.13) with (2.12), in view of the global estimates in [8], we have

$$\|u^j\|_{W^{2,1}_p(Q)} \leq C \left( \|f^j\|_{L^p(Q)} + \|\mu^j D u^j\|_{L^p(Q)} + \|\psi\|_{W^{2,1}_p(Q)} \right) \leq \|\mu^j\|_{L^p(Q)} \left\{ \varepsilon^{\alpha_1} \left( \|D^2 u^j\|_{L^p(Q)} + \|u^j\|_{L^p(Q)} \right) + C \varepsilon^{-\alpha_2}\|u^j\|_{L^p(Q)} \right\} + C \left( \|f^j\|_{L^p(Q)} + \|\psi\|_{W^{2,1}_p(Q)} \right). \tag{2.14}$$

Hence, for an appropriate $\varepsilon > 0$ (depending on $\|\mu^j\|_{L^p(Q)}$), using the ABP maximum principle, we obtain

$$\|u^j\|_{W^{2,1}_p(Q)} \leq C \left( \|f^j\|_{L^p(Q)} + \|\psi\|_{W^{2,1}_p(Q)} \right). \tag{2.15}$$

Since (by anisotropic Sobolev imbeddings) the functions $u^j$ are equicontinuous in $C(Q)$ and $u^j_{x_i}, i = 1, \ldots, n$, are locally equicontinuous, by taking a subsequence, we can assume that there exists $u \in W^{2,1}_p(Q) \cap C(Q)$ satisfying (2.9) and (2.15) such that $u^j \to u$ in $W^{2,1}_p(Q)$, $u^j \to u$ in $C(Q)$ and $u^j_{x_i} \to u_{x_i}$ locally uniformly. Thus $-\mu^j |Du^j| + f^j \to -\mu |Du| + f$ in $L^p_{\text{loc}}(Q)$. It is then standard by the techniques of [7] to obtain that $u$ is an $L^p$-viscosity solution and hence an $L^p$-strong solution of

$$u_t + \mathcal{P}^+(D^2 u) = g,$$

where $g = -\mu |Du| + f$, which concludes the proof.

### 3 The weak Harnack inequality

In what follows, we set $\Omega := (-10, 10)^n, T = 10$ and

$$Q := (-10, 10)^n \times (0, 10].$$

Although we need to suppose $\partial \Omega \in C^{1,1}$ to use Proposition 2.6, for the sake of simplicity of the presentation, we will assume that the boundary of cubes are $C^{1,1}$. Otherwise we
would have to use a smooth domain similar to $(-10, 10)^n$. We refer to [12] for such an argument.

In this section, we show the weak Harnack inequality for nonnegative $L^p$-viscosity supersolutions of

$$u_t + \mathcal{P}^+(D^2 u) + \mu |Du| + f = 0 \quad \text{in } Q,$$

where $f \in L^p_+(Q)$ and $\mu \in L^q_+(Q)$.

### 3.1 A restricted case

In order to show the weak Harnack inequality for nonnegative $L^p$-viscosity supersolutions of (3.1) with $f \in L^p_+(Q)$ and $\mu \in L^q_+(Q)$, we follow the standard argument as in [10] except for a new barrier function, which will be constructed in Lemma 3.7. However, for this purpose, we first have to show the weak Harnack inequality under a restricted setting.

**Theorem 3.1.** Assume that (2.4) holds, $f \in L^p_+(Q)$ and $\mu \in L^q_+(Q)$. Then, there exist constants $\varepsilon_0 = \varepsilon_0(n, \Lambda, \lambda, p, q) > 0$, $\delta_0 = \delta_0(n, \Lambda, \lambda, p, q) > 0$ and $C_0 = C_0(n, \Lambda, \lambda, p, q) > 0$ such that if

$$\|\mu\|_{L^p(Q)} \leq \delta_0,$$

then any nonnegative $L^p$-viscosity supersolution $u$ of (3.1) satisfies

$$\left( \int_{J_1} u^{\varepsilon_0} \, dx \, dt \right)^{\frac{1}{\varepsilon_0}} \leq C_0 \left( \inf_{J_2} u + \|f\|_{L^p(Q)} \right),$$

where

$$J_1 := (-1, 1)^n \times (0, 2^{-1}], \quad \text{and} \quad J_2 := (-1, 1)^n \times (9, 10].$$

We remark that the statement of Theorem 3.1 also holds for nonnegative $L^p$-strong supersolutions of (3.1).

In order to prove Theorem 3.1, we first construct a strong subsolution of an extremal equation. To this end, we use the following cubes:

$$K_1 := (-1, 1)^n \times (0, 1], \quad \text{and} \quad K_2 := (-3, 3)^n \times (1, 10].$$

We recall a barrier function from Lemma 2.4.16 of [10] (see also [22]). We can also construct one by the same manner as in Lemma 3.7 here.
Lemma 3.2. There exists a nonnegative function $\phi \in C^{2,1}(\overline{Q})$ and a function $g \in C(\overline{Q})$ such that

\[
\begin{cases}
\phi_t + \mathcal{P}^+(D^2\phi) \leq g(x,t) & \text{in } Q, \\
\phi \geq 2 & \text{in } K_2, \\
\phi = 0 & \text{on } \partial_p Q, \\
\supp g \subset K_1.
\end{cases}
\]

Letting $K_1$ as above, we denote by $\mathcal{C}_1$ the set of all $2^{n+2}$ cubes $(-1+i_1,i_1) \times \cdots \times (-1+i_n,i_n) \times (\frac{j}{4},\frac{j+1}{4}]$ for $i_k = 0,1$ ($k = 1,2,\ldots,n$), and $j = 0,1,2,3$. For each cube $L \in \mathcal{C}_1$, we divide it into $2^{n+2}$ cubes. We denote by $\mathcal{C}_2$ the set of such cubes constructed by the same procedure from each cube $L \in \mathcal{C}_1$. Inductively, we construct $\mathcal{C}_k$ whose elements have length $2^{-k+1}$ in each space direction and $4^{-k}$ in time. We call $L \in \bigcup_{k=1}^{\infty} \mathcal{C}_k$ a dyadic cube of $K_1$. When $L \in \mathcal{C}_k$ is constructed from an element of $\mathcal{C}_{k-1}$ by the above procedure, we denote by $\tilde{L} \in \mathcal{C}_{k-1}$ the predecessor of $L$.

For $L \in \mathcal{C}_k$ and its predecessor $\tilde{L} := J \times (\tau,\tau+\frac{1}{4^k-1}] \in \mathcal{C}_{k-1}$ for a cube $J = (\frac{a_1}{2^k-2},\frac{a_1+1}{2^k-2}) \times \cdots \times (\frac{a_n}{2^k-2},\frac{a_n+1}{2^k-2})$ with some integers $a_1,\ldots,a_n$, and $\tau \in [0,1)$, we define

\[
\tilde{L}^m := J \times (\tau + 4^{-k+1},\tau + 4^{-k+1}(m+1]],
\]

which is the union of $m$ cubes of the translated predecessor in the “future” direction.

We define $\mathcal{C} := \bigcup_{k=1}^{\infty} \mathcal{C}_k$. Moreover, for $m \in \mathbb{N}$, we define $\mathcal{C}(m) := \{L \in \mathcal{C} \mid \tilde{L}^m \subset Q\}$. Notice that when $1 \leq m \leq 36$, we have $\mathcal{C}(m) = \mathcal{C}$. 


We recall a parabolic version of the Calderón-Zygmund decomposition, which is a modification of Lemma 2.4.27 of [10]. Since fine cubes are needed in the proof of Lemma 2.4.27 of [10], we can follow the argument there to prove the next lemma.

**Lemma 3.3.** Let $m \geq 1$ be an integer, and $K_1 \subset \mathbb{R}^{n+1}$ be as above. Let measurable sets $A \subset B \subset K_1$ and $\sigma \in (0,1)$ satisfy

\[
\begin{cases}
(i) & |A| \leq \sigma |K_1|, \\
(ii) & \text{if } L \in \mathcal{C}(m) \text{ is such that } |A \cap L| > \sigma |L|, \text{ then } \tilde{L}^m \subset B,
\end{cases}
\]

where $\tilde{L}^m$ is from (3.4). Then, it follows that

\[
|A| \leq \sigma \frac{m+1}{m} |B|.
\]

**Proof of Theorem 3.1.** For $\varepsilon_1 > 0$, which will be fixed later, we set

\[
\tilde{u}(x,t) = N_0 u(x,t),
\]

where $N_0 = (\inf_{J_2} u + \varepsilon_1^{-1} \|f\|_{L^p(Q)} + \eta)^{-1}$ for $\eta > 0$, which will be sent to 0 at the end of the proof. By considering $\tilde{u}$ instead of $u$, it is enough to show that there are $\varepsilon_0, C_0 > 0$ such that

\[
\left( \int_{J_1} u^\varepsilon_0 \, dx \, dt \right)^{\frac{1}{\varepsilon_0}} \leq C_0
\]

under the assumptions

\[
\inf_{J_2} u \leq 1, \quad \text{and} \quad \|f\|_{L^p(Q)} \leq \varepsilon_1.
\]

Let $\phi$ be the function in Lemma 3.2. By letting $w := \phi - u$, it is immediate to see that $w$ is an $L^p$-viscosity subsolution of

\[
w_t + \mathcal{P}^{-}(D^2w) - \mu |Dw| - h = 0 \quad \text{in } Q,
\]

where $h := \mu |D\phi| + g + f$. In view of Proposition 2.5, we have

\[
\sup_Q w \leq C \|h\|_{L^p(Q+[w])}.
\]

Hence, by recalling supp $g \subset K_1$ in Lemma 3.2, it is easy to verify that this inequality implies

\[
1 \leq \sup_{J_2} w \leq C \left( \|g\|_{L^p(Q+[w])} + \varepsilon_1 + \delta_0 \|D\phi\|_{L^\infty(Q)} \right).
\]

Thus, since $\phi \in C^{2,1}(\bar{Q})$, for fixed $\varepsilon_1, \delta_0 > 0$, there is $\theta \in (0,1)$ such that $|\{(x,t) \in K_1 \mid w(x,t) > 0\}| \geq \theta |K_1|$. Hence, setting $M := \sup_{J_1} \phi$, ($M \geq 2$), we have

\[
|\{(x,t) \in K_1 \mid u(x,t) \geq M\}| \leq (1 - \theta)|K_1|.
\]
We next fix $\delta \in (1 - \theta, 1)$ and select large $m \in \mathbb{N}$ such that

$$1 - \theta < (1 - \theta) \frac{m + 1}{m} \leq \delta < 1. \quad (3.8)$$

Letting $J_1^k := (-1, 1)^n \times \left(0, \frac{1}{2} + \frac{m + 1}{9^m - 1}\right]$ for $k \geq 1$, we note that

$$J_1^{k+1} \subset J_1^k \quad (k \in \mathbb{N}), \quad \text{and} \quad \lim_{k \to \infty} J_1^k = J_1.$$

We choose $k_0 \in \mathbb{N}$ such that

$$\frac{m + 1}{9^{mk_0 - 3}(9^m - 1)} < \frac{1}{2} \quad (i.e. \ J_1^k \subset K_1 \ \text{for} \ k \geq k_0).$$

Finally, putting $\tilde{C}_0 := |J_1^{k_0}| \ (m(1 - \theta)^{-1}(m + 1)^{-1})^{k_0}$, by our choice of $\delta$ (i.e. (3.8)), we observe

$$|J_1^{k_0}| \leq \tilde{C}_0 \delta^{k_0}. \quad (3.9)$$

We will show that

$$|\{(x, t) \in J_1^k \mid u(x, t) \geq M^{km}\}| \leq \tilde{C}_0 \delta^k \quad (\forall k \geq k_0). \quad (3.10)$$

Notice that (3.9) yields (3.10) for $k = k_0$.

For any fixed $k \geq k_0 + 1$, we suppose that (3.10) holds for $k - 1$. Set

$$A = \{(x, t) \in J_1^k \mid u(x, t) \geq M^{km}\} \text{ and } B = \{(x, t) \in J_1^{k-1} \mid u(x, t) \geq M^{(k-1)m}\}.$$ 

It is immediate to see that $A \subset B \subset K_1$, and $|A| \leq (1 - \theta)|K_1|$ from (3.7) because $A \subset \{(x, t) \in K_1 \mid u(x, t) \geq M\}$.

If the hypotheses in Lemma 3.3 are satisfied for $A$, $B$ and $\sigma = 1 - \theta$, then using $|B| \leq \tilde{C}_0 \delta^{k-1}$, we have

$$|A| \leq (1 - \theta) \frac{m + 1}{m} \tilde{C}_0 \delta^{k-1}.$$ 

Hence, (3.10) holds for any $k \geq k_0$ by our choice of $\delta$ and $m$ in (3.8). Therefore, the standard argument implies

$$|\{(x, t) \in J_1 \mid u(x, t) \geq s\}| \leq A_0 s^{-\beta_0} \quad (s > 0), \quad (3.11)$$

where $A_0 = \tilde{C}_0 \delta^{-1}$ and $\beta_0 := -\frac{\log \delta}{m \log M} > 0$. We thus obtain (3.5) when $\varepsilon_0 \in (0, \beta_0)$.

In order to check (ii) in Lemma 3.3, we take a dyadic cube $L \in C(m)$ such that

$$|A \cap L| > (1 - \theta)|L|. \quad (3.12)$$

We can find $j \in \mathbb{N}$ and $(x_0, t_0) \in \overline{K}_1$ such that

$$L = (x_0, t_0) + (-2^{-j}, 2^{-j})^n \times (0, 2^{-2j}] .$$
We claim that if (3.12) holds then
\[
\inf_{N_\ell \cap (\mathbb{R}^n \times (0,10])} u > M^{km-\ell} \geq 1 \quad \text{for } \ell \in \{1, 2, \ldots, km\}. \tag{3.13}
\]

Here, we set \(N_1 := (x_0, t_0 + \frac{1}{4^1}) + \frac{3}{2^j}K_1, \ldots, N_\ell := (x_0, t_0 + \frac{9\ell - 1}{8} \cdot \frac{1}{8^j}) + \frac{3\ell}{2^j}K_1, \ldots\), where for \(\sigma > 0\),
\[
\sigma K_1 := (-\sigma, \sigma)^n \times (0, \sigma^2].
\]
We will prove this claim later.

One direct consequence of this claim for \(\ell = 1, 2, \ldots, m\) is the following assertion: under (3.12), it follows that
\[
u > M^{(k-1)m} \quad \text{in } \bigcup_{\ell=1}^{m} N_\ell \cap \{\mathbb{R}^n \times (0,10]\}. \tag{3.14}
\]

It is obvious from the definition that
\[
\tilde{L}^k \subset \Gamma_k \quad \text{for } k \in \mathbb{N}, \tag{3.15}
\]
where \(\Gamma_k := \bigcup_{\ell=1}^{k} N_\ell\) for \(k \in \mathbb{N}\). We also write \(\Gamma_\infty = \bigcup_{\ell \in \mathbb{N}} N_\ell\).

We easily verify the following inclusions:
\[
(x_0, t_0) + S^{-\frac{1}{4} , \infty} \subset \Gamma_\infty \subset (x_0, t_0) + S^{+\frac{1}{4} , \infty}. \tag{3.16}
\]
where for $0 \leq \alpha < \beta \leq \infty$, paraboloid type domains $S^\pm_{\alpha,\beta}$ are given by

$$S^-_{\alpha,\beta} := \{(x,t) \in \mathbb{R}^n \times (\alpha, \beta) \mid t > 2^{-3}(|x|_\infty^2 - 4^{-j})\},$$

$$S^+_{\alpha,\beta} := \{(x,t) \in \mathbb{R}^n \times (\alpha, \beta) \mid t > 2^{-3}(|x|_\infty^2 - 4^{-j})\}.$$

Here, $|x|_\infty := \max\{|x_1|, \ldots, |x_n|\}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For extreme cases when $(x_0, t_0) = (\hat{x}, 0)$ or $(x_0, t_0) = (\hat{x}, 1)$, where $\hat{x} = (1, 0, \ldots, 0)$, we observe

$$J_2 \subset (\hat{x}, 1) + S^-_{0,9} \text{ and } (\hat{x}, 0) + S^+_{0,10} \subset Q.$$

Hence, we obtain

$$J_2 \subset \Gamma_\infty \cap \{(\mathbb{R}^n \times (0, 10]\} \subset Q. \quad (3.17)$$

Now, assuming (3.12), we will prove $\tilde{L}^m \subset B$. To this end, by (3.14) and (3.15), it is enough to show that

$$\tilde{L}^m \subset J^{k-1}_1.$$

On the other hand, since (3.12) yields

$$|J^k_1 \cap L| > (1 - \theta)|L| > 0,$$

we have $(-1, 1)^n \times (0, \frac{1}{2} + \frac{m+1}{9mk-3(9m-1)}) \cap L \neq \emptyset$. By the definition of $\tilde{L}^m$, we have

$$\tilde{L}^m \subset (-1, 1)^n \times \left(0, \frac{1}{2} + \frac{m+1}{9mk-3(9m-1)} + \frac{m+1}{4j-1}\right). \quad (3.18)$$

Setting $\ell^* = \min\{k \in \mathbb{N} \mid L_{k+1} \cap \mathbb{R}^n \times (0, 10] = \emptyset\}$, we have

$$J_2 \subset \Gamma_\ell^* \cap \{(\mathbb{R}^n \times (0, 10]\}.$$

Since $\inf_{J_2} u \leq 1$, by (3.13) again for $\ell = 1, 2, \ldots, km$, we thus have

$$\inf_{\Gamma_{km} \cap \{(\mathbb{R}^n \times (0, 10]\}} u > 1 \geq \inf_{\Gamma_\ell^* \cap \{(\mathbb{R}^n \times (0, 10]\}} u$$

which implies $km < \ell^*$. Hence, noting

$$t_0 + 2^{-2j-3}(9^{\ell^*} - 1) \leq 10,$$

we have

$$2^{-2j} \leq \frac{80}{9km - 1}$$

which, together with (3.18), yields

$$\tilde{L}^m \subset (-1, 1)^n \times \left(0, \frac{1}{2} + \frac{m+1}{9mk-3(9m-1)} + \frac{320(m+1)}{9mk - 1}\right).$$

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Therefore, noting
\[
\frac{1}{9mk-3(9m-1)} + \frac{320}{9mk-1} \leq \frac{1}{9m(k-1)-3(9m-1)},
\]
we can apply Lemma 3.3 to conclude the proof.

It remains to show that (3.13) holds under (3.12).

By setting \( v(x, t) = M^{1-km}u(x_0 + \frac{1}{2^j}x, t_0 + \frac{1}{4}t) \), (3.12) implies
\[
\{(x, t) \in K_1 \mid v(x, t) \geq M\} > (1 - \theta)|K_1|.
\] (3.19)

However, we note that \( v \) is a nonnegative \( L^p \)-viscosity supersolution of
\[
v_i + \mathcal{P}^+(D^2v) + \tilde{\mu}|Dv| - \tilde{f} = 0 \quad \text{in} \ Q,
\]
where \( \tilde{\mu}(x, t) = \frac{1}{2^j}\mu(x_0 + \frac{1}{2^j}x, t_0 + \frac{1}{4}t), \tilde{f}(x, t) = \frac{1}{M^{km-1}}f(x_0 + \frac{1}{2^j}x, t_0 + \frac{1}{4}t) \). We notice that
\[
\|\tilde{f}\|_{L^p(Q)} \leq \varepsilon_1, \quad \text{and} \quad \|\tilde{\mu}\|_{L^q(Q)} \leq \|\mu\|_{L^q(Q)}
\]
because \( q > n + 2 \) and \( p > \frac{n+2}{2} \). Thus, if \( \inf_{K_2} v \leq 1 \) holds, then the same argument to obtain (3.7) yields
\[
\{(x, t) \in K_1 \mid v(x, t) \geq M\} \leq (1 - \theta)|K_1|,
\] (3.20)
which contradicts (3.19). Hence, we have \( v > 1 \) in \( K_2 \), namely, (3.13) holds for \( \ell = 1 \) by the definition.

Next, for \( \ell \geq 2 \), we suppose that (3.13) holds for \( \ell - 1 \). We may suppose that \( N_{\ell-1} \subset \mathbb{R}^n \times (0, 10] \) since otherwise \( N_{\ell} \cap \{\mathbb{R}^n \times (0, 10] \} = \emptyset \), which concludes (3.13) for \( \ell \).

Thus, since
\[
\inf_{N_{\ell-1}} u = \inf_{N_{\ell-1}\cap\{\mathbb{R}^n \times (0,10] \}} u > M^{km-\ell+1},
\]
we have a trivial inequality
\[
\{(x, t) \in N_{\ell-1} \mid u(x, t) \geq M^{km-\ell+1}\} = |N_{\ell-1}| > (1 - \theta)|N_{\ell-1}|.
\] (3.21)

Set \( w(x, t) := \frac{1}{M^{km-\ell+1}}u(x_0 + \frac{g^{\ell-1}}{2^j}x, t_0 + \frac{g^{\ell-1}}{8 \cdot 4^j} + \frac{g^{\ell-1}}{4^j} t) \). In view of (3.17), we easily see that
\[
(x_0, t_0 + \frac{g^{\ell-1}}{8 \cdot 4^j} + \left(-\frac{10 \cdot 3^{\ell-1}}{2^j}, \frac{10 \cdot 3^{\ell-1}}{2^j}\right)n \times \left(0, \frac{10 \cdot g^{\ell-1}}{4^j}\right) \subset Q.
\]

Hence, it follows that \( w > 1 \) in \( K_2 \) because, if \( \inf_{K_2} w \leq 1 \), then the above argument again implies (3.20) for \( w \) in place of \( v \), which contradicts (3.21) for \( w \). \( \square \)
3.2 A general case

In order to show the weak Harnack inequality without assuming (3.2), we use a new barrier function, which will be constructed in Lemma 3.7 below.

Theorem 3.4. Let (2.4) hold, $f \in L^p_+(Q)$ and $\mu \in L^q_+(Q)$. There exist $\epsilon_0 = \epsilon_0(n, \Lambda, \lambda, p, q, \|\mu\|_{L^q(Q)}) > 0$ and $C_0 = C_0(n, \Lambda, \lambda, p, q, \|\mu\|_{L^q(Q)}) > 0$ such that any nonnegative $L^p$-viscosity supersolution $u$ of (3.1) satisfies

$$\left(\int_{J_1} u^{\epsilon_0} \, dx dt\right)^{1/\epsilon_0} \leq C_0 \left(\inf_{J_2} u + \|f\|_{L^p(Q)}\right). \quad (3.22)$$

Remark 3.5. The constants $\epsilon_0, C_0$ above depend on $\|\mu\|_{L^q(Q)}$ in a sense that even if we consider a different $\hat{\mu} \in L^q(Q)$ such that $\|\hat{\mu}\|_{L^q(Q)} \leq \|\mu\|_{L^q(Q)}$ in place of $\mu$ in Theorem 3.4, the same conclusion holds true with the same constants as in Theorem 3.4.

Remark 3.6. When $\mu \in L^\infty(Q)$, $\phi$ in the next lemma can be given by a modified heat kernel from [22]. However, since we have unbounded $\mu$, it is not possible to construct such a precise function for $\phi$ below.

Lemma 3.7. Let $q > n + 2$ and $\mu \in L^q_+(Q)$. There exist a nonnegative function $\phi \in C(\overline{Q}) \cap W^{2,1}_q(Q)$ and a function $g \in L^q_+(Q)$ such that

$$\begin{cases} 
\phi_t + \mathcal{P}^+(D^2\phi) + \mu(x) \mid D\phi \mid \leq g(x, t) & \text{a.e. in } Q, \\
\phi \geq 2 & \text{in } K_2, \\
\phi = 0 & \text{on } \partial_p Q, \\
\text{supp } g \subset K_1.
\end{cases}$$

Proof. Choose a nonnegative function $\xi \in C^\infty(\overline{Q})$ such that $\xi = 0$ in $\overline{Q} \setminus K_{1/4}$, where $K_{1/4} := (-\frac{1}{2}, \frac{1}{2})^n \times (0, \frac{1}{4}]$ (see Fig 1), and $\xi(x, 0) > 0$ for $x \in (-\frac{1}{2}, \frac{1}{2})^n$. In view of Proposition 2.6, we can find a nonnegative function $\psi \in C(\overline{Q}) \cap W^{2,1}_q(Q)$ satisfying

$$\begin{cases} 
\psi_t + \mathcal{P}^+(D^2\psi) + \mu(D\psi) = 0 & \text{a.e. in } Q, \\
\psi = \xi & \text{on } \partial_p Q.
\end{cases}$$

We claim that there exists $\sigma > 0$ such that

$$\psi \geq \sigma \quad \text{in } K_2.$$ 

In fact, assuming $\psi(x_0, t_0) = 0$ for $(x_0, t_0) \in K_2$, we will obtain a contradiction.

For $r \in (0, \frac{1}{\sqrt{10}}]$, we set $v_0(x, t) = \psi(x_0 + rx, t_0 + r^2(t - 10)) = 0$ for $(x, t) \in Q$, $v_0(0, 10) = 0$ and the function $v_0$ is a solution of

$$(v_0)_t + \mathcal{P}^+(D^2v_0) + \hat{\mu}(Dv_0) = 0 \quad \text{in } Q,$$
where
\[ \hat{\mu}(x, t) = r\mu(x_0 + rx, t_0 + r^2(t - 10)). \]

Since it follows that
\[ \|\hat{\mu}\|_{L^p(Q)} \leq r^{1 - \frac{n+2}{q}} \|\mu\|_{L^q(Q)}, \]
if we choose \( r := \left( \delta_0 \|\mu\|_{L^p(Q)}^{-1} \right)^{\frac{n}{q} - (n+2)} \), where \( \delta_0 \) is from Theorem 3.1 for \( p = q \), then

Theorem 3.1 yields \( v_0 = 0 \) in \( J_1 \). To continue the proof we will assume (without loss of generality) that \( t_0 = 1 \). If \( x_0 \in [-\frac{1}{4}, \frac{1}{4}]^n \), then Theorem 3.1 implies \( \psi(x_0, 0) = 0 \), which contradicts our choice of \( \psi \). Thus, without loss of generality, it is enough to consider \( x_0 \in \partial (-3, 3)^n \). Therefore, we can choose \( x_1 \in (-3, 3)^n \) such that \( x_0 \in x_1 + \partial (-r, r)^n \).

Setting \( r_k = \frac{5}{2(k-1)} \) for \( k \geq 1 + \frac{5}{2r} \) (i.e. \( r_k \leq r \)), if we fix \( k \geq \max\{ \frac{253}{3}, 1 + \frac{5}{2r} \} \), then
\[ 10r_k^2(k - 1) \leq \frac{3}{4}. \]

Thus, using Theorem 3.1 finitely many times, we can find \( (x_k, 1 - 10(k-1)r_k^2) \in [-\frac{1}{2}, \frac{1}{2}]^n \times [\frac{1}{4}, 1] \) such that \( u(x_k, 1 - 10(k-1)r_k^2) = 0 \). See Fig 3 for this procedure. Hence, by Theorem 3.1 again, we arrive at a contradiction.
Therefore, for a large number $\tilde{M} > 0$, we verify that $\tilde{M} \psi \geq 2$ in $K_2$. Now, let $\eta \in C^\infty(Q)$ be a nonnegative function such that
\[ \eta = 1 \text{ in } Q \setminus K_1, \text{ and } \eta = 0 \text{ in } K_{1/4}. \]
It is easy to observe that $\phi := \tilde{M} \eta \psi$ satisfies the desired properties. In fact, we may choose $g = \tilde{M}[\psi \eta_t + \mathcal{P}^+(\psi D^2 \eta + 2D\eta \otimes D\psi) + \mu \psi |D\eta|]$.

**Remark 3.8.** We notice that the global $W^{2,1}_p(Q)$ estimate of Proposition 2.6 is necessary to verify that $g \in L^p(Q)$ in the final step of the above proof.

**Proof of Theorem 3.4.** For $\varepsilon_1 > 0$, which will be fixed later, we set
\[ \tilde{u}(x,t) = N_0 u(x,t), \]
where $N_0 = (\inf_{J_2} u + \varepsilon_1^{-1} \|f\|_{L^p(Q)} + \eta)^{-1}$ for $\eta > 0$, which will be sent to 0 at the end of the proof. As in the proof of Theorem 3.1, it is enough to show that there are $\varepsilon_0, C_0 > 0$ such that (3.5) holds under assumptions (3.6).

Let $\phi$ be the function from Lemma 3.7. By letting $w := \phi - u$, it is immediate to see that $w$ is an $L^p$-viscosity subsolution of
\[ w_t + \mathcal{P}^- (D^2 w) - \mu |Dw| - h = 0 \text{ in } Q, \]
where $h := g - f$. In view of Proposition 2.5, we have
\[ \sup_Q w \leq C \|h\|_{L^p(Q_+[w])}. \]
Hence, it is easy to verify that this inequality implies
\[ 1 \leq \sup_{J_2} w \leq C \|h\|_{L^p(Q_+[w])}. \]
Recalling that $\text{supp } g \subset K_1$ in Lemma 3.7, we can find $\tilde{C} = \tilde{C}(n, \Lambda, \lambda, p, q, \|\mu\|_{L^q(Q)}) > 0$ such that
\[ 1 \leq \tilde{C} \left( \|g\|_{L^p(Q_+[w] \cap K_1)} + \varepsilon_1 \right). \]
Thus, for some fixed $\varepsilon_1 > 0$, there is $\theta \in (0,1)$ such that $|\{(x,t) \in K_1 \mid w(x,t) > 0\}| \geq \theta |K_1|$. Hence, as before, we obtain (3.7).

We can follow the same arguments as those in the proof of Theorem 3.1 to conclude the proof. \qed

**Remark 3.9.** In the above proof, we have shown that there exist $A_0, \beta_0, \varepsilon_1 > 0$ such that if $u \in C(Q)$ is an $L^p$-viscosity supersolution of (3.1) satisfying
\[ \inf_{J_2} u \leq 1, \]
and if $\|f\|_{L^p(Q)} \leq \varepsilon_1$, then (3.11) holds true.
4 Applications

In this section, we consider $L^p$-viscosity solutions of

$$u_t + G(x, t, Du, D^2 u) - f(x, t) = 0 \text{ in } Q, \quad (4.1)$$

where $Q = (-10, 10)^n \times (0, 10]$, and $G : Q \times \mathbb{R}^n \times S^n \to \mathbb{R}$ and $f : Q \to \mathbb{R}$ are given. We assume the following hypotheses for $G$ and $f$:

1. there exists $\mu \in L^q(Q)$ for $q > n + 2$ such that
   $$|G(x, t, \xi, O)| \leq \mu(x, t) |\xi| \text{ for } (x, t) \in Q \text{ and } \xi \in \mathbb{R}^n,$$
   \hspace{1cm} (4.2)

2. $f \in L^p(Q)$ for $p \in (p_1, q]$, \hspace{1cm} (4.3)

Remark 4.1. We note that (4.2) yields

$$G(x, t, 0, O) = 0 \text{ for } (x, t) \in Q.$$  

Under (4.2) and (4.3), if we suppose that $G$ satisfies (2.2), then it is easy to observe that if $u \in C(Q)$ is an $L^p$-viscosity subsolution (resp., supersolution) of (4.1), then it is an $L^p$-viscosity subsolution (resp., supersolution) of

$$u_t + \mathcal{P}^-(D^2 u) - \mu |Du| - f = 0 \quad \text{(resp., } u_t + \mathcal{P}^+(D^2 u) + \mu |Du| - f = 0) \text{ in } Q. \quad (4.4)$$

Thus the properties of $L^p$-viscosity solutions of (4.1) discussed in this section will follow from the properties of $L^p$-viscosity sub/supersolutions of the extremal equations (4.4).

4.1 Hölder continuity

We show that the weak Harnack inequality for $L^p$-viscosity supersolutions of (3.1) yields the Hölder continuity of solutions of (4.1) under the above hypotheses. This was remarked in [12] for elliptic PDE.

For $r \in (0, 1)$, we set

$$Q_r := (-10r, 10r)^n \times (10 - 10r^2, 10].$$

Notice that $Q_{10r}(0, 10)$ defined in section 2 is slightly different from this $Q_r$.

Theorem 4.2. Let $G$ satisfy (2.2), (4.2) and (4.3). There exist $C > 0$ and $\alpha \in (0, 1)$ such that if $u \in C(Q)$ is an $L^p$-viscosity solution of (4.1), then

$$|u(x, t) - u(\hat{x}, \hat{t})| \leq C \left( |x - \hat{x}|^2 + |t - \hat{t}| \right)^{\frac{\alpha}{2}} \left( \|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)} \right) \text{ for } (x, t), (\hat{x}, \hat{t}) \in Q_{\frac{1}{2}}.$$
Proof. Working with extremal equations (4.4) and considering
\[ u := \frac{u}{\|u\|_{L^\infty(Q)} + \|f\|_{L^p(Q)}} \]
we can assume that \( \|u\|_{L^\infty(Q)} \leq 1 \) and \( \|f\|_{L^p(Q)} \leq 1 \).

Fix \( r \in (0, 1) \). Setting \( M_r := \sup_Q u \) and \( m_r := \inf_Q u \), we define
\[ \omega(r) := M_r - m_r \quad \text{for } r \in (0, 1). \]

It is easy to observe that for \( (x, t) \in Q \),
\[ v(x, t) := M_r - u(rx, 10 + r^2(t - 10)) \]
and
\[ w(x, t) := u(rx, 10 + r^2(t - 10)) - m_r \]
are nonnegative, \( L^p \)-viscosity supersolutions of (3.1).

Hence, in view of Theorem 3.4, we find constants \( \varepsilon_0, C_0 > 0 \) such that
\[
\left( \int_{J_1} U^\varepsilon_0 \, dx \, dt \right)^{1/\varepsilon_0} \leq C_0 \left( \inf_{J_2} U + r^2 \frac{n+2}{p} \|f\|_{L^p(Q)} \right)
\]
for \( U = v \) and \( U = w \),
where \( \varepsilon_0, C_0 > 0 \) are the constants from Theorem 3.4. Setting
\[ a_0 = 2 - \frac{n+2}{p} \]
and
\[ C_1 = 2^\max(0, \frac{1}{\varepsilon_0} - 1) C_0 |J_1|^{-\frac{1}{\varepsilon_0}}, \]
we have
\[
\omega(r) = \left( \frac{1}{|J_1|} \int_{J_1} \omega(r)^{\varepsilon_0} \, dx \, dt \right)^{1/\varepsilon_0} \leq C_1 \left( \inf_{J_2} v + \inf_{J_2} w + r^{\alpha_0} \right)
\leq C_1 \left( \omega(r) - \sup_{J_2} u + \inf_{J_2} u + r^{\alpha_0} \right),
\]
where \( J_2 := (-r, r)^n \times (10 - r^2, 10] \). Since we may suppose \( C_1 > 1 \), noting \( Q_{10} \subset J_2^r \), we have
\[
\omega \left( 10^{-1} r \right) \leq \sup_{J_2^r} u - \inf_{J_2^r} u \leq \gamma \omega(r) + r^{\alpha_0},
\]
where \( \gamma = \frac{C_1 - 1}{C_1} \). Therefore, in view of the standard argument (e.g. Lemma 8.23 in [9]),
setting \( \alpha = \min \left\{ -\frac{\log \gamma}{\log 10}, \alpha_0 \right\} \in (0, 1) \), we conclude the proof. \( \square \)

### 4.2 Harnack inequality

In order to prove the Harnack inequality we need the local maximum principle for \( L^p \)-viscosity subsolutions of
\[
 u_t + \mathcal{P}^- (D^2 u) - \mu |D u| - f = 0 \quad \text{in } Q. \quad (4.5)
\]
Following the arguments of [4], we show that the weak Harnack inequality implies the local maximum principle. We note that to show Proposition 4.5 below, we can apply the arguments of [10], which is based on the standard one (e.g. [9]).

In this paper, we present a parabolic version of the method of [4] (see also [14]). We first show a blow-up lemma.
Lemma 4.3. Let (2.4) hold and let \( f \in L^p_+(Q) \) and \( \mu \in L^q_+(Q) \). Assume that
\[
\|f\|_{L^p(\Omega)} \leq \varepsilon_1,
\]
where \( \varepsilon_1 > 0 \) is the constant in the proof of Theorem 3.4. Suppose that \( v \in C(Q) \) is an \( L^p \)-viscosity subsolution of (4.5) satisfying
\[
\{|(x,t) \in J_1 \mid v(x,t) \geq s\| \leq A_0 s^{-\beta_0} \quad (\forall s \geq 1)
\]
where \( \beta_0 > 0 \) and \( A_0 > 1 \). Then, there exist \( \nu = \nu(n,\Lambda,\lambda,p,q,\beta_0,A_0) > 1 \), \( n_0 = n_0(n,\Lambda,\lambda,p,q,\beta_0,A_0) \in \mathbb{N} \) and \( \ell_j = \ell_j(n,\Lambda,\lambda,p,q,\beta_0,A_0) \in (0,1) \) for \( j \in \mathbb{N} \) such that \( \sum_{j=1}^{\infty} \ell_j < \infty \), and if \( v \) satisfies \( v(x_0,t_0) \geq \nu^{j-1} \) for some \( j \geq n_0 \) and \( (x_0,t_0) \in \overline{J}_3 \), then it follows that
\[
\sup_{\hat{Q}_j} v \geq \nu^j,
\]
where \( J_3 = (-\frac{1}{2},\frac{1}{2})^n \times (\frac{1}{4},\frac{1}{2}] \) and \( \hat{Q}_j = (x_0,t_0) + (-\ell_j,\ell_j)^n \times (-\frac{\ell_j^2}{10},0] \).

Remark 4.4. The constants \( A_0 \) and \( \beta_0 \) in Lemma 4.3 will be those in Remark 3.9.

\[\text{Figure 4.}\]

Proof. We first fix \( \nu := \frac{\alpha}{\alpha-1} > 1 \) (i.e. \( \alpha = \frac{\nu}{\nu-1} \)), where
\[
\alpha := 2(2A_0)^{\frac{1}{\beta_0}} > 1.
\]
Assume \( \sup_{\hat{Q}_j} v \leq \nu^j \). We will arrive at a contradiction provided
\[
\ell_j := 2(2A_0)^{\frac{1}{\beta_0}+1} A_0^{\frac{1}{\nu-1} \beta_0} > 1.
\]
Choose \( j_0 \in \mathbb{N} \) such that \( \ell_j \leq \frac{1}{2^{\sqrt{10}}} \) for \( j \geq j_0 \). For \( j \geq j_0 \), setting
\[
w^j(x,t) = \frac{\nu}{\nu-1} \left\{ 1 - \nu^{-j} \left( x_0 + \ell_j x, t_0 + \ell_j^2 (t-10) \right) \right\} \geq 0 \quad \text{in} \ Q,
\]

we note that
\[
\inf_{J_2} w^j \leq w^j(0, 10) = \frac{\nu}{\nu - 1} \left\{ 1 - \nu^{-j} v(x_0, t_0) \right\} \leq 1
\]
and that \( w^j \) is an \( L^p \)-viscosity supersolution of
\[
w^j_t + P^+(D^2 w^j) + \hat{\mu}_j |Dw^j| - \hat{f}_j = 0 \quad \text{in } Q,
\]
where \( \hat{\mu}_j(x, t) = \ell_j \mu(x_0 + \ell_j x, t_0 + \ell_j^2 (t - 10)) \) and \( \hat{f}_j(x, t) = \frac{\ell_j^2}{(\nu - 1) \nu - 1} f(x_0 + \ell_j x, t_0 + \ell_j^2 (t - 10)) \). Since it follows that
\[
\|\hat{\mu}_j\|_{L^q(Q)} = \ell_j^{1 - \frac{n+2}{q}} \|\mu\|_{L^q(\hat{Q}_j)}, \quad \text{and} \quad \|\hat{f}_j\|_{L^p(Q)} = \frac{\nu^{1-j}}{\nu - 1} \ell_j^{2 - \frac{n+2}{p}} \|f\|_{L^p(\hat{Q}_j)},
\]
there exists an integer \( n_0 = n_0(n, \lambda, \Lambda, p, q, \beta_0, A_0) \geq j_0 \) such that \( \|\hat{\mu}_j\|_{L^q(Q)} \leq \|\mu\|_{L^q(\hat{Q}_j)} \) and \( \|\hat{f}_j\|_{L^p(Q)} \leq \varepsilon_1 \) for \( j \geq n_0 \).

In view of Remark 3.9, we thus have
\[
\left| \left\{ (x, t) \in J_1 \mid w^j(x, t) \geq \frac{1}{2} \right\} \right| \leq A_0 \left( \frac{2}{\alpha} \right)^{\beta_0} = \frac{1}{2},
\]
Hence, we have
\[
\left| \left\{ (y, s) \in \hat{C}_j \mid v(y, s) \leq \frac{1}{2} \right\} \right| \leq \frac{1}{2} \ell_j^{n+2},
\]
where
\[
\hat{C}_j = (x_0, t_0) + (-\ell_j, \ell_j)^n \times \left( -10 \ell_j^2, -\frac{19}{2} \ell_j^2 \right).
\]
On the other hand, since \( \hat{C}_j \subset J_1 \), by (4.6), we have
\[
\left| \left\{ (y, s) \in \hat{C}_j \mid v(y, s) \geq \frac{1}{2} \right\} \right| \leq A_0 \left( 2\nu^{-j} \right)^{\beta_0}.
\]
Thus, noting
\[
|\hat{C}_j| = 2^{n-1} \ell_j^{n+2} \leq \frac{1}{2} \ell_j^{n+2} + A_0 \left( 2\nu^{-j} \right)^{\beta_0},
\]
we have
\[
\ell_j \leq \left( 2^{\beta_0 + 1} A_0 \nu^{-j \beta_0} \right)^{\frac{1}{n+2}},
\]
which contradicts (4.7).

We can now show the local maximum principle for \( L^p \)-viscosity subsolutions.
Proposition 4.5. Let (2.4) hold and let $f \in L^p_+(Q)$ and $\mu \in L^q_+(Q)$. Then, for any $\varepsilon_0 \in (0, \beta_0)$, there exists a constant $C_3 = C_3(n, \Lambda, \lambda, p, q, \|\mu\|_{L^q(Q)}, \varepsilon_0) > 0$ such that any $L^p$-viscosity subsolution $u \in C(Q)$ of (4.5) satisfies

$$\sup_{J_3} u \leq C_3 \left( \|u^+\|_{L^p(J_1)} + \|f\|_{L^p(Q)} \right),$$

where $\beta_0 > 0$ is the constant in Remark 3.9.

Proof. Choose $(y_0, s_0) \in \mathcal{J}_3$ such that

$$\sup_{J_3} u = u(y_0, s_0).$$

Setting

$$N_0 = \left( A_0^{-1} \int_{J_1} (u^+) \cdot \varepsilon_0 \, dx \, dt \right)^{\frac{1}{\varepsilon_0}} + 2 \varepsilon_1^{-1} \|f\|_{L^p(Q)},$$

where $\varepsilon_1 > 0$ is from the proof of Theorem 3.4, we observe that $v := N_0^{-1}u$ is an $L^p$-viscosity subsolution of

$$v_t + \mathcal{P}^- (D^2 v) - \mu |Dv| - \frac{1}{N_0} f = 0 \quad \text{in } Q.$$

We note that for $s \geq 1$, we have

$$\left| \left\{ (x, t) \in J_1 \mid v(x, t) \geq s \right\} \right| \leq \frac{1}{s \varepsilon_0} \int_{J_1} v^\varepsilon_0 \, dx \, dt \leq A_0 s^{-\varepsilon_0}.$$

Let $\nu > 1$, $n_0 \in \mathbb{N}$ and $\ell_j > 0$ be the constants in Lemma 4.3 when $\beta_0 = \varepsilon_0$. There exists $n_1 \geq n_0$ such that

$$\sum_{j=n_1}^{\infty} \ell_j \leq \frac{1}{4}.$$

Now, suppose that there is $(y_0, s_0) \in \mathcal{J}_3$ such that

$$v(y_0, s_0) \geq \nu^{n_1-1}.$$

In view of Lemma 4.3, for $j \in \mathbb{N}$, we can find $(y_j, s_j) \in (y_{j-1}, s_{j-1}) + [-\ell_{j+n_1-1}, \ell_{j+n_1-1}]^n \times [-\frac{\ell_{j+n_1-1}}{10}, 0]$ such that

$$v(y_j, s_j) \geq \nu^{n_1+j-1}.$$

Because $(y_j, s_j) \in [-\frac{3}{4}, \frac{3}{4}]^n \times [\frac{1}{8}, \frac{1}{2}]$, this contradicts that $v \in C(Q)$. Therefore, we conclude the proof. \[\square\]

Using the weak Harnack inequality, together with Proposition 4.5, we can obtain the Harnack inequality which we state without proof.
Corollary 4.6. Let \((2.4)\) hold and let \(f \in L^p(Q)\) and \(\mu \in L^q(Q)\). There is a constant 
\[ C_4 = C_4(n, \Lambda, \lambda, p, q, \|\mu\|_{L^q(Q)}) > 0 \]
such that any nonnegative \(L^p\)-viscosity solution \(u \in C(\Omega)\) of \((4.1)\) satisfies 
\[ \sup_{J_3} u \leq C_4 \left( \inf_{J_2} u + \|f\|_{L^p(Q)} \right), \] 
where \(J_3 = (-\frac{1}{2}, \frac{1}{2})^n \times (\frac{1}{4}, \frac{1}{2})\) and \(J_2 = (-1, 1)^n \times (9, 10]\).

5 Remarks on the superlinear growth case

In this section, we exhibit several properties of \(L^p\)-viscosity solutions of \((4.1)\), where \(G\) satisfies \((2.2)\), \((4.3)\) and, in place of \((4.2)\), 
\[ \{ \begin{array}{l}
\text{there are } m > 1 \text{ and } \mu \in L^q_+(Q) \text{ for } q > n + 2 \text{ such that } \\
|G(x, t, \xi, O)| \leq \mu(x, t)|\xi|^m \text{ for } (x, t) \in Q \text{ and } \xi \in \mathbb{R}^n.
\end{array} \] 
(5.1)

More precisely, we present a remark on the ABP maximum principle in [11], and an 
existence result corresponding to that in [13], with which we show the weak Harnack 
inequality for \(L^p\)-viscosity supersolutions \((4.1)\) under \((5.1)\).

If \((2.2)\), \((4.3)\) and \((5.1)\) are satisfied then if \(u \in C(Q)\) is an \(L^p\)-viscosity subsolution 
(resp., supersolution) of \((4.1)\), then it is an \(L^p\)-viscosity subsolution (resp., supersolution) 
of 
\[ u_t + \mathcal{P}^- (D^2 u) - \mu |Du|^m - f = 0 \] 
(resp., \(u_t + \mathcal{P}^+ (D^2 u) + \mu |Du|^m - f = 0\)) in \(Q\).

To establish the ABP maximum principle and the weak Harnack inequality we only need 
to work with the above extremal inequalities.

5.1 A remark on the ABP maximum principle

In this section, to comply with the setup of [11], \(Q = \Omega \times (0, T]\), where \(0 < T \leq 1\) and 
the domain \(\Omega\) satisfies 
\[ \Omega \subset \{x \in \mathbb{R}^n \mid |x| < 1\}. \] 
(5.2)

We recall the ABP maximum principle from [11]. The estimates there seem a little 
complicated. However, if we carefully examine them, we can give simple statements as 
below.

Proposition 5.1. (Theorems 3.11 and 3.12 of [11]) Let \((2.4)\) hold with \(q < +\infty\). Let 
\((2.2)\), \((4.3)\) and \((5.1)\) be satisfied and let 
\[ p > \frac{(m - 1)q(n + 2)}{mq - n - 2}. \] 
(5.3)
There exist $\delta = \delta(n, \Lambda, \lambda, m, p, q) > 0$ and $C = C(n, \Lambda, \lambda, m, p, q) > 0$ such that if $u \in C(Q)$ is an $L^p$-viscosity subsolution of (4.1), and
\[
\|f\|^{m-1}_{L^p(Q)}\|\mu\|_{L^q(Q)} \leq \delta,
\] (5.4)
then
\[
\sup_{Q} u \leq \sup_{\partial_p Q} u + C\|f\|_{L^p(Q)}.
\]

**Remark 5.2.** We note that (5.3) is satisfied when $n + 2 \leq p \leq q, q > n + 2$, and (5.3) is equivalent to
\[
mg(n + 2 - p) < (n + 2)(q - p),
\] which is (iv) of (5.6). We also remark that when $q = +\infty$, the ABP maximum principle does not require any smallness condition and can be found in Theorems 3.7 and 3.8 of [11]. Condition (5.3) then reduces to $p > (m - 1)(n + 2)/m$, which is the inequality in (i) of (5.6).

We show here that the smallness condition (5.4) can be removed, however the estimate becomes more complicated.

**Theorem 5.3.** Let (2.4) hold with $q < +\infty$. Let (2.2), (4.3), (5.1) and (5.3) be satisfied. There exists $C = C(n, \Lambda, \lambda, m, p, q) > 0$ such that if $u \in C(Q)$ is an $L^p$-viscosity subsolution of (4.1), then
\[
\sup_{Q} u \leq \sup_{\partial_p Q} u + C\left(1 + \|f\|^{(m-1)q}_{L^p(Q)}\|\mu\|_{L^q(Q)}^{q}\right)^{\frac{p-1}{p}} \|f\|_{L^p(Q)}.
\]

**Proof.** By considering $u := u - \sup_{\partial_p Q} u$, we may assume that $\sup_{\partial_p Q} u \leq 0$. When (5.4) does not hold, it is easy to see that we can find a partition $0 = t_0 < t_1 < \ldots < t_k = T$ such that, setting $Q_i := \Omega \times [t_{i-1}, t_i], i = 1, \ldots, k,$ and $\hat{\delta} := \|f\|^{(m-1)q}_{L^p(Q)}\delta$, we have
\[
\|\mu\|_{L^q(Q_i)} \leq \hat{\delta} \quad \text{for} \quad i = 1, \ldots, k, \quad \text{where} \quad k \leq 1 + \delta^{-q}\|f\|^{(m-1)q}_{L^p(Q)}\|\mu\|_{L^q(Q)}.
\]
By Proposition 5.1, we then have
\[
\sup_{Q_i} u \leq \sup_{\partial_p Q_i} u + C\|f\|_{L^p(Q_i)} \quad \text{for} \quad i = 1, \ldots, k.
\]
Let $(\hat{x}, \hat{t}) \in Q_i$ satisfy $u = u(\hat{x}, \hat{t})$ for some $i \in \{1, \ldots, k\}$. Then
\[
\sup_{Q} u \leq \sup_{\partial_p Q_i} u + C\|f\|_{L^p(Q_i)} \leq \max(0, \sup_{Q_{i-1}} u) + C\|f\|_{L^p(Q_i)}.
\]
But
\[
\sup_{Q_{i-1}} u \leq \sup_{\partial_p Q_{i-1}} u + C\|f\|_{L^p(Q_{i-1})} \leq \max(0, \sup_{Q_{i-2}} u) + C\|f\|_{L^p(Q_{i-1})}.
\]
Therefore, continuing this procedure, we obtain

\[ \sup_Q u \leq C \sum_{i=1}^{k} \|f\|_{L^p(Q_i)}. \]

Now

\[ \sum_{i=1}^{k} \|f\|_{L^p(Q_i)} \leq k^{\frac{p-1}{p}} \|f\|_{L^p(Q)} \leq \left(1 + \delta^{-q} \|f\|_{L^p(Q)}^q \|\mu\|_{L^q(Q)}^{\frac{q}{2}} \right)^{\frac{p-1}{p}} \|f\|_{L^p(Q)}. \]

5.2 Existence of strong solutions

In this subsection, for the sake of simplicity, \( \Omega \) is as in (5.2) and we assume that \( \partial \Omega \) is \( C^{1,1} \). We discuss the existence of \( L^p \)-viscosity solutions of parabolic extremal PDE,

\[ u_t + \mathcal{P}^\pm (D^2 u) \pm \mu |Du|^m = f \quad \text{in } Q := \Omega \times (0,1], \tag{5.5} \]

where \( m > 1, f \in L^p(Q) \) and \( \mu \in L^q(Q) \). Since we do not know a precise proof of \( W_{p,1}^2 \)-estimates near \( \partial \Omega \) of [22], possibly for \( p \leq n + 1 \), (though it was mentioned in [22] without a proof), we will use global estimates for \( p > n + 1 \) from [8] to show a different type of estimates. Thus we will assume that \( p > n + 1 \). We first recall a global estimate for \( L^p \)-strong solutions of extremal PDE with no first derivative terms.

**Proposition 5.4.** (e.g. Theorem 1.1 of [8]) Let \( \partial \Omega \) be \( C^{1,1} \) and \( p > n + 1 \). Then, there exists a constant \( C = C(n, p, \Lambda, \lambda, \text{diam}(\Omega), \partial \Omega) > 0 \) such that for every \( f \in L^p(Q) \) and \( \psi \in W_{p,1}^2(Q), \) there exists a unique \( u \in C(\overline{Q}) \cap W_{p,1}^2(Q) \) such that

\[
\left\{\begin{array}{l}
    u_t + \mathcal{P}^\pm (D^2 u) = f \quad \text{a.e. in } Q, \\
    u = \psi \quad \text{on } \partial_p Q,
\end{array}\right.
\]

and

\[ \|u\|_{W_{p,1}^2(Q)} \leq C \left( \|f\|_{L^p(Q)} + \|\psi\|_{W_{p,1}^2(Q)} \right). \]

For the elliptic case, in [13], the existence of \( L^p \)-strong solutions of extremal PDE with superlinear growth in the first derivatives was obtained assuming that \( \|\mu\|_{L^q(Q)} \) is small enough. Following the idea of [13], we establish the corresponding existence result for \( L^p \)-strong solutions of (5.5).
Theorem 5.5. Let $\partial \Omega$ be $C^{1,1}$, $n + 1 < p \leq q$, $q > n + 2$, $f \in L^p(Q)$, $\mu \in L^q(Q)$ and $\psi \in W^{2,1}_p(Q)$. Assume that one of the following conditions holds:

\[
\begin{align*}
(i) & \quad n + 1 < p < n + 2, m(n + 2 - p) < n + 2, q = \infty, \\
(ii) & \quad p \geq n + 2, q = \infty, \\
(iii) & \quad n + 2 < p = q < \infty, \\
(iv) & \quad n + 1 < p < q, q > n + 2, mq(n + 2 - p) < (n + 2)(q - p).
\end{align*}
\]

(5.6)

Assume also that

\[
\begin{align*}
& \begin{cases}
  r = pm & \text{for (i), (ii)} \\
  r = \infty & \text{for (iii)}, \\
  r = \frac{mp}{q-p} & \text{for (iv)}.
\end{cases}
\end{align*}
\]

(5.7)

Then, there exists $\delta_1 = \delta_1(n, \Lambda, \lambda, p, q, m) > 0$ such that if

\[
\|\mu\|_{L^q(Q)} \left(\|f\|_{L^p(Q)} + \|\psi\|_{W^{2,1}_p(Q)}\right)^{m-1} \leq \delta_1,
\]

(5.8)

then there exist $L^p$-strong solutions $u \in W^{2,1}_p(Q)$ of

\[
\begin{align*}
& \begin{cases}
  u_t + \mathcal{P}^\pm(D^2u) \pm \mu|Du|^m = f & \text{a.e. in } Q, \\
  u = \psi & \text{on } \partial_p Q.
\end{cases}
\end{align*}
\]

(5.9)

Moreover, there exists $\hat{C} = \hat{C}(n, \Lambda, \lambda, p, q, m, \text{diam}(\Omega), \partial \Omega) > 0$ such that

\[
\|u\|_{W^{2,1}_p(Q)} \leq \hat{C} \left(\|f\|_{L^p(Q)} + \|\psi\|_{W^{2,1}_p(Q)}\right). 
\]

(5.10)

Remark 5.6. We note that in (iv) of (5.6), if $p \geq n + 2$, then the third inequality automatically holds.

Proof. We will do the proof only for the case of $\mathcal{P}^+$. For $r$ in (5.7), we define a mapping $K : W^{1,0}_r(Q) \to W^{2,1}_p(Q)$ in the following way. For $v \in W^{1,0}_r(Q)$, in view of Proposition 5.4, we find a unique solution $u := Kv \in W^{2,1}_p(Q)$ of

\[
\begin{align*}
& \begin{cases}
  u_t + \mathcal{P}^+(D^2u) + \mu|Du|^m = f & \text{a.e. in } Q, \\
  u = \psi & \text{on } \partial_p Q.
\end{cases}
\end{align*}
\]

(5.9)

Since $\|Kv\|_{W^{2,1}_p(Q)} \leq C(\|f\|_{L^p(Q)} + \|\mu|Du|^m\|_{L^p(Q)} + \|\psi\|_{W^{2,1}_p(Q)})$ holds for some $C > 0$, noting

\[
\|\mu|Du|^m\|_{L^p(Q)} \leq C\|\mu\|_{L^q(Q)}\|Du\|_{L^r(Q)}^m,
\]

we can argue like in the proof of Theorem 3.1 of [13] to find a sufficiently large $\alpha$ and small $\delta_1 > 0$ such that if $R = \alpha(\|f\|_{L^p(Q)} + \|\psi\|_{W^{2,1}_p(Q)})$, then $K : \mathcal{B}_R \to W^{2,1}_p(Q) \cap \mathcal{B}_R$ is a continuous map when (5.8) holds, where

\[
\mathcal{B}_R = \left\{ v \in W^{1,0}_r(Q) \mid \|v\|_{W^{1,0}_r(Q)} \leq R \right\}.
\]

Since $W^{2,1}_p(Q)$ is compactly imbedded in $W^{1,0}_r(Q)$ (see the next proposition), we conclude the proof by the Schauder fixed point theorem as in [13].
For the reader’s convenience, we provide a proof of compact imbeddings of parabolic Sobolev spaces. More general results for compact imbeddings of anisotropic Sobolev spaces can be found in [1] and [2] (see in particular Theorem 10.2 of [1] and Theorem 26.3.5 of [2]).

**Proposition 5.7.** Let \( \partial \Omega \) be Lipschitz. Assume that \( 1 \leq p \leq r \) satisfy one of the following conditions:

\[
\begin{align*}
(i) & \quad p < n + 2, p \leq r < \frac{p(n+2)}{n+2-p}, \\
(ii) & \quad p = n + 2 \leq r < \infty, \\
(iii) & \quad n + 2 < p < \infty, r = \infty.
\end{align*}
\]

(5.11)

Then, \( W^{2,1}_p(Q) \) is compactly imbedded in \( W^{1,0}_r(Q) \).

**Proof.** Under assumption (5.11), by Lemma 3.3 of [16], it follows that there exist \( \varepsilon' > 0 \) and \( C > 0 \) such that for any \( \varepsilon \in (0, \varepsilon') \), we have for \( u \in W^{2,1}_p(Q) \)

\[
\|u\|_{L^r(Q)} + \|Du\|_{L^r(Q)} \leq C\varepsilon^{\alpha} \left( \|D^2u\|_{L^p(Q)} + \|u_t\|_{L^p(Q)} \right) + C\varepsilon^{-(2-\alpha)}\|u\|_{L^p(Q)},
\]

(5.12)

where \( \alpha = 1 - \frac{n+2}{p} + \frac{n+2}{r} > 0 \) for \( r < \infty \), or \( \alpha = 1 - \frac{n+2}{p} > 0 \) for \( r = \infty \). Here, \( C \) is independent of \( u \) and \( \varepsilon \in (0, \varepsilon') \). (A better inequality is true for \( \|u\|_{L^r(Q)} \) but we do not need it here.)

In view of (5.12), it is thus enough to show that a bounded subset of \( W^{2,1}_p(Q) \) is compact in \( L^p(Q) \). However this is clear since \( W^{2,1}_p(Q) \subset W^{1,0}_p(Q) \) (when we consider \( Q \) as a subset of \( \mathbb{R}^{n+1} \)) and the mapping \( I : W^{1,0}_p(Q) \to L^p(Q) \) is compact by the standard compact Sobolev imbedding theorem (see e.g. Theorem 7.26 of [9]). \( \square \)

**Remark 5.8.** We remark that for case (5.11)-(iii) a stronger result is true, namely that \( W^{2,1}_p(Q) \) is compactly imbedded in the parabolic space \( C^{1+\alpha}(Q) \) for \( \alpha = 1 - \frac{n+2}{p} \).

### 5.3 Weak Harnack inequality

Using Theorem 5.5, we establish the weak Harnack inequality for \( L^p \)-viscosity supersolutions of uniformly parabolic PDE with superlinear growth in \( Du \). We refer to [13] for an analogous elliptic result.

In this subsection, we again set \( Q := (-10, 10)^n \times (0, 10] \). In what follows, we will utilize the same notation as that in Figure 1. We will construct a barrier function for (5.5) when \( m > 1 \). This will require a slightly more careful analysis than that in the elliptic case.

**Lemma 5.9.** Assume that (2.4) holds. Then, there exists \( \delta_2 = \delta_2(n, \Lambda, \lambda, q, m) > 0 \) such that if \( \mu \in L^q(Q) \) satisfies

\[
\|\mu\|_{L^q(Q)} \leq \delta_2,
\]

...
then there exist \( \phi \in W^{2,1}_q(Q) \cap C(\overline{Q}) \) and \( g \in L^q(Q) \) such that

\[
\begin{cases}
\phi_t + \mathcal{P}^+(D^2\phi) + \mu |D\phi|^m \leq g & \text{a.e. in } Q, \\
\phi \geq 2 & \text{in } K_2, \\
\phi = 0 & \text{in } \partial_p Q, \\
supp g \subset K_1.
\end{cases}
\] (5.13)

Proof. We first introduce a smooth, nonnegative \( \eta : \overline{Q} \rightarrow [0, 1] \) satisfying

\[
\begin{cases}
(i) \quad \eta(x, t) = 1 \text{ for } (x, t) \in \overline{Q} \text{ if } |x| \geq 1 \text{ or } t \geq 1, \\
(ii) \quad \eta(x, 0) = 0 \text{ for } |x| \leq \frac{1}{2}, \\
(iii) \quad \eta \in W^{2,1}_\infty(Q).
\end{cases}
\] (5.14)

We next choose a nonnegative function \( \xi_0 \in C^\infty(\mathbb{R}^n \times [0, \infty)) \) such that

\[
\begin{cases}
(i) \quad \xi_0 = 0 \text{ in } \mathbb{R}^n \times [0, \infty) \setminus \{(x, t) \in \mathbb{R}^n \times [0, \frac{1}{2}) \mid |x| < \frac{1}{2}\}, \\
(ii) \quad \xi_0(x, 0) > 0 \text{ for } |x| < \frac{1}{2}.
\end{cases}
\] (5.15)

As in the proof of Lemma 4.1 in [13], we claim that there exist \( \sigma > 0 \) such that if \( \mu \in L^q(Q) \) satisfies \( \|\mu\|_{L^q(Q)} \leq \delta_2 \), then the strong solution \( \psi \in W^{2,1}_q(Q) \) of

\[
\begin{cases}
\psi_t + \mathcal{P}^+(D^2\psi) + \mu' |D\psi|^m = 0 & \text{a.e. in } Q, \\
\psi = \xi_0 & \text{on } \partial_p Q
\end{cases}
\] (5.16)

satisfies

\[ \psi \geq \sigma \quad \text{in } K_2. \]

Indeed, otherwise, there are nonnegative \( \psi_k \in W^{2,1}_q(Q) \cap C(\overline{Q}) \) and \( \mu_k \in L^q(Q) \) such that \( \|\mu_k\|_{L^q(Q)} \leq \frac{1}{k} \) and \( \psi_k \) is a strong solution of (5.16) with \( \mu' \) replaced by \( \mu_k \), such that \( \inf_{K_2} \psi_k \leq \frac{1}{k} \), then (by (5.10)) a subsequence \( \{\psi_k\}_{j=1}^\infty \) converges uniformly in \( \overline{Q} \) to some \( \psi \in W^{2,1}_q(Q) \), and \( \inf_{K_2} \psi = 0 \). Since \( \psi \) is a strong solution of

\[
\begin{cases}
\psi_t + \mathcal{P}^+(D^2\psi) = 0 & \text{a.e. in } Q, \\
\psi = \xi_0 & \text{on } \partial_p Q
\end{cases}
\]

we can find \((\widehat{x}, \widehat{t}) \in K_2 \) such that \( \psi(\widehat{x}, \widehat{t}) = 0 \), which gives a contradiction as in the proof of Lemma 3.7.

We now choose \( \delta_2 > 0 \) small enough so that

\[ (4\sigma^{-1})^{m-1}\delta_2 \leq \delta_2^0 \quad \text{and} \quad (4\sigma^{-1})^{m-1}\delta_2 \|\xi_0\|^{m-1}_{W^{2,1}_q(Q)} \leq \delta_1, \]

where \( \delta_1 \) is from Theorem 5.5.
In view of Theorem 5.5 and the above choice of $\delta_2$, if $\mu \in L^q(Q)$ satisfies $\|\mu\|_{L^q(Q)} \leq \delta_2$, then there exists $\psi^0 \in C(\overline{Q}) \cap W^{2,1}_q(Q)$ such that

$$\begin{cases} 
\psi_t^0 + \mathcal{P}^+(D^2 \psi^0) + (4\sigma^{-1}\eta)^{m-1} \mu|D\psi^0|^m = 0 & \text{a.e. in } Q, \\
\psi^0 = \xi_0 & \text{on } \partial_p Q,
\end{cases}$$

and $\psi^0 \geq \sigma$ in $K_2$.

Setting $\psi = (2/\sigma)\psi^0$ and $\xi = (2/\sigma)\xi_0$, we observe that

$$\begin{cases} 
\psi_t + \mathcal{P}^+(D^2 \psi) + (2\eta)^{m-1} \mu|D\psi|^m = 0 & \text{a.e. in } Q, \\
\psi = \xi & \text{on } \partial_p Q.
\end{cases}$$

Furthermore, it is easy to check that $\phi := \eta \psi$ satisfies

$$\phi_t + \mathcal{P}^+(D^2 \phi) + \mu|D\phi|^m \leq g \quad \text{a.e. in } Q,$$

where $g = \psi_\eta + 2^{m-1}\mu|D\eta|^m + \mathcal{P}^+(D\eta \otimes D\psi + D\psi \otimes D\eta + \psi D^2 \eta)$, and $\phi$ and $g$ satisfy all the conditions required in (5.13). \hfill \Box

We will now show that the weak Harnack inequality holds under a smallness condition. Since we separate the weak Harnack inequality from the $L^\infty$-estimate, similarly to Theorem 4.2 in [13], we assume boundedness of supersolutions.

**Theorem 5.10.** Suppose that (5.6) holds and assume

$$1 < m < 2 - \frac{n+2}{q}, \quad (5.17)$$

Let $M > 0$, $f \in L^p_+(Q)$ and $\mu \in L^q(Q)$. Then, there exist $\delta_3 = \delta_3(n, \lambda, \Lambda, p, q, m, M) > 0$, $C = C(n, \lambda, \Lambda, p, q, m) > 0$ and $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda, p, q, m) > 0$ such that if

$$\|\mu\|_{L^q(Q)} \left(1 + \|f\|_{L^p(Q)}^{m-1}\right) < \delta_3, \quad (5.18)$$

and $u \in C(\overline{Q})$ is an $L^p$-viscosity supersolution of

$$u_t + \mathcal{P}^+(D^2 u) + \mu|Du|^m + f = 0 \quad \text{in } Q$$

satisfying $0 \leq u \leq M$ in $Q$, then

$$\left( \int_{J_1} u^{\varepsilon_0} \, dx \, dt \right)^{1/\varepsilon_0} \leq C \left( \inf_{J_2} u + \|f\|_{L^p(Q)} \right).$$

**Proof.** The proof follows the arguments of the proof of Theorem 4.2 of [13] so we just sketch it. We first reduce to the case of $f = 0$. Let $\delta_1$ be from Theorem 5.5 and let

$$\|\mu\|_{L^q(Q)} (2\|f\|_{L^p(Q)})^{m-1} < \delta_1.$$
Let \( w \in W^{2,1}_p(Q) \) be from Theorem 5.5 such that
\[
\begin{cases}
  w_t + \mathcal{P}^- (D^2 w) - 2^{m-1} \mu |Dw|^m - f = 0 & \text{a.e. in } Q, \\
  w = 0 & \text{on } \partial_p Q.
\end{cases}
\]

By Theorem 5.3, we have
\[
0 \leq w \leq C \|f\|_{L^p(Q)},
\]
and it is easy to see that \( v := u + w \) is an \( L^p \)-viscosity supersolution of
\[
v_t + \mathcal{P}^+ (D^2 v) + 2^{m-1} \mu |Dv|^m = 0 \quad \text{in } Q.
\]

Thus, if we can prove that
\[
\left( \int_{J_2} v^{\varepsilon_0} \, dx \, dt \right)^{\frac{1}{\varepsilon_0}} \leq C \inf_{J_2} v,
\]
the claim will follow using (5.19). Thus we can assume that \( f = 0 \).

We now set \( m_0 := \inf_{J_2} u \). We may suppose \( m_0 > 0 \) by adding a positive constant, which will be sent to 0 in the end. Considering \( v := m_0^{-1} u \), we verify that \( \inf_{J_2} v \leq 1 \), and it is an \( L^p \)-viscosity supersolution of
\[
v_t + \mathcal{P}^+ (D^2 v) + m_0^{m-1} \mu |Dv|^m = 0 \quad \text{in } Q.
\]

In view of Lemma 5.9, if
\[
(2M)^{m-1} \|\mu\|_{L^q(Q)} \leq \delta_2,
\]
where \( \delta_2 > 0 \) is from Lemma 5.9, we can find a strong solution \( \phi \in W^{2,1}_q(Q) \) of
\[
\begin{cases}
  \phi_t + \mathcal{P}^+ (D^2 \phi) + (2m_0)^{m-1} \mu |D\phi|^m \leq g & \text{a.e. in } Q, \\
  \phi = 0 & \text{on } \partial_p Q, \\
  \phi \geq 2 & \text{in } K_2, \\
  \text{supp } g \subset K_1.
\end{cases}
\]

Then \( w := \phi - v \) is an \( L^p \)-viscosity subsolution of
\[
w_t + \mathcal{P}^- (D^2 w) - (2m_0)^{m-1} \mu |Dw|^m - g = 0 \quad \text{in } Q.
\]

Hence, Theorem 5.3 yields
\[
1 \leq \sup_{J_2} w \leq \sup_Q w \leq C \|g\|_{L^p((x,t) \in K_1 | (\phi - v)(x,t) \geq 0))},
\]
where \( C \) is a constant which depends on various absolute constants, \( \delta_2 \), and \( \|g\|_{L^p(Q)} \), which is also bounded by various absolute constants. The above inequality now implies
\[
|\{(x,t) \in K_1 | v(x,t) > M\}| \leq (1 - \theta)|K_1| \text{ for some } M > 1 \text{ and } \theta \in (0,1). \]

The rest of the proof follows the arguments in the proof of Theorem 4.2 of [13].
References


