Viscosity solutions to HJB equations for boundary-noise and boundary-control problems

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Abstract

We consider a boundary-noise and boundary-control problem for an equation of parabolic type in a bounded domain in \( \mathbb{R}^d \), with a Neumann boundary condition. We study the Hamilton-Jacobi-Bellman equation associated to this problem and introduce an appropriate notion of viscosity solution. We prove comparison principle for viscosity subsolutions and viscosity supersolutions of the Hamilton-Jacobi-Bellman equation. We also prove various continuity properties of the value function, show that it satisfies the dynamic programming principle, and prove that the value function is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation.

Keywords: Hamilton-Jacobi-Bellman equation, viscosity solution, stochastic optimal control, boundary-noise and boundary-control problem.

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1 Introduction

We consider a boundary-noise and boundary-control problem for a state equation of parabolic type in a bounded domain \( O \subset \mathbb{R}^d \), with a Neumann boundary condition. In this problem the derivative of the unknown function on the boundary is equal to the sum of a control and a noise. Let \( T > 0 \).

The state equation for a problem on a time interval \( (t, T] \) is

\[
\begin{aligned}
\frac{\partial}{\partial s} y(s, \xi) &= \Delta_{\xi} y(s, \xi) + f(y(s, \xi)), (s, \xi) \in (t, T) \times O \\
\frac{\partial}{\partial n} y(s, \xi) &= h(\xi) \frac{dW_Q(s)}{ds} + a(s, \xi), (s, \xi) \in (t, T) \times \partial O \\
y(t, \xi) &= x(\xi),
\end{aligned}
\]

(1.1)

where \( f : \mathbb{R} \rightarrow \mathbb{R} \) is Lipschitz continuous, \( h \in L^\infty(\partial O) \), \( x \in L^2(O) \), \( W_Q \) is a \( Q \)-Wiener process in \( \Lambda = L^2(\partial O) \), and \( a(\cdot) : [t, T] \times \partial O \times \Omega \rightarrow \mathbb{R} \) is a control process on some probability space \( \Omega \). We assume that \( Q \geq 0 \) is a bounded, self-adjoint, trace class operator in \( \Lambda \). The goal is to minimize a cost functional

\[
I(t, x; a(\cdot)) = \mathbb{E} \left[ \int_t^T \left( \int_O \beta_1(\xi, y(s, \xi)) \, d\xi + \int_{\partial O} \beta_2(a(s, \xi)) \, d\xi \right) \, ds + \int_O \gamma(\xi, y(T, \xi)) \, d\xi \right]
\]

(1.2)
over all control processes, for some continuous functions \( \beta_1, \gamma : \mathcal{O} \times \mathbb{R} \to \mathbb{R}, \beta_2 : \mathbb{R} \to \mathbb{R} \). This will be made precise later. We first rewrite the problem in the infinite dimensional setup, see [4], Appendix C and Section 2.6.3.

Let \( H = L^2(\mathcal{O}) \). Throughout the manuscript \( \mathcal{O} \) is assumed to be a bounded domain in \( \mathbb{R}^d \) with smooth boundary. We define

\[
Ax = \Delta x, \quad D(A) = \left\{ x \in H^2(\mathcal{O}) : \frac{\partial x}{\partial n} = 0 \text{ on } \partial \mathcal{O} \right\},
\]

where \( n \) is the outward unit normal vector to \( \mathcal{O} \). Let \( \mathcal{N}_\lambda \) for \( \lambda > 0 \) be the Neumann operator, see e.g. [4], Appendix C.1. It is well known (see e.g. [4], Theorem C.5), that \( \mathcal{N}_\lambda : L^2(\partial \mathcal{O}) \to H^2(\mathcal{O}) \) is a bounded operator. Moreover (see e.g. [4], Proposition C.7), for every \( \lambda > 0 \), \( (\lambda I - A)^{-1} \) exists as a bounded operator in \( L^2(\mathcal{O}) \), and the semigroup \( e^{t(A-\lambda I)} \) is analytic for \( \lambda \geq 0 \). Thus (see e.g. [4], Proposition C.8), for \( \beta \in (1/4,1/2) \), \( G_\lambda := (\lambda I - A)^{1-\beta} \mathcal{N}_\lambda \) is a bounded operator from \( L^2(\partial \mathcal{O}) \) to \( L^2(\mathcal{O}) \).

We now define \( b : H \to H \) by \( b(x)(\cdot) := f(x(\cdot)) \), a bounded linear operator \( \sigma : \Lambda \to \Lambda \) by \( [\sigma z](\cdot) := h(\cdot)z(\cdot) \), \( a(\cdot) : [t,T] \times \Omega \to \Lambda \) by \( a(s) := a(s,\cdot) \), and set \( \beta \in (1/4,1/2), \lambda > 0 \). As it is explained in [4], Appendix C, the solution of (1.1) is defined to be the mild solution of the stochastic evolution equation

\[
\begin{cases}
    dX(s) = [Ax(s) + b(X(s)) + (\lambda I - A)\mathcal{N}_\lambda a(s)] \, ds + (\lambda I - A)\mathcal{N}_\lambda \sigma dW_Q(s) \\
    X(t) = x \in H,
\end{cases}
\]

which means that for \( s \in [t,T] \)

\[
X(s) = e^{(s-t)A}x + \int_t^s e^{(s-r)A}b(X(r))dr + \int_t^s (\lambda I - A)^{\beta}e^{(s-r)A}B_\lambda a(r)dr \\
+ \int_t^s (\lambda I - A)^{\beta}e^{(s-r)A}G_\lambda \sigma dW_Q(r) \quad \mathbb{P}\text{-a.e.}
\]

We now define rigorously the set of admissible controls. We fix \( K > 0 \). We say that \( \mu = (\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s \in [t,T]}, \mathbb{P}, W_Q) \) is a reference probability space if \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a complete probability space, \( W_Q \) is a \( Q \)-Wiener process in \( \Lambda \), \( W_Q(t) = 0 \), and \( \{\mathcal{F}_s\}_{s \in [t,T]} \) is the augmented filtration generated by \( W_Q \). We refer to [4], Section 1.2.4 and [1] for the notion of a \( Q \)-Wiener process, and to [4], Sections 1.3 and 2.2.1 for more on the concept of a reference probability space. For every reference probability space \( \mu \) on \( [t,T] \), we define \( \mathcal{U}_t^\mu \) to be the set of all progressively measurable processes \( a(\cdot) : [t,T] \to \Lambda \) with values in \( B^\Lambda_K := \{ a \in \Lambda : |a|_\Lambda \leq K \} \). We define the set of admissible controls to be

\[
\mathcal{U}_t := \bigcup_\mu \mathcal{U}_t^\mu,
\]

where the union is taken over all reference probability spaces \( \mu \). We now define \( l : H \times \Lambda \to \mathbb{R} \) by \( l(x,a) = l_1(x) + l_2(a) \), where

\[
l_1(x) = \int_\mathcal{O} \beta_1(\xi, x(\xi))d\xi, \quad l_2(a) = \int_{\partial \mathcal{O}} \beta_2(a(\xi))d\xi,
\]

and \( g : H \to \mathbb{R} \) by

\[
g(x) = \int_\mathcal{O} \gamma(\xi, x(\xi))d\xi.
\]
The functional $I$ in (1.2) can then be rewritten as

$$J(t, x; a(\cdot)) = \mathbb{E}\left\{\int_t^T l(X(s), a(s))ds + g(X(T))\right\}. \quad (1.7)$$

The value function for the problem is defined by

$$W(t, x) = \inf_{a(\cdot) \in \mathcal{U}_t} J(t, x; a(\cdot)). \quad (1.8)$$

The Hamilton-Jacobi-Bellman (HJB) equation associated to problem (1.3)–(1.7) is

$$\begin{cases}
  v_t + \frac{1}{2} \text{Tr} \left[ (\lambda I - A)N_\lambda \sigma Q^{\frac{1}{2}} \left[ (\lambda I - A)N_\lambda \sigma Q^{\frac{1}{2}} \right]^* D^2 v \right] + \langle Ax + b(x), Dv \rangle + F_1(x, Dv) = 0, \\
  v(T, x) = g(x), \quad x \in H,
\end{cases} \quad (1.9)$$

where the Hamiltonian $F_1$ is formally given by

$$F_1(x, p) = \inf_{|a|_{\lambda} \leq K} \left\{ \langle (\lambda I - A)N_\lambda a, p \rangle + l(x, a) \right\}.$$

The terms $\langle (\lambda I - A)N_\lambda a, Dv \rangle$ and $\left[ (\lambda I - A)N_\lambda \sigma Q^{\frac{1}{2}} \left[ (\lambda I - A)N_\lambda \sigma Q^{\frac{1}{2}} \right]^* D^2 v \right]$ in the HJB equation (1.9) are written here in a formal way and the equation must be interpreted properly. Thus the main difficulties in the HJB equation (1.9) stem from the fact that it is nonlinear and it contains unbounded first and second order terms.

Hamilton-Jacobi-Bellman equations associated to boundary-noise and boundary-control problems have been studied in the past, however the literature on the subject is rather limited. The main results use the approach through backward stochastic differential equations (BSDE) initiated in [2] and later exploited in [12, 13]. This approach uses a particular structure of the problems considered there that “the noise acts in the same way as the control”, which would be the case here if $\sigma = Q = I$. The technique of BSDE produces nice results but its applicability is restricted by such restrictions. In [2] a boundary-noise and boundary-control problem with Neumann boundary conditions on a bounded interval was considered and it was proved that the HJB equation for the problem has a unique mild solution which has a representation through a solution of an associated BSDE system. Similar results for boundary-noise and boundary-control problems with Neumann boundary conditions for stochastic delay partial differential equations on a bounded interval were obtained in [12, 13]. In [11] the existence of a unique mild solution of a stationary HJB equation associated to a one dimensional boundary-noise and boundary-control problem with Neumann boundary conditions was proved under a strong dissipativity assumption on the $b$ term, and the optimal feedback control was derived. A problem similar to the one considered in our paper was studied in [14], where the authors showed that the value function is a viscosity solution of the associated HJB equation. However the definition of viscosity solution used in [14] did not allow to prove any comparison theorem, and consequently uniqueness of viscosity solutions was not established there. HJB equations for boundary-noise and boundary-control problems with Dirichlet boundary conditions have also been studied. They are in a sense more difficult since if the Neumann operator $N_\lambda$ is replaced by the Dirichlet operator $D$, then we only have $(-A)^{1-\beta}D$ (where $A$ is the Laplace operator with the domain $H^2 \cap H^1_0$) is bounded only for $\beta \in \left(\frac{3}{4}, 1\right)$ (see [4], Sections 2.6.2 and 2.6.3 for more on this). One dimensional cases on a half-line have been studied in [3, 8]. Finally we mention that other results for optimal control problems with boundary-noise and boundary-control, like stochastic maximum principle, can be found for instance in [6, 7, 15].
A viscosity solution approach we propose in this manuscript uses a modified notion of viscosity solution from [5] to study HJB equations for boundary control problems. It is based on an appropriate change of variables which transforms the original HJB equation into a new HJB equation with fewer unbounded terms, more precisely without the “bad” unbounded terms. The new HJB equation is associated to a new stochastic optimal control problem, which is related to the old problem by the same change of variables. We only consider a HJB equation for a relatively simple boundary-noise and boundary-control problem with Neumann boundary condition. We do it so that we do not obscure the main ideas with complicated technicalities and so that we do not have to impose any additional continuity assumptions on the coefficients in norms other than the norm in \( H \). Thus our results apply to the most common and natural case where the nonlinearity \( b \) and the cost function \( l \) are the Nemytskii operators. Moreover, since we do not require any smoothing properties of the transition semigroup associated to equation (1.4), the boundary noise \( W_Q \) can be degenerate and \( O \) can be a subset of \( \mathbb{R}^d \) for \( d \geq 1 \). Our approach could also be adapted to the case of Dirichlet boundary condition or to other variants of (1.1). For instance, with some modifications we could add distributed controls to the problem and consider a problem with controls in the drift and the diffusion. Adding a dependence of \( b \) on a control parameter would not change any assumptions on \( b \), however the new diffusion coefficient would require some, perhaps artificial, \( B \)-continuity assumption, i.e. continuity in some weaker norm (see [4], Chapter 3 for more on this). The reader can experiment with these various extensions. The main contribution of the manuscript is the introduction of an appropriate notion of viscosity solution and the proof of comparison principle. Since we do not impose any additional continuity requirements on the coefficients, the proof needs to deal with rather delicate technical problems to handle the continuity issues around the terminal time \( T \). We also prove various continuity properties of the value function, argue that it satisfies the dynamic programming principle, and then prove that the value function satisfies the associated HJB equation. This is different from the method used in [5], where the fact that the value function is a viscosity solution was obtained by finite dimensional approximations, with the dynamic programming principle obtained as a byproduct of the procedure. Also, only a stationary HJB equation was considered in [5]. A Cauchy problem using the same approach as that of [5] was later considered in [10].

We refer the reader to [4], Chapter 3 for the overview of the theory of viscosity solutions in Hilbert spaces and extensive references, and to [4], Sections 2.6.2 and 2.6.3 for more on stochastic boundary-noise and boundary-control problems.

2 Notation

We will denote the norm and the inner product in \( H \) respectively by \( | \cdot | \) and \( \langle \cdot, \cdot \rangle \), and we will use \( | \cdot |_\Lambda \) to denote the norm in \( \Lambda \). We will write \( B_r \) for an open ball in \( H \) centered at 0 with radius \( r > 0 \), and we will use \( \rightharpoonup \) to denote the weak convergence in \( H \). For a Hilbert space \( Y \), we will denote by \( \mathcal{L}(Y) \) the space of bounded linear operators on \( Y \) with the operator norm \( \| \cdot \| \). We will denote by \( S(Y) \) the space of self-adjoint operators in \( \mathcal{L}(H) \) equipped with the usual partial order.

For \( \alpha \in \mathbb{R} \) we denote \( A_\alpha := (\lambda I - A)^\alpha \), the fractional powers of \( \lambda I - A \). Using this notation we have \( A_1 = \lambda I - A \). For \( \alpha > 0 \), \( D(A_\alpha) \) is a Hilbert space with the inner product \( \langle x, y \rangle_\alpha := \langle A_\alpha x, A_\alpha y \rangle \), and for \( x \in D(A_\alpha) \), we denote \( |x|_{2\alpha} := |A_\alpha x| \). If \( \alpha < 0 \) we also denote \( |x|_{2\alpha} := |A_\alpha x| \) for \( x \in H \). If \( 0 < \alpha < \gamma \) then there exists \( C > 0 \) such that \( |x|_{2\alpha} \leq C|x|_{2\gamma} \) for all \( x \in D(A_\gamma) \). Moreover, by a well known interpolation inequality (see e.g. [9], page 73), if \( 0 < \alpha < \gamma \) then, for every \( \eta > 0 \), there exists a constant \( C_\eta > 0 \) such that

\[
|x|_{2\alpha} \leq \eta |x| + C_\eta |x|_{2\gamma} \quad \text{for all } x \in D(A_\gamma),
\]  

(2.1)
and by a property of analytic semigroups (see e.g. [9], Theorem 6.13), for every $0 < t \leq T, \alpha > 0$, the operator $A_\alpha e^{-tA_1}$ is bounded, and
\[
\|A_\alpha e^{-tA_1}\| \leq M_\alpha t^{-\alpha}.
\]
(2.2)

The operators $A_\alpha$ are bounded, self-adjoint, and compact for $\alpha < 0$. We denote by $H_N, N \geq 1$, the space spanned by the eigenvectors of $A_{-1}$ corresponding to the eigenvalues greater than or equal to $1/N$. The spaces $H_N$ are finite-dimensional. We denote by $P_N$ the orthogonal projection in $H$ onto $H_N$, and we set $Q_N = I - P_N$. We have $P_N A_\alpha = A_\alpha P_N$ for every $\alpha \in \mathbb{R}$.

We say that $\mu = \left( \Omega, \mathcal{F}, \{ \mathcal{F}_s \}_{s \in [t,T]} ; \mathbb{P}, W_Q \right)$ is a generalized reference probability space if $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $\{ \mathcal{F}_s \}_{s \in [t,T]}$ is a right continuous and complete filtration in $\mathcal{F}$, and $W_Q$ is an $\mathcal{F}_t$-Wiener process in $\Lambda$ which does not necessarily starts at 0 (see [4], Section 1.2.4).

For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a separable Hilbert space $Y$, and $1 \leq p < \infty$, we denote by $L^p(\Omega, \mathcal{F}, \mathbb{P} ; Y)$ the space of measurable $Y$-valued random variables $\xi$ such that $\mathbb{E}|\xi|^p < \infty$. The Hilbert space $Y$ is equipped with the Borel $\sigma$-field. If $\mu$ is a generalized reference probability space, we denote by $M^p(\mu ; t,T;Y)$ the space of $Y$-valued progressively measurable processes $Z(\cdot)$ on $[t,T]$ such that
\[
|Z(\cdot)|_{M^p(\mu ; t,T;Y)} := \left( \mathbb{E} \left( \int_t^T |Y(s)|^p \, ds \right) \right)^{\frac{1}{p}} < \infty.
\]
$M^p(\mu ; t,T;Y)$ is a Banach space equipped with this norm.

If $X$ is a subset of a Hilbert space and $Y$ is another Hilbert space, we denote by $C(X,Y)$ the space of continuous functions on $X$ with values in $Y$. If $Y = \mathbb{R}$ we will just write $C(X)$. We denote by $UC(X,Y)$ the space of uniformly continuous functions on $X$ with values in $Y$, and by $UC_b(X,Y)$ the space of bounded functions in $UC(X,Y)$. We will write $C^{1,2}((0,T) \times H)$ to denote the space of functions $u : (0,T) \times H \to \mathbb{R}$ (writing $u(t,x)$) such that $u_t, Du, D^2 u$ are continuous on $(0,T) \times H$, where $Du$ and $D^2 u$ stand for the first and second Fréchet derivatives of $u$ with respect to the $x$-variable.

If $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i), i = 1,2$ are two probability spaces, $(\Omega, \mathcal{F})$ is a measurable space, and $\{X_1(s)\}_{s \in [t,T]} : (\Omega_1, \mathcal{F}_1, \mathbb{P}_1) \to (\Omega, \mathcal{F})$ are two processes, we say that $X_1(\cdot)$ and $X_2(\cdot)$ have the same finite-dimensional distributions on $[t,T]$ if there is a set $D$ of full measure on $[t,T]$ such that for any $t \leq t_1 < t_2 < \ldots < t_n \leq T, t_n \in D$ and $A \in \mathcal{F} \otimes \mathcal{F} \otimes \ldots \otimes \mathcal{F}$, $n$-times
\[
\mathbb{P}_1(w_1 : (X_1(t_1), \ldots, X_1(t_n))(\omega_1) \in A) = \mathbb{P}_2(w_2 : (X_2(t_1), \ldots, X_2(t_n))(\omega_2) \in A).
\]
In this case we write $L_{\mathbb{P}_1}(X_1(\cdot)) = L_{\mathbb{P}_2}(X_2(\cdot))$.

### 3 Viscosity solutions

We propose a definition of viscosity solution which is similar to that of [5] (see also [4], Section 3.12). We make the change of variables $x = A_\beta y$ and define the function
\[
u(y) := v(A_\beta y).
\]
Since $v$ should solve equation (1.9), the function $u$ should formally satisfy
\[
\begin{align*}
\begin{cases}
  u_t + (Ax, Du) + F(x, Du, D^2 u) &= 0, \\
u(T, x) &= f(x) := g(A_\beta x), \quad x \in D(A_\beta),
\end{cases}
\end{align*}
\]
(3.1)
where
\[
F(x, p, X) = \frac{1}{2} \text{Tr}[B_\lambda B_\lambda^* X] + \langle b(A_\beta x), A_\beta p \rangle + F_2(x, p),
\]
(3.2)

\[B_\lambda = G_\lambda \sigma Q^\lambda \frac{1}{2},\]
and
\[F_2(x, p) = \inf_{|a|_\lambda \leq K} \{ (G_\lambda a, p) + l(A_\beta x, a) \}.
\]

We observe that \(B_\lambda : \Lambda \to H\) is a Hilbert-Schmidt operator. This implies that the map
\[X \mapsto \frac{1}{2} \text{Tr}[B_\lambda B_\lambda^* X]
\]
is well defined and Lipschitz continuous from \(S(H)\) to \(\mathbb{R}\), which obviously satisfies
\[
\frac{1}{2} \text{Tr}[B_\lambda B_\lambda^* X] \leq \frac{1}{2} \text{Tr}[B_\lambda B_\lambda^* Y] \quad \text{for all } X, Y \in S(H), X \leq Y.
\]

In particular we have \(F : D(A_\beta) \times H \times S(H) \to \mathbb{R}\).

We will be using the following assumptions throughout the manuscript.

**Assumption 3.1.**

- \(\beta \in (1/4, 1/2), \lambda > 0, K > 0\) are fixed constants, and \(Q \in S(\Lambda), Q \geq 0\), is trace class.
- The operator \(\sigma \in \mathcal{L}(H)\), and there exists a constant \(L \geq 0\) such that
  \[|b(x) - b(y)| \leq L|x - y| \quad \text{for all } x, y \in H.\]
(3.3)

- The function \(l : H \times \Lambda \to \mathbb{R}\) is continuous, and there exists a modulus \(\rho\) such that
  \[|l(x, a) - l(y, a)| + |g(x) - g(y)| \leq \rho(|x - y|(1 + |x| + |y|)) \quad \text{for all } x, y \in H, a \in \Lambda.\]
(3.4)

- There exists a constant \(L_1 \geq 0\) such that
  \[\sup_{a \in B_\lambda^1} |l(0, a)| \leq L_1.\]
(3.5)

**Remark 3.1.** We notice that if Assumption 3.1 is satisfied then the Hamiltonian \(F_2\) satisfies
\[
|F_2(x, p)| \leq C(1 + |p| + |x|_{2\beta}^2) \quad \text{for all } x \in D(A_\beta), p \in H,
\]
(3.6)
\[
|F_2(x, p) - F_2(x, q)| \leq C|p - q| \quad \text{for all } x \in D(A_\beta), p, q \in H,
\]
(3.7)
for some constant \(C \geq 0\), and
\[
|F_2(x, p) - F_2(y, p)| \leq \rho(|x - y|_{2\beta}(1 + |x|_{2\beta} + |y|_{2\beta})) \quad \text{for all } x, y \in D(A_\beta), p \in H.
\]
(3.8)

**Definition 3.1.** We say that a function \(\psi\) is a test function if \(\psi(t, x) = \varphi(t, x) + \delta(t)(1 + |x|^2)^{\frac{1}{2}}\), where \(\delta \in C^1((0, T), \delta(t) > 0 \text{ on } (0, T), 2 \leq q < 1/\beta\) and

(i) \(\varphi \in C^{1,2}((0, T) \times H)\) and is weakly sequentially lower semicontinuous on \((0, T) \times H\).

(ii) \(D\varphi \in UC((\tau, T - \tau) \times H, H) \cap UC ((\tau, T - \tau) \times D(A_{1/2 - \epsilon}), D(A_{1/2/2})) \text{ for some } \epsilon = \epsilon(\varphi) > 0\) and all \(0 < \tau < T/2\).
(iii) $D^2 \varphi \in UC_b((\tau, T - \tau) \times H, S(H))$ for all $0 < \tau < T/2$.

**Definition 3.2.** We say that a function $u : ((0, T) \times H) \cup \{T\} \times D(A) \to \mathbb{R}$ is a viscosity subsolution of (3.1) if $u$ is weakly sequentially upper semi-continuous on $(0, T) \times H$, upper semi-continuous on $(0, T] \times D(A)$, $u(T, x) \leq f(x), x \in D(A)$, and whenever $u - \psi$ has a maximum at $(t, x)$ over $(t - \kappa, t + \kappa) \times H$ for some $\kappa > 0$ for a test function $\psi$, then

$$x \in D(A)$$

and

$$\psi_t(t, x) - \langle A_1 x, A_1 D \varphi(t, x) \rangle - q(t) |A_1 x|^2 (1 + |x|^2)^{\delta - 1}$$

$$+ \lambda(x, D \psi(t, x)) + F(x, D \psi(t, x), D^2 \psi(t, x)) \geq 0.$$

We say that a function $u : ((0, T) \times H) \cup \{T\} \times D(A) \to \mathbb{R}$ is a viscosity supersolution of (3.1) if $u$ is weakly sequentially lower semi-continuous on $(0, T) \times H$, lower semi-continuous on $(0, T] \times D(A)$, $u(T, x) \geq f(x), x \in D(A)$, and whenever $u + \psi$ has a minimum at $(t, x)$ over $(t - \kappa, t + \kappa) \times H$ for some $\kappa > 0$ for a test function $\psi$, then

$$x \in D(A)$$

and

$$-\psi_t(t, x) + \langle A_1 x, A_1 D \varphi(t, x) \rangle + q(t) |A_1 x|^2 (1 + |x|^2)^{\delta - 1}$$

$$- \lambda(x, D \psi(t, x)) + F(x, -D \psi(t, x), -D^2 \psi(t, x)) \leq 0.$$

A function $u$ is a viscosity solution of (3.1) if it is both a viscosity subsolution and a supersolution of (3.1).

It is standard to notice that the maxima and minima in Definition 3.2 can be made strict. We say that a function $v$ has a strict maximum at $(t, x)$ if $v$ has a maximum at $(t, x)$, and whenever $v(t_n, x_n) \to v(t, x)$ then $(t_n, x_n) \to (t, x)$.

**Definition 3.3.** We say that a continuous function $v : (0, T] \times H \to \mathbb{R}$ is a viscosity solution of (1.9) if the function $u$ such that $u(t, x) := v(t, A \beta x)$ for all $t \in (0, T], x \in D(A)$ is a viscosity solution of (3.1).

We remark that, since a viscosity solution $u$ of (3.1) is supposed to be continuous on $(0, T) \times H$, the relation $u(t, x) := v(t, A \beta x)$ for $t \in (0, T], x \in D(A)$, defines $u$ on $(0, T) \times H$ if $v$ is known on $(0, T] \times D(A)$. Also, if $u$ is known on $(0, T] \times D(A)$ then $v$ is known on $(0, T] \times H$.

The following result is the comparison theorem for our viscosity subsolutions and viscosity supersolutions.

**Theorem 3.1.** Let Assumption 3.1 be satisfied. Let $u$ be a viscosity subsolution of (3.1) and $v$ be a viscosity supersolution of (3.1). Suppose that for every $0 < \tau < T$ there exists a constant $C_{\tau}$ such that

$$u(t, x), -v(t, x) \leq C_{\tau} (1 + |x|^2) \quad \text{for all } x \in H, t \in (0, T - \tau),$$

and that

$$\lim_{t \to T} \left[ (u(t, x) - f(e^{-(T-t)A_1} x))^+ + (v(t, x) - f(e^{-(T-t)A_1} x))^- \right] = 0$$

uniformly on bounded subsets of $H$. Then $u \leq v$ on $(0, T] \times H$. 

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Proof. Let $0 < \eta < 1 - 2\beta$, $2 < q < \frac{1}{\beta}$ and $0 < \tau < T$. Suppose the claim is not true. Then

$$\sup_H \{ u - v \} \geq \nu$$

for some $\nu > 0$. For $\mu, \delta, \epsilon, \alpha, k > 0$ define the function

$\Phi(t, s, x, y) := u(t, x) - v(s, y) - \frac{|x - y|^2}{2\epsilon} - \delta e^{k(T-t)} (1 + |x|^2)^{\frac{q}{2}} - \delta e^{k(T-s)} (1 + |y|^2)^{\frac{q}{2}}$

$- \frac{(t-s)^2}{2\alpha} - \frac{\mu}{t} - \frac{\mu}{s}$

The constant $k$ will be fixed later and will depend on various absolute constants. The function $\Phi$ is weakly sequentially upper semicontinuous on $[0, T - \tau] \times [0, T - \tau] \times H \times H$ and

$$\lim_{|x|, |y| \to +\infty} \Phi(t, s, x, y) = -\infty$$

so it attains a global maximum at some point $(\bar{t}, \bar{s}, \bar{x}, \bar{y}) \in (0, T - \tau] \times (0, T - \tau] \times H \times H$. We will assume that the maximum is strict. Moreover we observe that

$$|\bar{x}|, |\bar{y}| \leq R_\delta$$

(3.11)

for some $R_\delta > 0$, uniformly for $\mu, \epsilon, \alpha, k$. We also observe that for small enough $\mu, \delta$ and $\tau$,

$$\Phi(\bar{t}, \bar{s}, \bar{x}, \bar{y}) > \frac{\nu}{2}.$$  

(3.12)

It is standard to notice (see e.g. [4], the proof of Theorem 3.50) that

$$\lim_{\alpha \to 0} \frac{(\bar{t} - \bar{s})^2}{\alpha} = 0 \quad \text{for every fixed } \epsilon, \delta, \mu > 0,$$

(3.13)

$$\lim_{\epsilon \to 0} \limsup_{\alpha \to 0} \frac{|\bar{x} - \bar{y}|^2}{\epsilon} = 0 \quad \text{for every fixed } \delta, \mu > 0.$$  

(3.14)

Then, defining for $N \geq 1$

$$u_1(t, x) = u(t, x) - \frac{\langle x, Q_N A_\eta Q_N (\bar{x} - \bar{y}) \rangle}{\epsilon} + \frac{\langle Q_N A_\eta Q_N (\bar{x} - \bar{y}), \bar{x} - \bar{y} \rangle}{2\epsilon} - \frac{|Q_N(x - \bar{x})|^2}{\epsilon} - \delta e^{k(T-t)} (1 + |x|^2)^{\frac{q}{2}} - \frac{\mu}{t},$$

$$v_1(s, y) = v(s, y) - \frac{\langle y, Q_N A_\eta Q_N (\bar{x} - \bar{y}) \rangle}{\epsilon} + \frac{|Q_N(y - \bar{y})|^2}{\epsilon} + \delta e^{k(T-s)} (1 + |y|^2)^{\frac{q}{2}} + \frac{\mu}{s},$$

it follows that

$$\bar{\Phi}(t, s, x, y) := u_1(t, x) - v_1(s, y) - \frac{|P_N(x - y)|^2}{2\epsilon} - \frac{(t-s)^2}{2\alpha} \leq \Phi(t, s, x, y)$$

and $\bar{\Phi}(\bar{t}, \bar{s}, \bar{x}, \bar{y}) = \Phi(\bar{t}, \bar{s}, \bar{x}, \bar{y})$, i.e. $\bar{\Phi}$ attains a strict global maximum at $(\bar{t}, \bar{s}, \bar{x}, \bar{y})$.

Suppose that $\bar{t}, \bar{s} < T - \tau$. We then apply Corollary 3.29 of [4] with $B = A_\eta$ on $(0, T - \tau] \times H$ to obtain functions $\varphi_n, -\psi_n$ satisfying conditions (i)-(iii) of Definition 3.1 such that

$$u_1(t, x) - \varphi_n(t, x)$$
has a global maximum at some point \((t^n, x^n)\),

\[ v_1(s, y) + \psi_n(s, y) \]

has a global minimum at some point \((s^n, y^n)\) and

\[
\begin{align*}
(t^n, x^n, u_1(t^n, x^n), (\varphi_n)(t^n, x^n), D\varphi_n(t^n, x^n), D^2\varphi_n(t^n, x^n)) & \\
\rightarrow & \\
\text{in } \mathbb{R} \times H \times \mathbb{R} \times \mathbb{R} \times D(A_1) \times \mathcal{L}(H),
\end{align*}
\]

\begin{align*}
(s^n, y^n, v_1(s^n, y^n), -(\psi_n)s(s^n, y^n), -D\psi_n(s^n, y^n), -D^2\psi_n(s^n, y^n)) & \\
\rightarrow & \\
\text{in } \mathbb{R} \times H \times \mathbb{R} \times \mathbb{R} \times D(A_1) \times \mathcal{L}(H),
\end{align*}

\quad (3.15) \quad \quad \quad (3.16)

for some \(X_N, Y_N \in S(H)\) such that \(X_N = P_NX_NP_N, Y_N = P_NY_NP_N, X_N \leq Y_N\) and \(\|X_N\| + \|Y_N\| \leq C_\epsilon\) for some constant \(C_\epsilon\). The fact that the above convergence of \(D\varphi_n(t^n, x^n)\) and \(D\psi_n(s^n, y^n)\) is in \(D(A_1)\) follows from the proof of Corollary 3.29 of [4] since it is easy to see that in our case the functions \(\varphi_n, \psi_n\) satisfy \(\varphi_n(t, x) = \varphi(t, P_Nx), \psi_n(s, y) = \psi(s, P_Ny)\).

We now have

\[
-\frac{\mu}{t_n} + (\varphi_n)t(t_n, x_n) - k\delta e^{k(T-t_n)} \left(1 + |x_n|^2\right)^{\frac{3}{2}}
\]

\[
-\left( A_{\frac{1}{2}} \bar{x}, A_{\frac{1}{2}} D\varphi_n(t_n, x_n) + \frac{A_{\frac{1}{2}} - \eta Q_N(\bar{x} - \bar{y})}{\epsilon} \right) - q\delta e^{k(T-t_n)}|A_{\frac{1}{2}} x_n|^2 \left(1 + |x_n|^2\right)^{\frac{3}{2}-1}
\]

\[
+ \lambda \langle x_n, D\varphi_n(t_n, x_n) + \frac{A_{\frac{1}{2}} - \eta Q_N(\bar{x} - \bar{y})}{\epsilon} + q\delta e^{k(T-t_n)} \left(1 + |x_n|^2\right)^{\frac{3}{2}-1} x_n \rangle
\]

\[
+ \frac{1}{2} \text{Tr} B_\lambda B_\lambda^\dagger \left( D^2\varphi_n(t_n, x_n) + 2A_{\frac{1}{2}} Q_N \right)
\]

\[
+ \delta e^{k(T-t_n)} \left( q(1 + |x_n|^2)^{\frac{3}{2}-1} + q(q - 2)(1 + |x_n|^2)^{\frac{3}{2}-2} x_n \otimes x_n \right)
\]

\[
+ \left( b(A_\beta x_n), A_{\frac{1}{2}} D\varphi_n(t_n, x_n) + \frac{A_{\frac{1}{2}} - \eta Q_N(\bar{x} - \bar{y})}{\epsilon} + q\delta e^{k(T-t_n)} \left(1 + |x_n|^2\right)^{\frac{3}{2}-1} x_n \right)
\]

\[
+ F_2 \left( x_n, D\varphi_n(t_n, x_n) + \frac{A_{\frac{1}{2}} - \eta Q_N(\bar{x} - \bar{y})}{\epsilon} + q\delta e^{k(T-t_n)} \left(1 + |x_n|^2\right)^{\frac{3}{2}-1} x_n \right) \geq 0.
\]

Using Cauchy-Schwarz inequality we now estimate

\[
\left| \left( b(A_\beta x_n), A_{\frac{1}{2}} \left( q\delta e^{k(T-t_n)} \left(1 + |x_n|^2\right)^{\frac{3}{2}-1} x_n \right) \right) \right|
\]

\[
\leq \frac{1}{2} q\delta e^{k(T-t_n)}|A_{\frac{1}{2}} x_n|^2 \left(1 + |x_n|^2\right)^{\frac{3}{2}-1} + C\delta e^{k(T-t_n)} \left(1 + |x_n|^2\right)^{\frac{3}{2}},
\]

where \(C\) is some absolute constant. Therefore, by (3.7) and elementary computations we obtain
that if $k$ is large enough (but independent of $\mu, \delta, \epsilon, \alpha, N, n$) we have

\[-\frac{\mu}{t_n^2} + (\varphi_n)_t(t_n, x_n) - \frac{1}{2}k\delta e^{k(T-t_n)}(1 + |x_n|^2)^{\frac{q}{2}}\]

\[-\left\langle A_{\frac{1}{2}} \bar{x}, A_{\frac{1}{2}} D\varphi_n(t_n, x_n) + \frac{A_{\frac{1}{2}-\eta} Q_N(\bar{x} - \bar{y})}{\epsilon} \right\rangle - \frac{1}{2}q\delta e^{k(T-t_n)}|A_{\frac{1}{2}} x_n|^2(1 + |x_n|^2)^{\frac{q-1}{2}}\]

\[+\lambda \left\langle x_n, D\varphi_n(t_n, x_n) + \frac{A_{-\eta} Q_N(\bar{x} - \bar{y})}{\epsilon} \right\rangle + \frac{1}{2} \text{Tr} \left[ B_\lambda B_\infty^* \left( D^2 \varphi_n(t_n, x_n) + 2A_{-\eta} Q_N \right) \right]\]

\[+ b(A_{\beta} x_n, A_{-\beta} \left( D\varphi_n(t_n, x_n) + \frac{A_{-\eta} Q_N(\bar{x} - \bar{y})}{\epsilon} \right))\]

\[+ F_2 \left( x_n, D\varphi_n(t_n, x_n) + \frac{A_{-\eta} Q_N(\bar{x} - \bar{y})}{\epsilon} \right) \geq 0.\]

It is easy to see using Assumption 3.1, Remark 3.1, (2.1), (3.11), (3.15), and the properties of test functions that, for $\mu, \delta, \epsilon, \alpha, N$ fixed, $\sup_n |A_{\frac{1}{2}} x_n| < \infty$. Since $A_{-\beta}$ is compact this implies that, up to a subsequence still denoted by $x_n$,

\[A_{\frac{1}{2}} x_n \rightarrow A_{\frac{1}{2}} \bar{x}, \quad A_{\beta} x_n \rightarrow A_{\beta} \bar{x} \quad \text{as} \quad n \rightarrow +\infty.\]

Thus, passing to the limsup as $n \rightarrow +\infty$ above and using (3.15) and the weak sequential lower semicontinuity of the norm, we obtain

\[-\frac{\mu}{\bar{t}} + \frac{\bar{t} - \bar{s}}{\alpha} - \frac{1}{2}k\delta e^{k(T-\bar{t})}(1 + |\bar{x}|^2)^{\frac{q}{2}}\]

\[-\left\langle A_{\frac{1}{2}} \bar{x}, A_{\frac{1}{2}-\eta} (\bar{x} - \bar{y}) \right\rangle - \frac{1}{2}q\delta e^{k(T-\bar{t})}|A_{\frac{1}{2}} \bar{x}|^2(1 + |\bar{x}|^2)^{\frac{q-1}{2}}\]

\[+\lambda \left\langle \bar{x}, \frac{A_{-\eta} (\bar{x} - \bar{y})}{\epsilon} \right\rangle + \frac{1}{2} \text{Tr} \left[ B_\lambda B_\infty^* X_N \right]\]

\[+ b(A_{\beta} \bar{x}, A_{-\beta} \left( \frac{A_{-\eta} (\bar{x} - \bar{y})}{\epsilon} \right)) + F_2 \left( \bar{x}, \frac{A_{-\eta} (\bar{x} - \bar{y})}{\epsilon} \right) \geq -\text{Tr} \left[ B_\lambda B_\infty^* A_{-\eta} Q_N \right] = -\rho_1(N),\]

where $\lim_{N \rightarrow +\infty} \rho_1(N) = 0$.

Arguing in the same way for the supersolution $v$ we obtain

\[\frac{\mu}{\bar{s}} + \frac{\bar{t} - \bar{s}}{\alpha} + \frac{1}{2}k\delta e^{k(T-\bar{s})}(1 + |\bar{y}|^2)^{\frac{q}{2}}\]

\[-\left\langle A_{\frac{1}{2}} \bar{y}, A_{\frac{1}{2}-\eta} (\bar{x} - \bar{y}) \right\rangle + \frac{1}{2}q\delta e^{k(T-\bar{s})}|A_{\frac{1}{2}} \bar{y}|^2(1 + |\bar{y}|^2)^{\frac{q-1}{2}}\]

\[+\lambda \left\langle \bar{y}, \frac{A_{-\eta} (\bar{x} - \bar{y})}{\epsilon} \right\rangle + \frac{1}{2} \text{Tr} \left[ B_\lambda B_\infty^* Y_N \right]\]

\[+ b(A_{\beta} \bar{y}, A_{-\beta} \left( \frac{A_{-\eta} (\bar{x} - \bar{y})}{\epsilon} \right)) + F_2 \left( \bar{y}, \frac{A_{-\eta} (\bar{x} - \bar{y})}{\epsilon} \right) \leq \rho_1(N),\]
Subtracting (3.17) from (3.18) we therefore get
\[
\begin{align*}
\frac{\mu}{T^2} + \frac{\mu}{s^2} + \frac{1}{2}k\delta e^{k(T-t)}(1 + |\bar{x}|^2)^{\frac{\gamma}{2}} + \frac{1}{2}k\delta e^{k(T-s)}(1 + |\bar{y}|^2)^{\frac{\gamma}{2}} \\
+ \frac{1}{2}q\delta e^{k(T-t)}|A_1\bar{x}|^2(1 + |\bar{x}|^2)^{\frac{\gamma}{2}-1} + \frac{1}{2}q\delta e^{k(T-s)}|A_2\bar{y}|^2(1 + |\bar{y}|^2)^{\frac{\gamma}{2}-1} \\
+ \frac{|\bar{x} - \bar{y}|^2_1}{\epsilon} - \lambda \frac{|\bar{x} - \bar{y}|^2_1}{\epsilon} - C_1|\bar{x} - \bar{y}|_{2\beta} \frac{|\bar{x} - \bar{y}|_{-\eta}}{\epsilon} \\
- \rho(|\bar{x} - \bar{y}|_{2\beta}(1 + |\bar{x}|_{2\beta} + |\bar{y}|_{2\beta})) \leq 2\rho_1(N).
\end{align*}
\]
(3.19)

Since \(2\beta < 1 - \eta\) we can estimate
\[
C_1|\bar{x} - \bar{y}|_{2\beta} \frac{|\bar{x} - \bar{y}|_{-\eta}}{\epsilon} \leq \frac{|\bar{x} - \bar{y}|^2_{1-\eta}}{2\epsilon} + C_2 \frac{|\bar{x} - \bar{y}|^2_{-\eta}}{\epsilon}.
\]

Let \(C_\mu\) be such that \(\sigma(r) \leq \frac{\mu}{T^2} + C_\mu r\). Then
\[
\rho(|\bar{x} - \bar{y}|_{2\beta}(1 + |\bar{x}|_{2\beta} + |\bar{y}|_{2\beta})) \leq \frac{\mu}{T^2} + \frac{|\bar{x} - \bar{y}|^2_{1-\eta}}{2\epsilon} + \epsilon C_\mu(1 + |\bar{x}|^2_1 + |\bar{y}|^2_1)
\]
for some constant \(C_\mu\). Plugging these estimates into (3.19) we thus obtain for \(\epsilon C_\mu \leq \delta\)
\[
\frac{\mu}{T^2} \leq C_3 \frac{|\bar{x} - \bar{y}|^2_{-\eta}}{\epsilon} + \epsilon C_\mu + 2\rho_1(N).
\]
(3.20)

This is impossible after we take \(\lim_{\delta \to 0} \lim_{\alpha \to 0} \lim_{N \to \infty}\) in (3.20) (where \(\mu, \delta\) are fixed and are sufficiently small) in light of (3.14).

Thus we must have either \(\bar{t} = T - \tau\) or \(\bar{s} = T - \tau\). Suppose without loss of generality that \(\bar{t} = T - \tau\). We then estimate
\[
u(\bar{t}, \bar{x}) - v(\bar{s}, \bar{y}) \leq (u(\bar{t}, \bar{x}) - f(e^{-(T-\bar{t})A_1\bar{x}}))^+ + |f(e^{-(T-\bar{t})A_1\bar{x}}) - f(e^{-(T-s)A_1\bar{y}})| + (v(\bar{s}, \bar{y}) - f(e^{-(T-s)A_1\bar{y}}))^+
\]
(3.21)

Using (3.10) and (3.13) we obtain
\[
(u(\bar{t}, \bar{x}) - f(e^{-(T-\bar{t})A_1\bar{x}}))^+ + (v(\bar{s}, \bar{y}) - f(e^{-(T-s)A_1\bar{y}}))^+ \leq \rho_\delta(\lambda, \alpha),
\]
(3.22)

where for fixed \(\delta, \epsilon\), \(\lim_{\alpha \to 0} \rho_\delta(\tau, \alpha) \leq \rho_\delta(\tau)\) for some moduli \(\rho_\delta\) which are independent of \(\epsilon\). Regarding the middle term of (3.21), we have
\[
|f(e^{-(T-\bar{t})A_1\bar{x}}) - f(e^{-(T-s)A_1\bar{y}})| \leq \rho \left( |e^{-(T-\bar{t})A_1\bar{x}} - e^{-(T-s)A_1\bar{y}}| + |e^{-(T-s)A_1A_\beta\bar{x}}| + |e^{-(T-s)A_1A_\beta\bar{y}}| \right).
\]
(3.23)

By (2.2) we now have
\[
|e^{-(T-\bar{t})A_1A_\beta\bar{x}}| \leq M_\beta \tau^{-\beta} |\bar{x}|, \quad |e^{-(T-s)A_1A_\beta\bar{y}}| \leq M_\beta \tau^{-\beta} |\bar{y}|.
\]
(3.24)

Since \(A_1\) satisfies the strong \(B\) condition with \(B = (\lambda I - A)^{-1}\) (see [4], Definition 3.10), we have, by [4], Lemma 3.19 (inequality (3.20)),
\[
|e^{-(T-\bar{t})A_1A_\beta\bar{x}} - e^{-(T-s)A_1A_\beta\bar{y}}| \leq C(\tau)|e^{-(T-s)A_1A_\beta\bar{y}} - A_\beta \bar{x}|_{-1}
\]
\[
\leq C(\tau)|e^{-(T-s)A_1A_\beta\bar{y}} - A_\beta \bar{y}|_{-1} + C(\tau)|A_\beta \bar{y} - A_\beta \bar{x}|_{-1}
\]
\[
\leq C(\tau)|e^{-(T-s)A_1A_\beta\bar{y}} - A_\beta \bar{y}|_{-1} + C(\tau)|\bar{y} - \bar{x}|_{-\eta}.
\]
(3.25)
Now, again by (2.2),
\[
|e^{-(\bar{t} - \bar{s})A_1}A_\beta \bar{y} - A_\beta \bar{y}|_{-1} = \left| \int_0^{\bar{t} - \bar{s}} e^{-\eta A_1} A_{1/2 + \beta} \bar{y} d\eta \right| \leq M_\beta \int_0^{\bar{t} - \bar{s}} \eta^{-1/2 - \beta} |\bar{y}| d\eta = M_\beta |\bar{y}|(\bar{t} - \bar{s})^{1/2 - \beta}. \tag{3.26}
\]

Using (3.22)-(3.26) in (3.21) and recalling (3.12), we thus obtain
\[
\frac{\nu}{2} \leq u(\bar{t}, \bar{x}) - v(\bar{s}, \bar{y}) \leq \rho_\alpha(\tau, \alpha) + \rho\left( (C(\tau)M_\beta |\bar{y}|(\bar{t} - \bar{s})^{1/2 - \beta} + C(\tau)|\bar{y} - \bar{x}|_{-1} \right) \left( 1 + M_\beta \tau^{-\beta}|\bar{x}| + M_\beta \tau^{-\beta}|\bar{y}| \right).
\tag{3.27}
\]

Taking \(\lim_{\epsilon \to 0} \limsup_{\alpha \to 0}\) in (3.27) we now arrive at
\[
\frac{\nu}{2} \leq \rho_\alpha(\tau)
\]
which is impossible if \(\tau\) is small. This completes the proof. \(\square\)

4 Optimal control problem

The transformed HJB equation (3.1) is associated with a new control problem. The new state equation is
\[
\begin{cases}
    dY(s) = [AY(s) + A_{-\beta} b(A_\beta Y(s)) + G_\lambda a(s)] ds + G_\lambda \sigma dW_Q(s) \\
    Y(t) = y \in H.
\end{cases}
\tag{4.1}
\]

It is more convenient to write its mild form using the operator \(A_1 = \lambda I - A\), i.e.
\[
Y(s) = e^{-(s-t)A_1}y + \int_t^s e^{-(s-r)A_1} \left[ \lambda Y(r) + A_{-\beta} b(A_\beta Y(r)) + G_\lambda a(r) \right] dr \\
\quad + \int_t^s e^{-(s-r)A_1} G_\lambda \sigma dW_Q(r), \quad r \geq t. \tag{4.2}
\]

In the new problem we are minimizing the cost functional
\[
J(t, y; a(\cdot)) = \mathbb{E} \left[ \int_t^T l(A_\beta Y(s; t, y, a(\cdot)), a(s)) ds + g(A_\beta Y(T; t, y, a(\cdot))) \right] \tag{4.3}
\]
and define the value function
\[
V(t, y) = \inf_{a(\cdot) \in \mathcal{U}_t} J(t, y; a(\cdot)). \tag{4.4}
\]

It will be clear from the results of this section, in particular Lemma 4.2, that \(Y(\cdot)\) satisfies (4.2) with initial condition \(Y(t) = y = A_\beta x\) if and only if \(Y(\cdot) = A_\beta X(\cdot; t, x, a(\cdot))\), where \(X(\cdot; t, x, a(\cdot))\) is the mild solution of (1.3) with \(X(t) = x\). Therefore the old value function \(W\) defined by (1.8) and the new value function \(V\) satisfy the relation \(W(t, A_\beta x) = V(t, x)\) for all \(t \in [0, T], x \in D(A_\beta)\).
4.1 Estimates

Using the same arguments as those in the proof of Proposition 3.121 of [4], we easily obtain the following proposition.

**Proposition 4.1.** Let Assumption 3.1 be satisfied. Let \( \beta \leq \eta < \frac{1}{2} \), and \( 2 \leq q < \frac{1}{\eta} \). Let \( \mu = (\Omega, \mathcal{F}, \{\mathcal{F}_s^t\}_{s \geq t}, \mathbb{P}, W_Q) \) be a generalized reference probability space, \( T > 0 \), \( a(\cdot) \in \mathcal{U}_t^\mu \), and \( \xi \in L^q(\Omega, \mathcal{F}_T^t, \mathbb{P}; H) \). Then equation (4.1) with \( Y(t) = \xi \) has a unique mild solution \( Y(\cdot) = Y(\cdot; t, \xi, a(\cdot)) \) among all processes which have \( dt \otimes \mathbb{P} \) equivalent versions in \( M^2_t(t, T; D(A_\eta)) \). The solution has continuous trajectories in \( H \).

Repeating the arguments in the proof of Lemma 3.124 of [4] we also easily obtain that if \( Y_N(\cdot) \) is the solution of

\[
\begin{aligned}
\begin{cases}
dY_N(s) = [P_N AY_N(s) + A_\beta P_N b(A_\beta Y_N(s)) + P_N G_\lambda a(s)] ds + P_N G_\lambda \sigma dW_Q(s) \\
Y(t) = P_N \xi
\end{cases}
\end{aligned}
\tag{4.5}
\]

then, under the assumptions of Proposition 4.1,

\[
\lim_{N \to +\infty} |Y_N(\cdot) - Y(\cdot)|_{M^2(t, T; D(A_\eta))} = 0. \tag{4.6}
\]

If the assumptions of Proposition 4.1 are satisfied then also the assumptions of Hypothesis 3.99 of [4] are satisfied and so, by Theorem 3.102 of [4], equation (4.1) with \( Y(t) = \xi \) has a unique variational solution which must coincide with \( Y(\cdot) \), since it is obtained as the limit of the solutions \( Y_N(\cdot) \).

**Lemma 4.1.** Let Assumption 3.1 be satisfied. Let \( 2 \leq p < 1/\beta, \beta \leq \eta < \frac{1}{2}, 2 \leq q < \frac{1}{\eta} \) and \( 0 \leq t < T \). Then there exist absolute constants \( C \) such that for every generalized reference probability space \( \mu = (\Omega, \mathcal{F}, \{\mathcal{F}_s^t\}_{s \geq t}, \mathbb{P}, W_Q) \), \( a(\cdot) \in \mathcal{U}_t^\mu \), \( x, y \in H \), the unique mild/variational solution \( Y(\cdot) = Y(\cdot; t, y, a(\cdot)) \) of (4.1) satisfies:

(i) \[
\mathbb{E}|Y(s; t, y, a(\cdot))|^2 \leq C(1 + |y|^q)(s - t)^{-\eta}, \quad t \leq s \leq T. \tag{4.7}
\]

Moreover

\[
\mathbb{E} \sup_{t \leq s \leq T} |Y(s; t, y, a(\cdot))|^p \leq C(1 + |y|^p) \tag{4.8}
\]

and there is a modulus \( \sigma_y \), independent of \( a(\cdot) \in \mathcal{U}_t^\mu \) such that

\[
\mathbb{E} \sup_{t \leq \tau \leq s} |Y(\tau; t, y, a(\cdot)) - y|^p \leq \sigma_y(s - t), \quad t \leq s \leq T. \tag{4.9}
\]

(ii) \[
\mathbb{E} \int_t^T |Y(s; t, y, a(\cdot))|^2 (1 + |Y(s; t, y, a(\cdot))|^p - 2) ds \leq C(1 + |y|^p). \tag{4.10}
\]

(iii) If \( 2\beta < \eta < 1 \) then

\[
\mathbb{E}|Y(s; t, x, a(\cdot)) - Y(s; t, y, a(\cdot))|_{2\beta}^2 \leq C|x - y|_{2\beta-\eta}^2(s - t)^{-\eta}, \quad t < s \leq T. \tag{4.11}
\]
Lemma 4.2. Let Assumption 3.1 be satisfied. Let $\nu = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W^1_{\mathcal{Q}})$ be another generalized reference probability space and $a_1(\cdot) \in \mathcal{U}$. Let $\xi, \xi_1$ be respectively $\mathcal{F}_t^\nu$- and $\mathcal{F}_t^{1,t}$-measurable, $H$-valued random variables such that $\mathbb{E}^\nu[|\xi|^p] < +\infty$ and $\mathbb{E}^{1,t}[|\xi_1|^p] < +\infty$. If

$$\mathcal{L}_\mathcal{F}(\xi, a(\cdot), W_Q) = \mathcal{L}_\mathcal{F}(\xi_1, a_1(\cdot), W_Q)$$

then

$$\mathcal{L}_\mathcal{F}(a(\cdot), Y(\cdot; \xi, a(\cdot))) = \mathcal{L}_\mathcal{F}(a_1(\cdot), Y(\cdot; \xi_1, a_1(\cdot))).$$

Proof. (i): Estimate (4.7) is proved similarly to estimate (3.41) in Proposition 3.122 of [4]. Estimates (4.8) and (4.9) are easy consequences of (4.7).

(ii): Estimate (4.10) follows applying Itô’s formula to the function $\varphi(r) = (1 + r)^{\frac{p}{2}}$ and the standard energy identity

$$[Y(s)]^2 = [y]^2 - 2 \int_t^s [Y(r)]^2 dr + 2 \int_t^s (A_\delta b(A_\delta Y(r)) + \lambda Y(r) + G_\lambda a(r), Y(r)) dr$$

$$+ \int_t^s \text{Tr}(B_\lambda B_\lambda^*) dr + 2 \int_t^s \langle G_\lambda \sigma dW_Q(r), Y(r) \rangle. \quad (4.12)$$

We then get

$$\mathbb{E}(1 + |Y(s)|^2)^{\frac{p}{2}} = (1 + |y|^2)^{\frac{p}{2}} - p\mathbb{E} \int_t^s |Y(r)|^2 (1 + |Y(r)|^2)^{\frac{p}{2}-1} dr$$

$$+ p\mathbb{E} \int_t^s (A_\delta b(A_\delta Y(r)) + \lambda Y(r) + G_\lambda a(r), Y(r)) (1 + |Y(r)|^2)^{\frac{p}{2}-1} dr$$

$$+ \frac{p}{2} \mathbb{E} \int_t^s \text{Tr}(B_\lambda B_\lambda^*) (1 + |Y(r)|^2)^{\frac{p}{2}-1} dr$$

$$+ p(\frac{p}{2} - 1) \int_t^s |B_\lambda Y(r)|^2 (1 + |Y(r)|^2)^{\frac{p}{2}-2} dr. \quad (4.13)$$

We conclude using (4.7).

(iii): Denote $Y(s) = Y(s; t, x, a(\cdot)), X(s) = Y(s; t, x, a(\cdot))$. We easily obtain

$$\mathbb{E}|A_\beta(X(s) - Y(s))|^2 \leq 2|A_\beta e^{-(s-t)A_1}(x - y)|^2 + C \int_t^s \mathbb{E}|A_\beta(X(r) - Y(r))|^2 dr$$

$$\leq 2|A_\beta e^{-(s-t)A_1}A_\beta(x - y)|^2 + C \int_t^s \mathbb{E}|A_\beta(X(r) - Y(r))|^2 dr$$

$$\leq 2|x - y|^2 2\beta_\beta^{-\eta}(s-t)^{\eta} + C \int_t^s \mathbb{E}|A_\beta(X(r) - Y(r))|^2 dr,$$

where we used (2.2) to get the last line above. Estimate (4.11) now follows from a version of Gronwall’s lemma, see e.g. [4], Proposition D.30.

(iv): The claim follows using the same arguments as those in the proof of Proposition 1.137 of [4].

Lemma 4.2. Let Assumption 3.1 be satisfied. Let $2 \leq p < 1/\beta$ and $0 \leq t < T$. We denote $Y(\cdot) = Y(\cdot; t, y, a(\cdot))$. 


(i) For every $a(\cdot) \in \mathcal{U}_t, y \in D(A_\eta), \beta \leq \eta < \frac{1}{2}$, the solution $Y(\cdot; t, y, a(\cdot))$ has a.e. trajectories in $C([t, T], D(A_\eta))$ and there exist a modulus $\sigma_0^\eta$ and a constant $C$, independent of $a(\cdot) \in \mathcal{U}_t$, such that

\[ \mathbb{E} \sup_{t \leq \tau \leq s} |Y(\tau)|^{p_2\eta} \leq C(1 + |y|_{2\eta}^p), \]  

(4.14)

\[ \mathbb{E} \sup_{t \leq \tau \leq s} |Y(\tau) - y|_{2\eta}^p \leq \sigma_0^\eta(s - t), \quad t \leq s \leq T. \]  

(4.15)

If $y \in H$ then the solution $Y(\cdot; t, y, a(\cdot))$ has a.e. trajectories in $C((t, T], D(A_\eta))$.

(ii) For every $R > 0$ there exist a modulus $\rho_R$, independent of $a(\cdot) \in \mathcal{U}_t$, such that if $y \in H, |y| \leq R$, then

\[ \mathbb{E} \sup_{t \leq \tau \leq s} |Y(\tau) - e^{-(\tau-t)A_\eta}y|_{2\beta}^p \leq \rho_R(s - t), \quad t \leq s \leq T. \]  

(4.16)

(iii) If $x, y \in D(A_\beta)$ then there exists a constant $C$ such that for every $a(\cdot) \in \mathcal{U}_t$

\[ \mathbb{E} \sup_{t \leq \tau \leq s} |X(\tau) - Y(\tau)|_{2\beta}^p \leq C|x - y|_{2\beta}, \]  

(4.17)

where $X(s) = Y(s; t, x, a(\cdot))$.

Proof. (i): We have

\[ A_\eta Y(s) = e^{-(s-t)A_1}A_\eta y + \int_t^s A_\eta e^{-(s-r)A_1}A_1 \left[ \lambda A_\beta Y(r) + b(A_\beta Y(r)) \right] dr \]
\[ + \int_t^s A_\eta e^{-(s-r)A_1}G_\lambda a(r)dr + \int_t^s A_\eta e^{-(s-r)A_1}G_\lambda \sigma dW_Q(r). \]  

(4.18)

The first integral in the right-hand side of (4.18) has a.e. continuous trajectories by Lemma 1.115-(ii) of [4] applied with $\alpha = 1$ and $p = 2$. The second integral has a.e. continuous trajectories by Lemma 1.115-(ii) of [4] applied with $\alpha = 1$ since $G_\lambda a(r)$ has a.e. bounded trajectories. The stochastic integral has a.e. continuous trajectories by Proposition 1.116 of [4] applied with $A_1 := A_\eta$ and $0 < \alpha < \frac{1}{2} - \eta$ there. (We point out that the operator $A_1$ in Proposition 1.116 of [4] is different from the operator $A_1$ used here.)

It now follows from (4.18) and (2.2) that, for some absolute constant $C$,

\[ \mathbb{E} \sup_{t \leq \tau \leq s} |Y(\tau) - y|_{2\eta}^p \leq C \left[ \sup_{t \leq \tau \leq s} |e^{-(\tau-t)A_1}A_\eta y - A_\eta y|^p \right. \]
\[ + \mathbb{E} \sup_{t \leq \tau \leq s} \left. \left| \int_t^\tau (\tau - r)^{\beta-\eta}(1 + |A_\beta Y(r)|)dr \right|^p \right. \]
\[ + \sup_{t \leq \tau \leq s} \left. \left| \int_t^\tau (\tau - r)^{-\eta}dr \right|^p \right. + \mathbb{E} \sup_{t \leq \tau \leq s} \left. \left| \int_t^\tau A_\eta e^{-(\tau-r)A_1}G_\lambda \sigma dW_Q(r) \right|^p \right]. \]  

(4.19)

Now, using Hölder’s inequality and (4.7),

\[ \mathbb{E} \sup_{t \leq \tau \leq s} \left| \int_t^\tau (\tau - r)^{\beta-\eta}(1 + |A_\beta Y(r)|)dr \right|^p + \sup_{t \leq \tau \leq s} \left| \int_t^\tau (\tau - r)^{-\eta}dr \right|^p \]
\[ \leq \mathbb{E} \left( \int_t^s (s - r)^{\frac{p-1}{p}((\beta-\eta)}dr \right)^{p-1} \int_t^s (1 + |A_\beta Y(r)|)^pdr + \left( \int_t^s (s - r)^{-\eta}dr \right)^p \]
\[ \leq C(s - t)^{p(1-\eta)}(1 + |y|^p). \]
To estimate the last term we use factorization method. It follows from Lemma 1.114 of [4] that for $0 < \alpha < \frac{1}{2} - \eta$

$$\int_t^\tau A_\eta e^{-(r-t)A_1}G_\lambda \sigma dW_Q(r) = \frac{\sin(\alpha \pi)}{\pi} \int_t^\tau (\tau - r)^{\alpha - 1} e^{-(r-t)A_1}Y_\alpha(r)dr,$$

where

$$Y_\alpha(r) = \int_t^\tau (r-h)^{-\alpha} A_\eta e^{-(r-h)A_1}G_\lambda \sigma dW_Q(h).$$

Thus, if $\frac{1}{q} < \alpha$, $q \geq p$, using Hölder’s and Jensen’s inequalities

$$\mathbb{E} \sup_{t \leq \tau \leq s} \left| \int_t^\tau A_\eta e^{-(r-t)A_1}G_\lambda \sigma dW_Q(r) \right|^p \leq C \left( \int_t^s (s-r)^{\frac{q}{p} \chi_1(\alpha-1)}dr \right)^{\frac{q}{q-1}} \left( \int_t^s \mathbb{E}|Y_\alpha(r)|^q dr \right)^{\frac{p}{q}}.$$

By Burkholder-Davis-Gundy inequality (see e.g. Theorem 1.111 of [4]), and (2.2),

$$\mathbb{E}|Y_\alpha(r)|^q \leq C \mathbb{E} \left( \int_t^\tau (r-h)^{-2(\alpha+\eta)}dr \right)^{\frac{q}{2}},$$

and the result follows since $2(\alpha + \eta) < 1$ and $\frac{q}{q-1}(\alpha - 1) > -1$.

Estimate (4.14) follows from the same calculation.

(ii): We now have in place of (4.19),

$$\mathbb{E} \sup_{t \leq \tau \leq s} |Y(\tau) - e^{-(r-t)A_1}Y(\tau)|^{2\beta} \leq C \left[ \mathbb{E} \sup_{t \leq \tau \leq s} \left| \int_t^\tau (1 + |A_\beta Y(r)|)dr \right|^p + \mathbb{E} \sup_{t \leq \tau \leq s} \left| \int_t^\tau (\tau - r)^{-\beta}dr \right|^p + \mathbb{E} \sup_{t \leq \tau \leq s} \left| \int_t^\tau A_\beta e^{-(r-t)A_1}G_\lambda \sigma dW_Q(r) \right|^p \right]$$

and we conclude arguing as in the proof of part (i).

(iii) Inequality (4.17) is a standard consequence of Gronwall’s lemma. \(\square\)

We remark that if we rewrote (4.18) as an equation satisfied by $Z(s) = A_\eta Y(s)$ and looked at estimates satisfied by solutions of this equation, it would be possible to obtain (4.15) for every $p \geq 2$. However we will not need such estimates here.

### 4.2 Continuity of the value function

**Assumption 4.1.**

- There exists a modulus $\rho$ such that

$$|g(x) - g(y)| \leq \rho(|x - y|) \quad \text{for all } x, y \in H. \quad (4.20)$$

**Proposition 4.2.** Let Assumptions 3.1 and 4.1 be satisfied and let $0 < \eta < 1 - 2\beta$. Then:

(i) For every $R > 0$, $0 < \tau < T$, there exists a modulus $\sigma_{R,\tau}$ such that for every $x, y \in H, |x| \leq R, |y| \leq R, t \in [0, T - \tau]$ and $a(\cdot) \in \mathcal{U}_t$,

$$|J(t, x; a(\cdot)) - J(t, y; a(\cdot))| \leq \sigma_{R,\tau}(|x - y|^{-\eta}). \quad (4.21)$$
(ii) For every $0 < \tau < T$ there exists $C_\tau > 0$ such that for every $x \in H, t \in [0, T - \tau]$ and $a(\cdot) \in U_t$,

$$|J(t, x; a(\cdot))| \leq C_\tau (1 + |x|^2).$$  \hfill (4.22)

(iii) For every $R > 0$, there exists a modulus $\sigma_R$ such that for every $x \in H, |x| \leq R, t \in [0, T]$ and $a(\cdot) \in U_t$,

$$|J(t, x; a(\cdot)) - f(e^{-(T-t)A_1}x)| \leq \sigma_R (T - t).$$  \hfill (4.23)

Proof. (i) Denote $Y(s) = Y(s; t, y, a(\cdot)), X(s) = Y(s; t, x, a(\cdot))$. Using Assumptions 3.1 and 4.1, the fact that $\rho$ can be assumed to be concave, Jensen’s inequality, and finally (4.7) and (4.11), we have

$$|J(t, x; a(\cdot))| \leq \mathbb{E} \left[ \int_t^T \rho(|A_\beta X(s) - A_\beta Y(s)|(l + |A_\beta X(s)| + |A_\beta Y(s)|))ds \right].$$

$$\leq C \rho \left( \int_t^T \mathbb{E} |A_\beta X(s) - A_\beta Y(s)|(l + |A_\beta X(s)| + |A_\beta Y(s)|)ds \right)$$

$$\leq C \rho \left( \int_t^T \mathbb{E} |A_\beta X(s) - A_\beta Y(s)|^2 (\mathbb{E} (l + |A_\beta X(s)| + |A_\beta Y(s)|)^2)^{1/2} ds \right)$$

$$\leq C \rho \left( \int_t^T |x - y|^{\eta}(s-t)^{-\beta/2}(1 + |x| + |y|)(s - t)^{-\beta} ds \right)$$

$$\leq C \rho \left( |x - y|^{\eta}(1 + 2R) + \rho \left(|x - y|^{\eta}(t - \tau)^{\beta/2} \right)^{1/2} \right).$$

(ii) Inequality (4.22) follows easily from (4.7), (3.4), (3.5), and (4.20), which imply that

$$|l(A_\beta x)| \leq C \left(1 + |A_\beta x|^2\right), \quad |g(A_\beta x)| \leq C \left(1 + |A_\beta x|\right), \quad x \in D(A_\beta) \quad x \in D(A_\beta).$$

(iii) Denoting $X(s) = Y(s; t, x, a(\cdot))$, we have using (4.7) and (4.16),

$$|J(t, x; a(\cdot)) - f(e^{-(T-t)A_1}x)| \leq \mathbb{E} \int_t^T |l(A_\beta X(s), a(\cdot))|ds + \mathbb{E}|g(A_\beta X(T)) - g(A_\beta e^{-(T-t)A_1}x)|$$

$$\leq \int_t^T C(1 + |x|^2)(s - t)^{-2\beta} ds + \mathbb{E} \rho \left(|X(T) - e^{-(T-t)A_1}x|^{2\beta} \right)$$

$$\leq C(1 + R^2)(T - t)^{1-2\beta} + \rho \left((\rho R(T - t))^{1/2} \right).$$

The following corollary is an immediate consequence of Proposition 4.2.

**Corollary 4.1.** Let Assumptions 3.1 and 4.1 be satisfied and let $0 < \eta < 1 - 2\beta$. Let the moduli $\sigma_{R, \tau}, \sigma_R$, and the constants $C_\tau$ be from Proposition 4.2. Then
(i) For every $R > 0$, $0 < \tau < T$, $x, y \in H$, $|x| \leq R$, $|y| \leq R$, $t \in [0, T - \tau]$,
\[ |V(t, x) - V(t, y)| \leq \sigma_{R, \tau}(|x - y| - \eta). \quad (4.24) \]

(ii) For every $0 < \tau < T$, $x \in H$, $t \in [0, T - \tau]$,
\[ |V(t, x)| \leq C_{\tau}(1 + |x|^2). \quad (4.25) \]

(iii) For every $R > 0$ and $x \in H$, $|y| \leq R$, $t \in [0, T]$,
\[ |V(t, x) - f(e^{-(T-t)A_\beta}x)| \leq \sigma_R(T - t). \quad (4.26) \]

**Theorem 4.1.** Let Assumptions 3.1 and 4.1 be satisfied. Then for every $0 \leq t < s < T, x \in H$,
\[ V(t, x) = \inf_{a(\cdot) \in U_t} \mathbb{E} \left\{ \int_t^s l(A_\beta Y(r; t, x, a(\cdot)), a(r)) dr + V(s, Y(s; t, x, a(\cdot))) \right\}. \quad (4.27) \]

**Proof.** It is easy to see that the assumptions of Theorem 2.24 of [4] are satisfied. In particular, the
fact that Hypothesis 2.12 of [4] is satisfied is proved similarly to the proof of Proposition 2.16 of
[4] and condition (A1) in Hypothesis 2.12 follows from Lemma 4.1-(iv).

**Proposition 4.3.** Let Assumptions 3.1 and 4.1 be satisfied. Then $V \in C([0, T] \times H) \cap C([0, T] \times
D(A_\beta))$.

**Proof.** In light of (4.24), it is enough to prove the continuity of $V$ is the $t$ variable. Let $0 \leq t < s < T - \tau$ for some $\tau > 0$, let $x \in H$ and let $2 < p < \frac{1}{2}$. Denote $Y(r) = Y(r; t, x, a(\cdot))$. By the
dynamic programming principle, Assumptions 3.1, 4.1, and (4.7), we have
\[ |V(t, x) - V(s, x)| \leq \sup_{a(\cdot) \in U_t} \left\{ \mathbb{E} \int_t^s [l(A_\beta Y(r; t, x, a(\cdot)), a(r)) dr + \mathbb{E}|V(s, Y(s)) - V(s, x)|] \right\} \]
\[ \leq C(1 + |x|^2)(s - t)^{1-2\beta} + \mathbb{E}|V(s, Y(s)) - V(s, x)|. \quad (4.28) \]

Denote $R = |x| + 1$ and $\Omega_s = \{ \omega : |Y(s)| > R \}$. By (4.9), we have
\[ \mathbb{P}(\Omega_s) \leq \gamma_x(s - t) \]
for some modulus $\gamma_x$ which is independent of $a(\cdot) \in U_t$. Now, using (4.24), (4.25) and (4.9),
\[ \mathbb{E}|V(s, Y(s)) - V(s, x)| \leq C_{\tau} \int_{\Omega_s} (2 + |Y(s)|^{2} + |x|^{2}) d\mathbb{P}(\omega) + \mathbb{E}\sigma_{R, \tau}(C|Y(s) - x|) \]
\[ \leq C (\gamma_x(s - t))^{p-2} \mathbb{E}(2 + |Y(s)|^{p} + |x|^{p})^{\frac{2}{p}} + \mathbb{E}\sigma_{R, \tau} \left( C(\mathbb{E}|Y(s) - x|^{2})^{\frac{1}{2}} \right) \]
\[ \leq C_R (\gamma_x(s - t))^{p-2} + \mathbb{E}\sigma_{R, \tau} \left( C(\sigma_x(s - t))^{\frac{2}{2}} \right). \]

This shows that $V \in C([0, T) \times H)$. To prove that $V \in C([0, T] \times D(A_\beta))$ it is enough to show that if $t \to T$, and $x, y \in A_\beta, A_\beta y \to A_\beta x$, then $V(t, y) \to f(x)$. Denote $Y(s) = Y(s; t, y, a(\cdot)), X(s) = Y(s; t, x, a(\cdot))$. Using (4.14), (4.15), and (4.17), we have
\[ |V(t, y) - f(x)| \leq \sup_{a(\cdot) \in U_t} \left\{ \mathbb{E} \int_t^T |l(A_\beta Y(r; t, x, a(\cdot)), a(r)) dr + \mathbb{E}|g(A_\beta Y(T)) - g(A_\beta x)| \right\} \]
\[ \leq \sup_{a(\cdot) \in U_t} \left\{ \mathbb{E} \int_t^T C(1 + |Y(r)|^{2}) |r| dr + \mathbb{E}\rho(|Y(T) - X(T)|_{2\beta}) + \mathbb{E}\rho(|X(T) - x|_{2\beta}) \right\} \]
\[ \leq C(1 + |y|^{2})(T - t) + \rho(C|y - x|_{2\beta}) + \rho \left( (\sigma_\rho(T - t))^{\frac{1}{2}} \right). \quad \Box \]
By Itô’s formula, satisfies the same estimates as \( Y \) the solution of (4.5) with space \( \mu \).

In particular, \( V \) is weakly sequentially continuous on \([0, T) \times H\).

We will need Dynkin’s formulas for test functions \( \psi(s, y) = \varphi(s, y) + \delta(s)(1 + |y|^2)^{\frac{\eta}{2}} \). Dynkin’s formula for \( \delta(s)(1 + |y|^2)^{\frac{\eta}{2}} \) follows applying Itô’s formula to the function \( \Psi(s, r) = \delta(s)(1 + r)^{\frac{\eta}{2}} \) and the energy identity (4.12) (see also (4.13)). Dynkin’s formula for the function \( \varphi \) is proved below.

**Lemma 4.3.** Assume that \( b \) satisfies (3.3). Let \( \varphi \) be a test function satisfying conditions (i)-(iii) of Definition 3.1. Let \( 0 < t < s < T, y \in H, \) and \( a(\cdot) \in U_0^\mu \) for some generalized reference probability space \( \mu \). Then, denoting \( Y(s) = Y(s; t, y, a(\cdot)) \),

\[
\mathbb{E}\varphi(s, Y(s)) = \varphi(t, y) + \mathbb{E} \int_t^s \varphi_t(r, Y(r))dr - \mathbb{E} \int_t^s \langle A_2 Y(r), A_1 D\varphi(r, Y(r)) \rangle dr \\
+ \mathbb{E} \int_t^s \langle A_\beta b(A_\beta Y(r)) + \lambda Y(r) + G_\lambda a(r), D\varphi(r, Y(r)) \rangle dr \\
+ \mathbb{E} \int_t^s \frac{1}{2} \text{Tr}(B_\lambda B_\lambda^* D^2\varphi(r, Y(r)))dr.
\]

(4.29)

**Proof.** Let \( \eta = \frac{1}{2} - \epsilon(\varphi) \), where \( \epsilon(\varphi) \) is from Definition 3.1-(ii). Let \( Y_N(s) = Y_N(s; t, P_N, Y, a(\cdot)) \) be the solution of (4.5) with \( Y_N(t) = P_N Y \). We know that \( Y_N(\cdot) \rightharpoonup Y(\cdot) \in M^2(t, T; D(A_2)) \), \( Y_N(\cdot) \) satisfies the same estimates as \( Y(\cdot) \) in Lemma 4.1, and by (4.6), we have

\[
\lim_{N \to +\infty} |Y_N(\cdot) - Y(\cdot)|_{M^2(t, T; D(A_\eta))} = 0.
\]

By Itô’s formula,

\[
\mathbb{E}\varphi(s, Y_N(s)) = \varphi(t, P_N y) + \mathbb{E} \int_t^s \varphi_t(r, Y_N(r))dr - \mathbb{E} \int_t^s \langle A_2 Y_N(r), A_1 D\varphi(r, Y_N(r)) \rangle dr \\
+ \mathbb{E} \int_t^s \langle A_\beta P_N b(A_\beta Y_N(r)) + \lambda Y_N(r) + P_N G_\lambda a(r), D\varphi(r, Y_N(r)) \rangle dr \\
+ \mathbb{E} \int_t^s \frac{1}{2} \text{Tr}(P_N B_\lambda B_\lambda^* P_N D^2\varphi(r, Y_N(r)))dr.
\]

(4.30)

Identity (4.29) now follows by letting \( N \to +\infty \) in (4.30) after some rather standard calculations.

\[\square\]

### 4.3 Existence of viscosity solutions

**Theorem 4.2.** Let Assumptions 3.1 and 4.1 be satisfied. Then the value function \( W \) is the unique viscosity solution of (1.9), i.e. the value function \( V \) is the unique viscosity solution of (3.1). Uniqueness holds within the class of continuous functions \( w : [0, T] \times H \to \mathbb{R} \) such that the functions \( v : (0, T] \times H \to \mathbb{R} \), related to \( w \) by \( v(t, x) = w(t, A_\beta x) \) for \( t \in (0, T], x \in D(A_\beta) \), are weakly sequentially continuous on \((0, T) \times H\), continuous on \([0, T] \times D(A_\beta) \), and satisfy (4.25), and (4.26).
Proof. The uniqueness follows from Theorem 3.1 and the continuity and growth estimates of Corollary 4.1 and Proposition 4.3. We will only prove that \( V \) is a viscosity supersolution as the proof that \( V \) is a viscosity subsolution is similar and easier.

Let \( 2 < q < 1/\beta \) and \( \psi(s,y) = \varphi(s,y) + \delta(s)(1 + |y|^2)^{\frac{q}{2}} \), be a test function such that \( w + \psi \) has a minimum at \((t,x)\) over \((t-\kappa,t+\kappa) \times H\) for some \( \kappa > 0 \). Denote \( h_n = \frac{1}{n}, n = 1, 2, \ldots \) By Theorem 4.1, for every \( n \) there exists \( a_n(\cdot) \in \mathcal{U}_t \) such that, denoting \( Y_n(s) = Y(s; t, x, a_n(\cdot)) \),

\[
V(t,x) + h_n^2 \geq \mathbb{E} \left\{ \int_t^{t+h_n} l(\beta Y_n(s),a_n(s))ds + V(t+h_n,Y_n(t+h_n)) \right\}. \tag{4.31}
\]

We can assume that \( a_n(\cdot) \) is \( \mathcal{F}_s^{t,0} \)-predictable (see e.g. Lemma 1.99 of [4]) and then, by Corollary 2.21 of [4] and Lemma 4.1-(iv), we can assume that all processes \( a_n(\cdot) \) are defined on the same reference probability space \( \mu = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{s \geq t}, \mathbb{P}, \mathbb{W}_Q) \).

We have from (4.31),

\[
-h_n^2 + \mathbb{E} \int_t^{t+h_n} l(\beta Y_n(s),a_n(s))ds \leq \mathbb{E} \left[ V(t,x) - V(t+h_n,Y_n(t+h_n)) \right] \leq \mathbb{E} \left[ \psi(t+h_n,Y_n(t+h_n)) - \psi(t,x) \right].
\]

It therefore follows from (4.13) and (4.29) that for sufficiently large \( n \),

\[
-h_n + \frac{1}{h_n} \mathbb{E} \int_t^{t+h_n} l(\beta Y_n(s),a_n(s))ds &\leq \mathbb{E} \left\{ \int_t^{t+h_n} \left[ \varphi(s,Y_n(s)) + \delta(s)(1 + |Y_n(s)|^2)^{\frac{q}{2}} \right] ds \right. \\
&- \mathbb{E} \left\{ \int_t^{t+h_n} \langle A_{\frac{1}{2}} Y_n(s), A_{\frac{1}{2}} D\varphi(s,Y_n(s)) \rangle ds - q \mathbb{E} \left[ \int_t^{t+h_n} \delta(s)|Y_n(s)|^2(1 + |Y_n(s)|^2)^{\frac{q}{2} - 1} ds \right]

\]

\[
+ \mathbb{E} \left[ \int_t^{t+h_n} \langle A_{-\beta} b(\beta Y_n(s)) + \lambda Y_n(s) + G_n a_n(s), D\psi(s,Y_n(s)) \rangle ds \right]

+ \mathbb{E} \left[ \int_t^{t+h_n} \frac{1}{2} \text{Tr}(B_n B_n^* D^2\psi(s,Y_n(s)))ds \right].
\]

Since \(|A_{\frac{1}{2}} D\varphi(s,Y_n(s))| \leq C(1 + |Y_n(s)|^{\frac{1}{2} - v(\varphi)})\), it easily follows from (4.32) that for some constant \( C \),

\[
\frac{1}{h_n} \int_t^{t+h_n} \mathbb{E} |Y_n(s)|^2(1 + |Y_n(s)|^2)^{\frac{q}{2} - 1} ds \leq C. \tag{4.33}
\]

This implies that there is a sequence \( s_n \to t \) such that \( \mathbb{E} |Y_n(s_n)|^2 \leq C \), and thus, up to a subsequence, still denoted by \( s_n \), we have \( Y_n(s_n) \to Y \) in \( L^2(\Omega, \mathcal{F}, \mathbb{P}; D(A_{\frac{1}{2}})) \) for some \( Y \). Since, by (4.9), \( Y_n(s_n) \to x \) in \( L^2(\Omega, \mathcal{F}, \mathbb{P}; H) \), we thus obtain \( x = Y \in D(A_{\frac{1}{2}}) \).

We now need to pass to the limit as \( n \to +\infty \) in (4.32). This is standard for most of the terms if we use estimates of Lemmas 4.1 and 4.2. We will thus just discuss the first two terms of the right hand side of (4.32).

We denote

\[
Z_n := \frac{1}{h_n} \int_t^{t+h_n} \sqrt{\delta(s)A_{\frac{1}{2}} Y_n(s)(1 + |Y_n(s)|^2)^{\frac{q}{2}}} ds.
\]

It follows from (4.33) that

\[
\mathbb{E} |Z_n|^2 \leq \mathbb{E} \left[ \int_t^{t+h_n} \delta(s)|Y_n(s)|^2(1 + |Y_n(s)|^2)^{\frac{q}{2} - 1} ds \right] \leq C. \tag{4.34}
\]
Therefore, up to a subsequence, still denoted by \( h_n \), we have \( Z_n \to \tilde{Y} \) in \( L^2(\Omega, \mathcal{F}, \mathbb{P}; H) \) for some \( \tilde{Y} \). However, using (4.8) and (4.9), we easily see that
\[
A_{-\frac{1}{2}}Z_n \to \sqrt{\delta(t)}x(1 + |x|^2)^{\frac{\eta - 2}{2}} \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}; H),
\]
which implies that \( \tilde{Y} = \sqrt{\delta(t)}A_{\frac{1}{2}}x(1 + |x|^2)^{\frac{\eta - 2}{2}} \). Thus, using the first inequality of (4.34) and the lower semicontinuity of the norm with respect to the weak sequential convergence, we obtain
\[
\limsup_{n \to +\infty} \left[ -\mathbb{E} \frac{1}{h_n} \int_t^{t+h_n} \delta(s)|Y_n(s)|^2(1 + |Y_n(s)|^2)^{\frac{\eta - 2}{2} - 1} ds \right] \leq -\delta(t)|x|^2(1 + |x|^2)^{\frac{\eta - 2}{2} - 1}.
\]

As regards the first term in the second line of (4.32), arguing similarly as above we show that, up to a further subsequence,
\[
\frac{1}{h_n} \int_t^{t+h_n} A_{\frac{1}{2}}Y_n(s)ds \to A_{\frac{1}{2}}x \text{ in } L^2(\Omega, \mathcal{F}, \mathbb{P}; H). \quad (4.35)
\]

Now
\[
\mathbb{E} \frac{1}{h_n} \int_t^{t+h_n} (A_{\frac{1}{2}}Y_n(s), A_{\frac{1}{2}}D\varphi(s, Y_n(s)))ds - (A_{\frac{1}{2}}x, A_{\frac{1}{2}}D\varphi(t, x))
= \mathbb{E} \frac{1}{h_n} \int_t^{t+h_n} (A_{\frac{1}{2}}Y_n(s), A_{\frac{1}{2}}D\varphi(s, Y_n(s)) - A_{\frac{1}{2}}D\varphi(t, x))ds
+ \mathbb{E} \left( \frac{1}{h_n} \int_t^{t+h_n} A_{\frac{1}{2}}Y_n(s)ds - A_{\frac{1}{2}}x, A_{\frac{1}{2}}D\varphi(t, x) \right).
\]
The second term of the right hand side of the equality above converges to 0 by (4.35). Regarding the first term, using the properties of the test function \( \varphi \), if \( \eta = \frac{1}{2} - \epsilon(\varphi) \), we have
\[
|A_{\frac{1}{2}}D\varphi(s, Y_n(s)) - A_{\frac{1}{2}}D\varphi(t, x)| \leq \rho(|s - t| + |Y_n(s) - x|_{2\eta})
\]
for some modulus of continuity \( \rho \). Let for \( \tau > 0 \), the constants \( C_\tau \) be such that \( (\rho(r))^2 \leq \tau + C_\tau r^2 \) for \( r \geq 0 \). Then, using Cauchy-Schwarz inequality,
\[
\left| \mathbb{E} \frac{1}{h_n} \int_t^{t+h_n} (A_{\frac{1}{2}}Y_n(s), A_{\frac{1}{2}}D\varphi(s, Y_n(s)) - A_{\frac{1}{2}}D\varphi(t, x))ds \right|
\leq \left( \mathbb{E} \frac{1}{h_n} \int_t^{t+h_n} |Y_n(s)|^2 ds \right)^{\frac{1}{2}} \left( \mathbb{E} \frac{1}{h_n} \int_t^{t+h_n} (\rho(|s - t| + |Y_n(s) - x|_{2\eta})^2 ds \right)^{\frac{1}{2}}
\leq C \left( \mathbb{E} \left( \rho(h_n + \sup_{t \leq s \leq t+h_n} |Y_n(s) - x|_{2\eta}) \right) \right)^{\frac{1}{2}} \leq C \inf_{\tau > 0} \left( \tau + 2C_\tau \left( h_n^2 + \mathbb{E} \sup_{t \leq s \leq t+h_n} |Y_n(s) - x|_{2\eta}^2 \right) \right)^{\frac{1}{2}},
\]
and the last expression goes to 0 as \( n \to +\infty \) because of (4.15).
Therefore we can rewrite (4.32) as

\[
\rho_1(n) \geq \mathbb{E} \frac{1}{h_n} \int_t^{t+h_n} \left[ - \varphi(t,x) - \delta'(t)(1 + |x|^2)^{q} \\
+ (A_{1/2} x, A_{1/2} D \varphi(t,x)) + q \delta(t)|x|^2(1 + |x|^2)^{q-1} \\
+ (A_{-\beta} b(A_{\beta} x) + \lambda x, -D \psi(t,x)) + \frac{1}{2} \text{Tr}(B_{\Lambda} B_{\Lambda}^*( -D^2 \psi(t,x))) \\
+ (G_\Lambda a_n(s), -D \psi(t,x)) + l(A_{\beta} x, a_n(s)) \right] ds \\
\geq \mathbb{E} \frac{1}{h_n} \int_t^{t+h_n} \left[ - \psi(t,x) + (A_{1/2} x, A_{1/2} D \varphi(t,x)) + q \delta(t)|x|^2(1 + |x|^2)^{q-1} \\
- \lambda(x, D \psi(t,x)) + F(x, -D \psi(t,x) - D^2 \psi(t,x)) \right] ds \\
= - \psi(t,x) + (A_{1/2} x, A_{1/2} D \varphi(t,x)) + q \delta(t)|x|^2(1 + |x|^2)^{q-1} \\
- \lambda(x, D \psi(t,x)) + F(x, -D \psi(t,x) - D^2 \psi(t,x)),
\]

where \( \lim_{n \to +\infty} \rho_1(n) = 0 \). We conclude by sending \( n \to +\infty \).

The subsolution property of \( V \) is proved by similar arguments applied to solutions \( Y(s; t, x, a(\cdot)) \) with control processes \( a(\cdot) \equiv a \) on some reference probability space, for all \( a \in B^A_R \).

**Example 4.1.** Let \( f : \mathbb{R} \to \mathbb{R} \) be Lipschitz continuous, and \( h \in L^\infty(\partial O) \). It is then obvious that the function \( b : H \to H, b(x)(\cdot) := f(x(\cdot)) \) satisfies (3.3), and the linear operator \( \sigma : \Lambda \to \Lambda \) defined by \( [\sigma z](\cdot) := h(\cdot) z(\cdot) \) belongs to \( L(\Lambda) \). Suppose that the functions \( \beta_1, \gamma : \partial O \times \mathbb{R} \to \mathbb{R}, \beta_2 : \mathbb{R} \to \mathbb{R} \) are continuous and satisfy

\[
|\beta_1(\xi, r_1) - \beta_1(\xi, r_2)| \leq \sigma_1(|r_1 - r_2|(1 + |r_1| + |r_2|)), \quad \text{for all } \xi \in \partial O, r_1, r_2 \in \mathbb{R},
\]

\[
|\gamma(\xi, r_1) - \gamma(\xi, r_2)| \leq \sigma_2(|r_1 - r_2|), \quad \text{for all } \xi \in \partial O, r_1, r_2 \in \mathbb{R},
\]

for some moduli \( \sigma_1, \sigma_2 \),

\[
\sup_{\xi \in \partial O} (|\gamma(\xi, 0)| + |\beta_1(\xi, 0)|) < +\infty,
\]

and for some \( 0 \leq q \leq 2 \),

\[
|\beta_2(r)| \leq C(1 + |r|^q) \quad \text{for all } r \in \mathbb{R}.
\]

Under these conditions, it is easy to see using Jensen’s and Cauchy-Schwarz inequalities, that the functions \( l, g \) defined by (1.5) and (1.6) satisfy Assumptions 3.1 and 4.1. The fact that \( l_1 \) is continuous follows from standard uniform integrability arguments.

**References**


