VISCOSITY SOLUTIONS OF DYNAMIC PROGRAMMING EQUATIONS FOR THE OPTIMAL CONTROL OF 2-D NAVIER-STOKES EQUATIONS

Fausto Gozzi
Dipartimento di Matematica per le Decisioni 
Economiche, Finanziarie e Assicurative
Università di Roma “La Sapienza”
Via del Castro Laurenziano 9, 00161 Roma, Italy

S. S. Sritharan
Space & Naval Warfare Systems Center, Code D73H
San Diego, CA 92152-5001, U.S.A.

Andrzej Święch *
School of Mathematics, Georgia Institute of Technology
Atlanta, GA 30332, U.S.A.

Abstract
In this paper we study the infinite dimensional Hamilton-Jacobi equation associated with the optimal feedback control of viscous hydrodynamics. We resolve the global unique solvability problem of this equation by showing that the value function is the unique viscosity solution.

Key words: Optimal control of fluids, Navier-Stokes equations, Hamilton-Jacobi equations, viscosity solutions.

2000 Mathematics Subject Classification: 49L25, 35R15, 49L20.

* A. Święch was supported by NSF grant DMS 0098565.
1 Introduction

Optimal control of Navier-Stokes equations has many applications in traditional aero-
dynamic and hydrodynamic drag reduction, lift enhancement, turbulence control and
combustion control. Data assimilation in meteorology and oceanography involves the task
of finding best initial data and unknown boundary and body forces by optimal control
methodology [40]. During the past decade several fundamental advances have been made
in developing an organized mathematical theory to this subject [16, 19, 32]. One of the
open problems identified in [33] was the global unique solvability of the infinite dimen-
sional Hamilton-Jacobi equation associated with the feedback synthesis for Navier-Stokes
equation. This problem is resolved in this paper. In this respect, this paper is at the cross
roads between two theoretical developments in the past several years, namely flow con-
trol theory and the theory of viscosity solutions for infinite dimensional Hamilton-Jacobi
equations. Local solvability for the Hamilton-Jacobi equation associated to the feedback
control of Navier-Stokes equation has been achieved in the past in the context of $H^\infty$-
control theory [4] and time periodic control [1]. Global solvability for mild solutions (in
the semigroup formulation) of a semilinear Hamilton-Jacobi equation associated with the
stochastic control of 2-d Navier-Stokes equations is given in [12] where the existence of
smooth solutions is proven in some cases. For the impulse control problem Hamilton-Jacobi
equation becomes a quasi-variational inequality and this is again resolved in a mild sense
in [26]. Viscosity solution method which provides another notion of a generalized solution
was first introduced in the context of finite dimensional Hamilton-Jacobi equations by M.
G. Crandall and P. L. Lions [9] and was generalized to infinite dimensions by a number of
authors. Of particular relevance to this paper are [10, Parts IV,V,VII] and [7, 17, 18, 22].
For other literature on infinite-dimensional Hamilton-Jacobi equations we refer the reader
to [2, 3, 6, 23, 25, 35, 36], [10, PartVI] and the references therein. This paper deals with
Hamilton-Jacobi-Bellman (HJB) equations associated to optimal control of Navier-Stokes
equations in two dimensional domains under homogeneous boundary conditions. Periodic
boundary conditions can also be treated by our methods but we do not do it here as this
case is easier.

The approach presented here allows natural extensions to second order HJB equations
of this type. We think that modifications of the methods developed here can be applied
to prove comparison results for such equations. However connecting these second order
HJB equations to optimal control of stochastic Navier-Stokes equations seems to be an
challenging and difficult problem (see [12]).

Finally we mention that after this paper had been submitted we have learnt about a
manuscript [31] that deals with abstract infinite dimensional Hamilton-Jacobi equations
and uses a different notion of viscosity solution. The results of [31] can be applied to some
HJB equations coming from the optimal control of Navier-Stokes equations.

In the rest of this section we will describe the mathematical problem. In section 2
we establish various continuous dependence theorems for the controlled Navier-Stokes
equation that are needed in the paper. In section 3 we describe a suitable notion of the
viscosity solution used in this paper. In section 4 we prove the uniqueness theorem
for these viscosity solutions for time independent and time dependent Hamilton-Jacobi
equations. Section 5 is devoted to the application of these abstract PDE results to an optimal control problem. In section 5.1 we show that the value function has the required continuity properties, and in section 5.2 we prove that it is indeed a viscosity solution in the sense of the definition given in section 3. In section 5.3 we briefly discuss the applicability of our results.

Let us now describe a typical optimal control problem for the abstract Navier-Stokes equation [32, Ch. I]. We will assume throughout that the the kinematic viscosity is equal to 1. Let \( \Omega \subset \mathbb{R}^2 \) be an open and bounded set with smooth boundary. Let \( H = \) the closure of \( \{ x \in \mathcal{D}(\Omega; \mathbb{R}^2), \text{div} \ x = 0 \} \) in \( L^2(\Omega; \mathbb{R}^2) \)

\[ V = \text{the closure of } \{ x \in \mathcal{D}(\Omega; \mathbb{R}^2), \text{div} \ x = 0 \} \text{ in } H^1(\Omega; \mathbb{R}^2), \]

and let \( P_H \) be the orthogonal projection in \( L^2(\Omega; \mathbb{R}^2) \) onto \( H \). Denote by \( A x = -P_H \Delta x \), and by \( B(x, y) = P_H[(x \cdot \nabla)y] \). Let \( t \geq 0 \) be the initial time and \( T \geq t \) be the terminal time. Our state variable is the velocity field \( X: [t, T] \times \Omega \to \mathbb{R}^2 \) that satisfies the equation

\[
\begin{aligned}
\frac{dX(s)}{ds} &= -Ax(s) - B(X(s), X(s)) + f(s, a(s)) \quad \text{in } (t, T) \times H, \\
X(t) &= x \in H,
\end{aligned}
\tag{1.1}
\]

where \( f: [0, T] \times \Theta \to V \), \( \Theta \) is a complete metric space (the control set), and \( a(\cdot): [0, T] \to \Theta \) is a measurable function which plays the role of a control strategy. We will denote the set of control strategies by \( U \). The minimization, over all controls \( a(\cdot) \in U \), of a cost functional

\[
J(t, x; a) = \int_t^T l(s, X(s), a(s))ds + g(X(T))
\]

formally leads to the Hamilton-Jacobi-Bellman (HJB) partial differential equation in \( (0, T) \times H \)

\[
\begin{aligned}
u_t - \langle Ax + B(x, x), Du \rangle + \inf_{a \in \Theta} \{ f(t, a), Du \} + l(t, x, a) &= 0 \quad \text{in } (0, T) \times H, \\
u(T, x) &= g(x) \quad \text{in } H.
\end{aligned}
\tag{1.2}
\]

This equation should be satisfied by the value function

\[ \mathcal{V}(t, x) = \inf_{a(\cdot) \in U} J(t, x; a). \]

Equation (1.2) falls into a general class of infinite dimensional Hamilton-Jacobi equations

\[
\begin{aligned}
u_t - \langle Ax + B(x, x), Du \rangle + F(t, x, Du) &= 0 \quad \text{in } (0, T) \times H, \\
u(T, x) &= g(x) \quad \text{in } H
\end{aligned}
\tag{1.3}
\]

that will be the object of our study. Likewise we will be investigating stationary versions of (1.3),

\[
\lambda u + \langle Ax + B(x, x), Du \rangle + F(x, Du) = 0 \quad \text{in } H, \tag{1.4}
\]

that are related to the infinite horizon optimal control problems.
2 Preliminaries on the 2-D controlled Navier-Stokes equation

2.1 Abstract spaces and the Stokes operator

Throughout the paper Ω will be an open and bounded subset of $\mathbb{R}^2$ with smooth boundary. We denote by $W^{m,p}(\Omega; \mathbb{R}^2)$ (or simply $W^{m,p}(\Omega)$) the Sobolev space of order $0 \leq m \in \mathbb{R}$ and power $p \geq 1$ of functions with values in $\mathbb{R}^2$ (which can be seen as a product space $W^{m,p}(\Omega; \mathbb{R}^2) = [W^{m,p}(\Omega; \mathbb{R})]^2$). The norm of $x \in W^{m,p}(\Omega; \mathbb{R}^2)$ will be denoted by $\|x\|_{m,p}$.

We will use the notation $L^p$ for $W^{0,p}$. Moreover, when $p = 2$, we will write $H^m$ for $W^{m,2}$.

We will also be using the negative Sobolev spaces $H^{-m}$. The space $H$ can be alternatively defined by

$$H = \left\{ x \in L^2(\Omega; \mathbb{R}^2), \text{ div } x = 0 \text{ in } D'(\Omega; \mathbb{R}^2), \text{ } x \cdot n = 0 \text{ in } H^{-\frac{1}{2}}(\partial \Omega) \right\},$$

where $n$ is the outward normal to the boundary, see [37, Ch.1]. The operator $A = -P_H \Delta$ with the domain $D(A) = H^2(\Omega; \mathbb{R}^2) \cap V$ is called the Stokes operator. It is well known that $A$ is positive definite, self-adjoint, $A^{-1}$ is compact, and $A$ generates an analytic semigroup.

For $\gamma > 0$ we denote by $V_\gamma$ the domain of $A^{\frac{1}{2}}$, $D(A^{\frac{1}{2}})$, equipped with the norm

$$\|x\|_{\gamma} = \|A^{\frac{1}{2}}x\|_{0,2}. \quad (2.5)$$

For $\gamma < 0$ the space $V_\gamma$ is defined as the completion of $H$ under the norm (2.5). If $\gamma > -1/2$ the norm of $V_\gamma$ is equivalent to the norm of $H^\gamma$ (see [16, Lemma 4.5]). Moreover, the space $V_1$ coincides with $V$. Identifying $H$ with its dual, the space $V_{-\gamma}$ is the dual of $V_\gamma$ for $\gamma > 0$. We will also be using the customary notation $V'$ for the dual of $V$ and the duality pairing between $V'$ and $V$ will be denoted by $\langle \cdot, \cdot \rangle$. The same symbol will also be used to denote the inner product in $H$ if both entries are in $H$.

Finally the operators $A^{\gamma}$ are isometries between $V_{\delta} = V_{\delta-\gamma}$ for each real $\gamma$ and $\delta$. We recall below Sobolev imbeddings and other inequalities that we will need in the sequel.

- **Sobolev imbedding type inequalities:** If $m \geq 0$, $mp \leq 2$ and $p \leq q \leq \frac{2p}{2-mp}$ then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, i.e.

  $$\|x\|_{0,q} \leq C \|x\|_{m,p}, \quad \text{for } x \in W^{m,p}(\Omega),$$

  (reminding that when $mp = 2$ then the imbedding holds for all $q < +\infty$). Combining the above with the equivalence of norms of $V_\gamma$ and $H^\gamma$, we obtain that for $\gamma \in (0,1]$ and $q \in \left[2, \frac{2}{1-\gamma}\right]$ ($q \in [2, +\infty)$ if $\gamma = 1$) $V_\gamma \hookrightarrow L^q(\Omega)$, i.e.

  $$\|x\|_{0,q} \leq C \|x\|_{\gamma}, \quad \text{for } x \in V_\gamma. \quad (2.6)$$
• Gagliardo Nirenberg type inequality:

$$\|x\|_{0,4} \leq C \|x\|_{0}^{1/2} \|x\|_{1}^{1/2}, \quad \text{for} \quad x \in V. \quad (2.7)$$

• Interpolation inequality: Recall that if an operator $S$ generates an analytic semigroup then there exists a constant $C$ such that for every $x \in D(S)$ and $0 \leq \gamma \leq 1$

$$\|S^\gamma z\|_0 \leq C \|Sz\|_0 \|z\|_0^{1-\gamma}. \quad (2.8)$$

2.2 The Euler operator (inertia term) $B$

We define the trilinear form $b(\cdot, \cdot, \cdot) : V \times V \times V \to \mathbb{R}$ as

$$b(x, y, z) = \int_\Omega z(\xi) \cdot (x(\xi) \cdot \nabla \xi)y(\xi) \, d\xi$$

and the bilinear operator $B(\cdot, \cdot) : V \times V \to V'$ as

$$\langle B(x, y), z \rangle = b(x, y, z)$$

for all $z \in V$. This is just another way to introduce the operator $B$ that we have already used in the Navier-Stokes equation (1.1). We will write $B(x)$ for $B(x, x)$ when no confusion arises. This operator can be extended in different topologies. By the incompressibility condition we have

$$b(x, y, y) = 0, \quad b(x, y, z) = -b(x, z, y). \quad (2.9)$$

Moreover we have the following estimates for $b(x, y, z)$, with $x, y, z \in V$:

1. By Hölder inequality, for $p, q > 1, \frac{1}{p} + \frac{1}{q} + \frac{1}{2} = 1$

$$|b(x, y, z)| \leq C \|x\|_{0,p} \|y\|_1 \|z\|_{0,q}. \quad (2.10)$$

2. By (2.7) we obtain

$$|b(x, y, z)| \leq C \|x\|_{0,4} \|y\|_1 \|z\|_{0,4} \leq C \|x\|_0^{1/2} \|x\|_1^{1/2} \|y\|_1 \|z\|_0^{1/2} \|z\|_1^{1/2} \quad (2.11)$$

which gives when $x = z$

$$|b(x, y, x)| \leq C \|x\|_0^2 \|y\|_1 \leq C \|x\|_0 \|x\|_1 \|y\|_1. \quad (2.12)$$

2.3 The controlled Navier-Stokes equation: existence, uniqueness and continuous dependence

In this section we derive various estimates for solutions of the Navier-Stokes equation (1.1). We begin with the definition of solution (see for instance [37, 38]).
Definition 2.1 (i) A weak solution of the equation (1.1) is a map
\[ X \in C([t,T];H) \cap L^2(t,T;V) \text{ with } X' \in L^2(t,T;V_{-1}) \]
such that (1.1) is satisfied as an equality in \( V_{-1} \) i.e. for every \( z \in V \) and almost every \( s \in (t,T) \), we have
\[ \langle X'(s), z \rangle = \langle -AX(s) - B(X(s), X(s)) + f(s, a(s)), z \rangle \]

(ii) A strong solution of the equation (1.1) is a map
\[ X \in C([t,T];V) \cap L^2(t,T;D(A)) \text{ with } X' \in L^2(t,T;H) \]
such that (1.1) is satisfied as an equality in \( H \) for almost every \( s \in (t,T) \).

Given \( 0 \leq t \leq T \), an admissible control strategy \( a(\cdot) \in \mathcal{U} \) and an initial datum \( x \in H \), a weak (or strong) solution of the state equation (1.1) will be denoted by \( X(\cdot; t, x, a) \). We will often denote it simply by \( X(\cdot) \) when there is no possibility of confusion. We first recall the standard existence and uniqueness results [37] about the Navier-Stokes equation (1.1).

Theorem 2.2 Let \( f : [0,T] \times \Theta \to V_{-1} \) be bounded and continuous. Let \( 0 \leq t \leq T \), and let \( a(\cdot) \in \mathcal{U} \).

(i) Given an initial datum \( x \in H \) there exists a unique weak solution \( X(\cdot; t, x, a) \) of the state equation (1.1). Moreover we have the estimate
\[ \sup_{t \leq s \leq T} \|X(s)\|_0^2 + \int_t^T \|X(s)\|_1^2 ds \leq \|x\|_0^2 + \int_t^T \|f(s, a(s))\|_{-1}^2 ds. \quad (2.13) \]

(ii) Given an initial datum \( x \in V \) and assuming moreover that \( f : [0,T] \times \Theta \to H \) is bounded and continuous there exists a unique strong solution \( X(\cdot; t, x, a) \) of equation (1.1). Moreover we have the estimate
\[ \sup_{t \leq s \leq T} \|X(s)\|_1^2 + \int_t^T \|AX(s)\|_0^2 ds \leq \|x\|_1^2 \exp(E(t, T)) + \int_t^T \|f(s, a(s))\|_0^2 \exp(E(s, T)) ds \quad (2.14) \]
where
\[ E(t, \tau) := C \int_t^\tau \|X(r)\|_0^2 \|X(r)\|_1^2 dr \leq C \sup_{t \leq r \leq \tau} \|X(r)\|_0^2 \int_t^\tau \|X(s)\|_1^2 ds. \]

The next proposition gathers continuous dependence estimates for solutions of equation (1.1). The negative norm estimates given here generalize the results given in [32, Ch. I]].
Proposition 2.3 Let $f : [0, T] \times \Theta \to V_{-1}$ be bounded and continuous. Then

(i) Let $1/2 \leq \alpha \leq 1$. There exist constants $C > 0$, depending only on the indicated variables and independent of $a(\cdot) \in U$, such that for every initial conditions $x, y \in H$

$$
\|X(s) - Y(s)\|_{-\alpha} \leq C \left( \|x\|_0, \|y\|_0, \sup_{a(\cdot)\in U} \int_t^T \|f(r, a(r))\|_{-1}^2 \, dr \right) \|x - y\|_{-\alpha},
$$

(2.15)

$$
\int_t^T \|X(s) - Y(s)\|_{1-\alpha}^2 \leq C \left( \|x\|_0, \|y\|_0, \sup_{a(\cdot)\in U} \int_t^T \|f(r, a(r))\|_{-1}^2 \, dr \right) \|x - y\|_{-\alpha}^2,
$$

(2.16)

$$
\|X(s) - x\|_{-\alpha} \leq C \left( \|x\|_0, \sup_{a(\cdot)\in U} \int_t^T \|f(r, a(r))\|_{-1}^2 \, dr \right) (s - t)^{1/4},
$$

(2.17)

and there is a modulus $\sigma$, independent of the strategies $a(\cdot) \in U$, such that

$$
\|X(s) - x\|_0 \leq \sigma(s - t).
$$

(2.18)

(ii) For every initial condition $x \in V$ there exists a constant $C > 0$ independent of the strategies $a(\cdot) \in U$ such that

$$
\|X(s) - x\|_0 \leq C \left( \|x\|_1, \sup_{s \in [t, T], a \in \Theta} \|f(s, a)\|_{-1}^2 \right) (s - t)^{1/2},
$$

(2.19)

and if in addition $f : [0, T] \times \Theta \to H$ is bounded and continuous then

$$
\|X(s) - x\|_1 \leq \sigma(s - t)
$$

(2.20)

for some modulus $\sigma$ independent of the strategies $a(\cdot) \in U$.

**Proof.** Proof of (i). We first prove (2.15). Let $X(s)$ and $Y(s)$ be respectively the solutions of (1.1) with initial conditions $x$ and $y$ respectively. Let $Z(s) = X(s) - Y(s)$. Then taking the inner products of (1.1) for $X(s)$ and $Y(s)$ with $A^{-\alpha}Z(s)$ (i.e. using the definition of weak solution since $A^{-\alpha}Z(s) \in V$) we get, for $s \in [t, T]$,

$$
\begin{cases}
\langle X'(s), A^{-\alpha}Z(s) \rangle = \langle -AX(s) - B(X(s)) + f(s, a(s)), A^{-\alpha}Z(s) \rangle \\
X(t) = x,
\end{cases}
$$

(2.21)

$$
\begin{cases}
\langle Y'(s), A^{-\alpha}Z(s) \rangle = \langle -AY(s) - B(Y(s)) + f(s, a(s)), A^{-\alpha}Z(s) \rangle \\
Y(t) = y.
\end{cases}
$$

(2.22)

Subtracting the resulting equalities we obtain

$$
\begin{cases}
\langle Z'(s), A^{-\alpha}Z(s) \rangle = -\langle AZ(s), A^{-\alpha}Z(s) \rangle + \langle B(Y(s)) - B(X(s)), A^{-\alpha}Z(s) \rangle \\
Z(t) = x - y,
\end{cases}
$$

(2.23)
so
\[
\frac{1}{2} \frac{d}{ds} \left| |A^{-\frac{\alpha}{2}} Z(s)||_0^2 + |A^{1-\frac{\alpha}{2}} Z(s)||_0^2 \right| = -b \left( X(s), X(s), A^{-\alpha} Z(s) \right) + b \left( Y(s), Y(s), A^{-\alpha} Z(s) \right).
\]

(2.24)

Now, by linearity, adding and subtracting \(b(Y(s), X(s), A^{-\alpha} Z(s))\),
\[
- b \left( X(s), X(s), A^{-\alpha} Z(s) \right) + b \left( Y(s), Y(s), A^{-\alpha} Z(s) \right) = - b(Z(s), X(s), A^{-\alpha} Z(s)) - b(Y(s), Z(s), A^{-\alpha} Z(s)).
\]

By Hölder inequality we have (see (2.10))
\[
|b(Z(s), X(s), A^{-\alpha} Z(s))| \leq C \|Z(s)\|_{0, \frac{2}{\alpha}} \|X(s)\|_1 \|A^{-\alpha} Z(s)\|_{0, \frac{1}{1-\alpha}}.
\]

Sobolev imbeddings give us (see (2.6))
\[
\|Z(s)\|_{0, \frac{2}{\alpha}} \leq C \|Z(s)\|_{1-\alpha}, \quad \text{and} \quad \|A^{-\alpha} Z(s)\|_{0, \frac{2}{1-\alpha}} \leq C \|A^{-\alpha} Z(s)\|_{\alpha} = C \|Z(s)\|_{-\alpha}.
\]

Therefore we obtain that
\[
|b(Z(s), X(s), A^{-\alpha} Z(s))| \leq C \|Z(s)\|_{1-\alpha} \|X(s)\|_1 \|Z(s)\|_{-\alpha} \leq \frac{1}{4} \|Z(s)\|_1^2 + C \|X(s)\|_1^2 \|Z(s)\|_{-\alpha}^2
\]

(2.25)

for some constant \(C > 0\). The second term is a little more difficult to estimate. Similarly
as before we have
\[
|b(Y(s), Z(s), A^{-\alpha} Z(s))| = |b(Y(s), A^{-\alpha} Z(s), Z(s))|
\leq C \|Y(s)\|_{0, \frac{2}{1-\alpha}} \|A^{-\alpha} Z(s)\|_1 \|Z(s)\|_{0, \frac{2}{\alpha}}
= C \|Y(s)\|_{0, \frac{2}{1-\alpha}} \|A^{\frac{1}{2}-\alpha} Z(s)\|_0 \|Z(s)\|_{0, \frac{2}{\alpha}}.
\]

Estimates (2.6) and (2.8) give
\[
\|Y(s)\|_{0, \frac{2}{1-\alpha}} \leq C \|A^{\frac{1}{2}} Y(s)\|_0 = C \|(A^{\frac{1}{2}})^\alpha Y(s)\|_0 \leq C \|Y(s)\|_1 \|Y(s)\|_{0, 1-\alpha},
\]
\[
\|A^{\frac{1}{2}-\alpha} Z(s)\|_0 = \|(A^{\frac{1}{2}})^{1-\alpha} (A^{-\frac{\alpha}{2}} Z(s))\|_0 \leq C \|A^{\frac{1}{2}-\alpha} Z(s)\|_{0, 1-\alpha} \|A^{-\frac{\alpha}{2}} Z(s)\|_0,
\]
\[
\|Z(s)\|_{0, \frac{2}{\alpha}} \leq C \|A^{\frac{1}{2}-\alpha} Z(s)\|_0.
\]

and, by (2.13)
\[
\sup_{s \in [t, T]} \|Y(s)\|_{0, 1-\alpha} \leq \left( \|y\|_0^2 + \int_t^T \|f(s, a(s))\|_{-1}^2 ds \right)^{\frac{1-\alpha}{2}}.
\]

Therefore collecting these estimates and using Young’s inequality we obtain
\[
|b(Y(s), Z(s), A^{-\alpha} Z(s))| \leq C \|Y(s)\|_1 \|Y(s)\|_{0, 1-\alpha} \|A^{\frac{1}{2}-\alpha} Z(s)\|_{0, 2-\alpha} \|A^{-\frac{\alpha}{2}} Z(s)\|_0^3
\leq C \|Y(s)\|_1 \|A^{\frac{1}{2}-\alpha} Z(s)\|_{0, 2-\alpha} \|A^{-\frac{\alpha}{2}} Z(s)\|_0^3
\leq \frac{1}{4} \|Z(s)\|_{1-\alpha}^2 + C \|Y(s)\|_1^2 \|Z(s)\|_{-\alpha}^2
\]

(2.26)
Using (2.25) and (2.26) in (2.24) produces
\[
\frac{1}{2} \frac{d}{ds} \|Z(s)\|_{-\alpha}^2 + \frac{1}{2} \|A^{-\alpha} Z(s)\|_0^2 \leq C(\|X(s)\|_1^2 + \|Y(s)\|_0^2) \|Z(s)\|_{-\alpha}^2. \tag{2.27}
\]

Gronwall’s inequality and (2.14) now yield
\[
\|X(s) - Y(s)\|_{-\alpha}^2 = \|Z(s)\|_{-\alpha}^2 \leq e^{C \int_t^s (\|X(r)\|_1^2 + \|Y(r)\|_0^2)dr} \|x - y\|_{-\alpha}^2 \\
\leq C \left(\|x\|_0, \|y\|_0, \int_t^s \|f(r, a(r))\|_{-1}^2 \, dr\right) \|x - y\|_{-\alpha}^2 \tag{2.28}
\]
which gives us (2.15). Then putting (2.28) into (2.27) we get (2.16). To prove (2.17) we set \(Y(s) = X(s) - x\) so that
\[
Y'(s) = X'(s) = -AX(s) - B(X(s)) + f(s, a(s)) \tag{2.29}
\]
Taking the inner product of (2.29) with \(A^{-\alpha} Y(s)\) we get
\[
\frac{1}{2} \frac{d}{ds} \|Y(s)\|_{-\alpha}^2 = -\left\langle A^{1-\alpha} X(s), Y(s) \right\rangle \\
-\langle b(X(s), X(s), A^{-\alpha} Y(s)) \rangle + \langle f(s, a(s)), A^{-\alpha} Y(s) \rangle. \tag{2.30}
\]
Since \(\alpha \geq 1/2\) we have, using also (2.13),
\[
\left|\langle A^{1-\alpha} X(s), Y(s) \rangle\right| \leq \|A^{1-\alpha} X(s)\|_0 \|Y(s)\|_0 \leq C \|X(s)\|_1.
\]
Moreover by (2.12) and (2.13)
\[
|b(X(s), X(s), A^{-\alpha} Y(s))| \leq C \|X(s)\|_0 \|A^{-\alpha} Y(s)\|_1 \|X(s)\|_1 \\
\leq C \|X(s)\|_1 \|Y(s)\|_0 \|X(s)\|_0 \leq C \|X(s)\|_1.
\]
Finally
\[
\left|\langle f(s, a(s)), A^{-\alpha} Y(s) \rangle\right| \leq \|f(s, a(s))\|_{-1} \|A^{-\alpha} Y(s)\|_1 \\
\leq C \|f(s, a(s))\|_{-1} \|Y(s)\|_0 \leq C \|f(s, a(s))\|_{-1}.
\]
Putting these inequalities in (2.30) produces
\[
\frac{1}{2} \frac{d}{ds} \|Y(s)\|_{-\alpha}^2 \leq C \left(\|X(s)\|_1 + \|f(s, a(s))\|_{-1}\right).
\]
Integrating and using the Cauchy-Schwartz inequality we then get
\[
\|X(s) - x\|_{-\alpha}^2 \leq C \left(\int_t^s \|X(r)\|_1^2 \, dr \right)^{\frac{1}{2}} + \left(\int_t^T \|f(r, a(r))\|_{-1}^2 \, ds \right)^{\frac{1}{2}} (s - t)^{\frac{1}{2}}
\]
and the claim follows. To prove the fourth estimate we argue by contradiction. We want to show that
\[
\sup_{s \in [t, t+\varepsilon], a(\varepsilon) \in U} \|X(s; t, x, a) - x\|_0 \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.
\]
Assume that this is false. Then there exist sequences $s_n, a_n(\cdot)$ such that
\[ s_n \to t, \quad \|X(s_n; t, x, a_n) - x\|_0 \geq \varepsilon. \]

However by the boundedness of $\|X(s_n; t, x, a_n)\|_0$ and the third estimate of (i) we have
\[ X(s_n; t, x, a_n) \to x \text{ strongly in } V_{-\alpha} \text{ and weakly in } H. \]

Moreover weak convergence in $H$ implies that
\[ \lim_{n \to \infty} \|X(s_n; t, x, a_n)\|_0 \geq \|x\|_0, \]
and from the energy inequality (2.13) we can conclude that
\[ \limsup_{n \to \infty} \|X(s_n; t, x, a_n)\|_0 \leq \|x\|_0. \]

Combining these two results gives convergence of the norms and this, along with the weak convergence in $H$, yields that $X(s_n; t, x, a_n) \to x$ strongly in $H$ which is a contradiction.

**Proof of (ii)** Let $X(s) = X(s; t, x, a)$ be the weak solution of (1.1) with initial conditions $x$ at time $t$. Set as before $Y(s) = X(s) - x$ so that $Y(s)$ satisfies (2.29). Taking the inner product of (2.29) with $Y(s)$ we obtain
\[ \frac{1}{2} \frac{d}{ds} \|Y(s)\|_0^2 = - \langle Ax(s), Y(s) \rangle - b(X(s), X(s), Y(s)) \]
\[ \quad + \langle f(s, a(s)), Y(s) \rangle \quad (2.31) \]
\[ = - \langle Ay(s), Y(s) \rangle - \langle Ax, Y(s) \rangle \]
\[ - b(X(s), X(s), Y(s)) + \langle f(s, a(s)), Y(s) \rangle. \]

Now by the tri-linearity of $b(\cdot, \cdot, \cdot)$ and the orthogonality relations (2.9)
\[ b(X(s), X(s), Y(s)) = b(Y(s), X(s), Y(s)) + b(x, X(s), Y(s)) \]
\[ = b(Y(s), X(s), Y(s)) + b(x, x, Y(s)), \]
which yields, when plugged into (2.31),
\[ \frac{1}{2} \frac{d}{ds} \|Y(s)\|_0^2 + \|Y(s)\|_1^2 = - \langle Ax, Y(s) \rangle - b(Y(s), X(s), Y(s)) \]
\[ + b(x, x, Y(s)) + \langle f(s, a(s)), Y(s) \rangle. \]

Now we have
\[ |\langle Ax, Y(s) \rangle| = |\langle A^{1/2}x, A^{1/2}Y(s) \rangle| \leq \|x\|_1 \|Y(s)\|_1 \leq \frac{1}{8} \|Y(s)\|_1^2 + C. \quad (2.33) \]

Moreover, by (2.12),
\[ |b(Y(s), X(s), Y(s))| \leq C \|Y(s)\|_0 \|Y(s)\|_1 \|X(s)\|_1 \leq \frac{1}{8} \|Y(s)\|_1^2 + C \|Y(s)\|_0^2 \|X(s)\|_1^2 \quad (2.34) \]
\[ |b(x, x, Y(s))| \leq C \|x\|_0 \|x\|_1 \|Y(s)\|_1 \leq \frac{1}{8} \|Y(s)\|_1^2 + C, \quad (2.35) \]

and finally
\[ |\langle f(s, a(s)), Y(s) \rangle| \leq \|f(s, a(s))\|_{-1} \|Y(s)\|_1 \leq \frac{1}{8} \|Y(s)\|_1^2 + C \|f(s, a(s))\|_{-1}^2. \quad (2.36) \]

Using (2.33)–(2.36) in (2.32) produces
\[ \frac{1}{2} \frac{d}{ds} \|Y(s)\|_0^2 + \frac{1}{2} \|Y(s)\|_1^2 \leq C \left[ 1 + \|Y(s)\|_0^2 \|X(s)\|_1^2 + \|f(s, a(s))\|_{-1}^2 \right]. \]

Therefore it follows by Gronwall’s inequality that
\[ \|X(s) - x\|_0^2 \leq C \left[ (s - t) + \int_t^s \|f(r, a(r))\|_{-1}^2 dr \right] e^{C \int_t^s \|X(r)\|_1^2 dr} \]
from which our claim (2.19) easily follows. Moreover we note that, setting \( \omega_1(r - t) = \|X(s) - x\|_0 \), we also have
\[ \int_t^s \|X(r) - x\|_0^2 dr \leq C \left[ (s - t) + \int_t^s \|f(r, a(r))\|_{-1}^2 dr \right] + C \int_t^s \omega_1(r - t) \|X(r)\|_1^2 dr. \]

The proof of (2.20) is the same as in case (i) (estimate (2.18)) once we know that \( \limsup_{s \to t} \|X(s)\|_0 \leq \|x\|_0 \) which follows from the assumption that \( f \) is bounded in \( H \) and from (2.14).

\section{Viscosity solutions of the HJB equation}

The definition of a viscosity solution that we propose here has its predecessors in [10] and [7, 17, 18]. We just have to use appropriate test functions.

\textbf{Definition 3.1} A function \( \psi \) is a test function for equation (1.3) if \( \psi = \varphi + \delta(t) \|x\|_0^2 \), where
\begin{itemize}
  \item \( \varphi \in C^1((0,T) \times H) \), \( A^\frac{1}{2} \varphi \) is continuous, and \( \varphi \) is weakly sequentially lower-semicontinuous,
  \item \( \delta \in C^1((0,T)) \) is such that \( \delta > 0 \) on \( (0,T) \).
\end{itemize}

\textbf{Definition 3.2} A function \( \psi \) is a test function for equation (1.4) if \( \psi = \varphi + \delta \|x\|_0^2 \), where
\begin{itemize}
  \item \( \varphi \in C^1(H) \), \( A^\frac{1}{2} \varphi \) is continuous, and \( \varphi \) is weakly sequentially lower-semicontinuous,
  \item \( \delta > 0 \).
\end{itemize}
In the definition below we assume that $F : [0, T] \times \mathbf{V} \times \mathbf{V} \to \mathbb{R}$.

**Definition 3.3** A function $u \in C((0, T) \times \mathbf{H})$ is a viscosity subsolution (supersolution) of (1.3) if for every test function $\psi$, whenever $u - \psi$ has a local maximum (respectively $u + \psi$ has a local minimum) at $(t, x)$ then $x \in \mathbf{V}$ and

$$\psi_t(t, x) - \langle Ax + B(x, x), D\psi(t, x) \rangle + F(t, x, D\psi(t, x)) \geq 0$$

(respectively

$$-(\psi_t(t, x) - \langle Ax + B(x, x), D\psi(t, x) \rangle) + F(t, x, -D\psi(t, x)) \leq 0.)$$

A function is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

**Definition 3.4** A function $u \in C(\mathbf{H})$ is a viscosity subsolution (supersolution) of (1.4) if for every test function $\psi$, whenever $u - \psi$ has a local maximum (respectively $u + \psi$ has a local minimum) at $x$ then $x \in \mathbf{V}$ and

$$\lambda u(x) + \langle Ax + B(x, x), D\psi(x) \rangle + F(x, D\psi(x)) \leq 0$$

(respectively

$$\lambda u(x) - \langle Ax + B(x, x), D\psi(x) \rangle + F(x, -D\psi(x)) \geq 0.)$$

A function is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

The somehow arbitrary choice of the function $\| \cdot \|^2_0$ as part of the test function was made because of simplicity and since in this paper we only deal with linearly growing solutions. Other radial functions can be chosen to handle more general cases.

## 4 Comparison principles

In this section we prove comparison principles for viscosity solutions that in turn imply the uniqueness of solutions of (1.3) and (1.4) under certain conditions. We will assume the following hypothesis.

**Hypothesis 4.1** $F : [0, T] \times \mathbf{V}_\beta \times \mathbf{V} \to \mathbb{R}$ and there exist a modulus of continuity $\omega$, $0 < \beta < \frac{1}{2}$, and moduli $\omega_R$ such that for every $R > 0$ we have

$$|F(t, x, p) - F(t, y, p)| \leq \omega_R (||x - y||_\beta + \omega (||x - y||_\beta ||p||_0), \text{ if } ||x||_0, ||y||_0 \leq R, (4.37)$$

$$|F(t, x, p) - F(t, x, q)| \leq \omega ((1 + ||x||_0)||p - q||_1), \text{ (4.38)}$$

$$|F(t, x, p) - F(s, x, p)| \leq \omega_R (|t - s|), \text{ if } ||x||_0, ||y||_0, ||p||_0 \leq R. \text{ (4.39)}$$
Theorem 4.2 Let Hypothesis 4.1 hold. Let \( u, v : (0, T) \times \mathbb{H} \rightarrow \mathbb{R} \) be respectively a viscosity subsolution, and a viscosity supersolution of (1.3). Let \( u(t, x), -v(t, x), |g(x)| \leq C(1 + ||x||_0) \) and let \( u(t, \cdot), v(t, \cdot), g \) be locally Lipschitz continuous in \( || \cdot ||_{\beta - 1} \) norm on bounded subsets of \( \mathbb{H} \), uniformly for \( t \) in closed subsets of \((0, T)\). Let moreover

\[
\begin{align*}
(i) \quad & \lim_{t \uparrow T} (u(t, x) - g(x))^+ = 0 \\
(ii) \quad & \lim_{t \uparrow T} (v(t, x) - g(x))^+ = 0
\end{align*}
\]

uniformly on bounded subsets of \( \mathbb{H} \). Then \( u \leq v \).

Theorem 4.3 Let Hypothesis 4.1 hold. Let \( u, v : \mathbb{H} \rightarrow \mathbb{R} \) be respectively a viscosity subsolution, and a viscosity supersolution of (1.4). Let \( u, -v \) be bounded from above and be Lipschitz continuous in \( || \cdot ||_{\beta - 1} \) norm on bounded subsets of \( \mathbb{H} \). Then \( u \leq v \).

Proof of Theorem 4.3. Set \( \alpha = 1 - \beta \) and assume for simplicity that \( \lambda = 1 \). For \( \epsilon, \delta > 0 \) we consider the function

\[
\Phi(x, y, \epsilon, \delta) := u(x) - v(y) - \frac{||x - y||^2_{\alpha}}{2\epsilon} - \delta(||x||^2_0 + ||y||^2_0).
\]

Because of the continuity assumptions on \( u, v \) and the fact that \( A^{-1} \) is compact, the function above is weakly sequentially upper-semicontinuous and so it attains a global maximum at some points \( \bar{x}, \bar{y} \in V \) (by the definition of viscosity solution). For a fixed \( \delta \) we will show (as for instance in [22]) that the points \( \bar{x} \) and \( \bar{y} \) are bounded independently of \( \epsilon \),

\[
\lim_{\delta \to 0} \limsup_{\epsilon \to 0} \delta(||\bar{x}||^2_0 + ||\bar{y}||^2_0) = 0, \quad (4.41)
\]

and

\[
\lim_{\epsilon \to 0} \frac{||\bar{x} - \bar{y}||^2_{-\alpha}}{2\epsilon} = 0 \quad \text{for fixed } \delta. \quad (4.42)
\]

(Please notice that we have reversed the order in which we pass to the limits in the above expressions compared with [22]). To see this we set

\[
m_1(\epsilon, \delta) = \sup_{x, y \in \mathbb{H}} \Phi(x, y, \epsilon, \delta),
\]

\[
m_2(\delta) = \sup_{x, y \in \mathbb{H}} \left\{ u(x) - v(y) - \delta \left(||x||^2_0 + ||y||^2_0\right) \right\}
\]

and note that we have

\[
m = \lim_{\delta \downarrow 0} m_2(\delta) \quad \text{and} \quad m_2(\delta) = \lim_{\epsilon \downarrow 0} m_1(\epsilon, \delta).
\]

Now,

\[
m_1(\epsilon, \delta) = \Phi(\bar{x}, \bar{y}, \epsilon, \delta) = u(\bar{x}) - v(\bar{y}) - \frac{||\bar{x} - \bar{y}||^2_{-\alpha}}{2\epsilon} - \delta(||\bar{x}||^2_0 + ||\bar{y}||^2_0)
\]

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and for fixed $\delta$,

$$m_1(\epsilon, \delta) + \frac{||\bar{x} - \bar{y}||^2_{-\alpha}}{4\epsilon} = u(\bar{x}) - v(\bar{y}) - \frac{||\bar{x} - \bar{y}||^2_{-\alpha}}{4\epsilon} - \delta(||\bar{x}||^2_0 + ||\bar{y}||^2_0) \leq m_1(2\epsilon, \delta).$$

Thus

$$\frac{||\bar{x} - \bar{y}||^2_{-\alpha}}{4\epsilon} \leq m_1(2\epsilon, \delta) - m_1(\epsilon, \delta).$$

This gives (4.42). Similarly,

$$m_1(\epsilon, \delta) + \frac{\delta}{2} (||\bar{x}||^2_0 + ||\bar{y}||^2_0) = u(\bar{x}) - v(\bar{y}) - \frac{||\bar{x} - \bar{y}||^2_{-\alpha}}{2\epsilon} - \frac{\delta}{2} (||\bar{x}||^2_0 + ||\bar{y}||^2_0) \leq m_1(\epsilon, \frac{\delta}{2})$$

which gives

$$\frac{\delta}{2} (||\bar{x}||^2_0 + ||\bar{y}||^2_0) \leq m_1(\epsilon, \delta) - m_1(\epsilon, \frac{\delta}{2}),$$

from which we obtain (4.41). Using the definition of viscosity solution and $b(x, x, x) = 0$ we now have

$$u(\bar{x}) + 2\delta \langle A\bar{x}, \bar{x} \rangle + \frac{1}{\epsilon} \langle A\bar{x}, A^{-\alpha}(\bar{x} - \bar{y}) \rangle$$

$$+ \frac{1}{\epsilon} b(\bar{x}, \bar{x}, A^{-\alpha}(\bar{x} - \bar{y})) + F(\bar{x}, \frac{1}{\epsilon} A^{-\alpha}(\bar{x} - \bar{y}) + 2\delta \bar{x}) \leq 0$$

and

$$v(\bar{y}) - 2\delta \langle A\bar{y}, \bar{y} \rangle + \frac{1}{\epsilon} \langle A\bar{y}, A^{-\alpha}(\bar{x} - \bar{y}) \rangle$$

$$+ \frac{1}{\epsilon} b(\bar{y}, \bar{y}, A^{-\alpha}(\bar{x} - \bar{y})) + F(\bar{y}, \frac{1}{\epsilon} A^{-\alpha}(\bar{x} - \bar{y}) - 2\delta \bar{y}) \geq 0.$$
where we took $p$ and $q$ such that $1/p + 1/q = 1$, and later $p$ will be sufficiently small. To continue we will use inequality (2.8). We choose $\tau$ such that $0 < \tau < \alpha - \frac{1}{2}$. To estimate $\|A^{-\alpha + \frac{1}{2}}(\bar{x} - \bar{y})\|_{0, 2p}$ we first notice that if $p$ is sufficiently close to 1, Sobolev imbedding (2.6) guarantee that

\[ \|A^{-\alpha + \frac{1}{2}}(\bar{x} - \bar{y})\|_{0, 2p} \leq C\|A^{-\alpha + \frac{1}{2} + \frac{\tau}{2}}(\bar{x} - \bar{y})\|_{0}. \]

We now set $S = A^{\frac{1}{2}}$, and $z = A^{-\frac{\tau}{2}}(\bar{x} - \bar{y})$ in (2.8). Then

\[ \|A^{-\alpha + \frac{1}{2} + \frac{\tau}{2}}(\bar{x} - \bar{y})\|_{0} = \|S^{1-\alpha + \tau}z\|_{0} \leq C\|Sz\|_{0}^{1-\alpha + \tau}\|z\|_{0}^{-\tau} = C\|A^{\frac{1}{2}+\frac{\tau}{2}}(\bar{x} - \bar{y})\|_{0}^{1-\alpha + \tau}\|\bar{x} - \bar{y}\|_{0}^{-\alpha - \tau}. \]

To estimate $\|\bar{x} - \bar{y}\|_{0, 2p}$ we again use the Sobolev imbedding

\[ \|\bar{x} - \bar{y}\|_{0, 2p} \leq C\|\bar{x} - \bar{y}\|_{\alpha} \]

that holds if $p$ is small enough, and then set $S = A^{\frac{1}{2+\alpha}}$ and $z = \bar{x} - \bar{y}$ in (2.8). We then obtain

\[ \|\bar{x} - \bar{y}\|_{\alpha} = \|S^{1}z\|_{0} \leq C\|Sz\|_{0}^{1-\alpha + \tau}\|z\|_{0}^{-\tau} = C\|A^{\frac{1}{2}+\frac{\tau}{2}}(\bar{x} - \bar{y})\|_{0}^{1-\alpha + \tau}\|\bar{x} - \bar{y}\|_{0}^{-\alpha - \tau}. \] (4.45)

Therefore, plugging (4.45) and (4.45) into (4.44), and estimating further

\[ \|\bar{x}\|_{0, \alpha} \leq C\|\bar{x}\|_{1} \]

we get

\[ \frac{1}{\epsilon}b(\bar{x} - \bar{y}, \bar{x}, A^{-\alpha}(\bar{x} - \bar{y})) \leq C\|\bar{x}\|_{1}\|A^{\frac{1}{2}+\frac{\tau}{2}}(\bar{x} - \bar{y})\|_{0}^{1-\alpha + \tau}\|\bar{x} - \bar{y}\|_{0}^{-\alpha - \tau}\|\bar{x} - \bar{y}\|_{0}^{-\tau} \]

which, upon using $\|\bar{x} - \bar{y}\|_{0} \leq C\|A^{\frac{1}{2}+\frac{\tau}{2}}(\bar{x} - \bar{y})\|_{0}$, yields

\[ \frac{1}{\epsilon}b(\bar{x} - \bar{y}, \bar{x}, A^{-\alpha}(\bar{x} - \bar{y})) \leq C\|\bar{x}\|_{1}\frac{\|\bar{x} - \bar{y}\|_{1-\alpha}}{\sqrt{\epsilon}} \frac{\|\bar{x} - \bar{y}\|_{0}^{1-\alpha - \tau}}{\sqrt{\epsilon}} \|\bar{x} - \bar{y}\|_{0}^{1-\alpha + \tau}. \] (4.46)

We now notice that

\[ \Phi(\bar{x}, \bar{y}, \epsilon, \delta) \geq \{\Phi(\bar{x}, \bar{x}, \epsilon, \delta), \Phi(\bar{y}, \bar{y}, \epsilon, \delta)\}. \] (4.47)

We use the fact that $u$ and $v$ are locally Lipschitz continuous in $\|\cdot\|_{-\alpha}$ norm and $\|\bar{x}\|_{0}, \|\bar{y}\|_{0} \leq R_{\delta}$ independently of $\epsilon$ for a fixed $\delta$ to deduce from (4.47) that

\[ \frac{\|\bar{x} - \bar{y}\|_{-\alpha}^{2}}{\epsilon} \leq K_{\delta}\|\bar{x} - \bar{y}\|_{-\alpha} \]

for some $K_{\delta} > 0$. This implies that

\[ \frac{\|\bar{x} - \bar{y}\|_{-\alpha}^{2}}{\sqrt{\epsilon}} \leq \sqrt{K_{\delta}}. \]
Therefore
\[
\frac{||x - y||_{\alpha-\tau}}{\sqrt{\epsilon}} = \frac{||x - y||_{1-\alpha}^{2}}{\sqrt{\epsilon}} ||x - y||_{\alpha-\tau} \leq \sqrt{K_{\delta} ||x - y||_{\alpha-\tau}} \to 0
\]
as \(\epsilon \to 0\) by (4.42) since \(\alpha - \frac{1}{2} + \tau > 0\). Using this in (4.46) we thus obtain
\[
\frac{1}{\epsilon} |b(x - y, x, A^{-\alpha}(x - y))| \leq ||x||_{1} \frac{||x - y||_{1-\alpha}}{\sqrt{\epsilon}} \sigma_{1}(\epsilon, \delta) \\
\leq \delta ||x||_{1}^{2} + \frac{||x - y||_{1-\alpha}^{2}}{2\epsilon} \sigma_{2}(\epsilon, \delta) \tag{4.48}
\]
for some local moduli \(\sigma_{1}\) and \(\sigma_{2}\). Similarly we obtain
\[
\frac{1}{\epsilon} |b(y, x, y, A^{-\alpha}(x - y))| \leq \delta ||y||_{1}^{2} + \frac{||x - y||_{1-\alpha}^{2}}{2\epsilon} \sigma_{2}(\epsilon, \delta) . \tag{4.49}
\]
We can now proceed with estimating (4.43). Using (4.48), (4.49), and Hypothesis 4.1 in (4.43) (notice that \(\beta = 1 - \alpha\)) it follows that
\[
u(x) - v(y) + \frac{1}{\epsilon} ||\bar{x} - \bar{y}||_{2-\alpha}^{2} - \frac{1}{\epsilon} ||\bar{x} - \bar{y}||_{1-\alpha}^{2} \sigma_{2}(\epsilon, \delta) + \delta(||\bar{x}||_{1}^{2} + ||\bar{y}||_{1}^{2})
\]
\[\leq \omega(\delta(1 + ||\bar{x}||_{0})||\bar{x}||_{1}) + \omega(\delta(1 + ||\bar{y}||_{0})||\bar{y}||_{1}). \tag{4.50}\]
For fixed \(\mu, \delta > 0\) we now choose \(C_{\mu}\) and \(C_{\mu, \delta}\) so that \(\omega(s) \leq \mu + C_{\mu}s\) and \(\omega_{R_{\delta}}(s) \leq \mu + C_{\mu, \delta}s\). Then
\[
\omega(\delta(1 + ||\bar{x}||_{0})||\bar{x}||_{1}) + \omega(\delta(1 + ||\bar{y}||_{0})||\bar{y}||_{1})
\]
\[\leq 2\mu + C_{\mu, \delta}(1 + ||\bar{x}||_{0})||\bar{x}||_{1} + (1 + ||\bar{y}||_{0})||\bar{y}||_{1}) \leq 2\mu + \delta(||\bar{x}||_{1}^{2} + ||\bar{y}||_{1}^{2}) + D_{\mu, \delta} (1 + ||\bar{x}||_{0}^{2} + ||\bar{y}||_{0}^{2}) . \tag{4.51}\]
Moreover
\[
\omega_{R_{\delta}} (||\bar{x} - \bar{y}||_{1-\alpha}) + \omega \left( ||\bar{x} - \bar{y}||_{1-\alpha} \frac{||\bar{x} - \bar{y}||_{\alpha-\tau}}{\epsilon} \right)
\]
\[\leq 2\mu + C_{\mu} ||\bar{x} - \bar{y}||_{1-\alpha} \frac{||\bar{x} - \bar{y}||_{\alpha-\tau}}{\epsilon} + C_{\mu, \delta} ||\bar{x} - \bar{y}||_{1-\alpha} \leq 2\mu + \frac{||\bar{x} - \bar{y}||_{2-\alpha}^{2}}{2\epsilon} + C_{\mu, \delta} ||\bar{x} - \bar{y}||_{1-\alpha} + 2C_{\mu} \frac{||\bar{x} - \bar{y}||_{2-\alpha}^{2}}{\epsilon} . \tag{4.52}\]
Putting (4.51) and (4.52) into (4.50), and using (4.41) and (4.42) we therefore obtain
\[
u(x) - v(y) + \left( \frac{1}{2} - \sigma_{2}(\epsilon, \delta) \right) \frac{||\bar{x} - \bar{y}||_{2-\alpha}^{2}}{\epsilon} - C_{\mu, \delta} ||\bar{x} - \bar{y}||_{1-\alpha} \leq 4\mu + \sigma_{3}(\epsilon, \delta; \mu)
\]
for some function $\sigma_3$ such that for every fixed $\mu$

$$\limsup_{\delta \to 0} \limsup_{\epsilon \to 0} \sigma_3(\epsilon, \delta; \mu) = 0. \quad (4.53)$$

We now observe that

$$\liminf_{\epsilon \to 0 \atop r > 0} \left( \frac{r^2}{4\epsilon} - C_{\mu, \delta} \right) = 0. \quad (4.54)$$

This yields

$$u(\bar{x}) - v(\bar{y}) \leq 4\mu + \sigma_4(\epsilon, \delta; \mu) \quad (4.55)$$

for some function $\sigma_4$ satisfying (4.53). Finally (4.55) yields that

$$u(t, x) - v(t, x) - 2\epsilon \leq \epsilon \leq u(\bar{x}) - v(\bar{y}) \leq 4\mu + \sigma_4(\epsilon, \delta; \mu).$$

If we now let $\epsilon \to 0$, $\delta \to 0$ and then $\mu \to 0$ we obtain that $u \leq v$. \quad \Box

**Proof of Theorem 4.2.** Given $\mu > 0$, define

$$u_\mu(t, x) = u(t, x) - \frac{\mu}{t}, \quad v_\mu(t, x) = v(t, x) + \frac{\mu}{t}.$$ 

Then $u_\mu$ and $v_\mu$ satisfy respectively

$$(u_\mu)_t - \langle Ax + B(x, x), Du_\mu \rangle + F(x, Du_\mu) \geq \frac{\mu}{T^2}$$

and

$$(v_\mu)_t - \langle Ax + B(x, x), Dv_\mu \rangle + F(x, Dv_\mu) \leq -\frac{\mu}{T^2}.$$ 

Let $C_\mu$ be such that $\omega(s) \leq \frac{\mu}{2T^2} + C_\mu s$, and let $K_\mu = 2C_\mu^2$. For $\epsilon, \delta, \gamma > 0$, and $0 < T_\delta < T$, we consider the function

$$u_\mu(t, x) - v_\mu(s, y) - \frac{||x - y||_0^2}{2\epsilon} - \delta e^{K_\mu(T-t)}||x||_0^2 - \delta e^{K_\mu(T-s)}||y||_0^2 - \frac{(t-s)^2}{2\gamma}$$

on $(0, T_\delta] \times H$. This function has a global maximum at $\tilde{t}, \tilde{s}, \bar{x}, \bar{y}$, where $0 < \tilde{t}, \tilde{s}$, and $\bar{x}, \bar{y}$ are bounded independently of $\epsilon$ for a fixed $\delta$. Moreover $\bar{x}, \bar{y} \in V$. Similarly to the stationary case we have

$$\limsup_{\delta \to 0} \limsup_{\epsilon \to 0} \limsup_{\gamma \to 0} \delta(\|\bar{x}\|_0^2 + \|\bar{y}\|_0^2) = 0, \quad (4.56)$$

$$\limsup_{\epsilon \to 0} \limsup_{\gamma \to 0} \frac{||\bar{x} - \bar{y}||_0^2}{2\epsilon} = 0 \quad \text{for fixed } \delta, \quad (4.57)$$

and

$$\limsup_{\gamma \to 0} \frac{(\tilde{t} - \tilde{s})^2}{2\gamma} = 0 \quad \text{for fixed } \delta, \epsilon. \quad (4.58)$$
If $u \not= v$ it then follows from (4.57), (4.58) and (4.40) that for small $\mu$ and $\delta$, and for $T_\delta$ sufficiently close to $T$ we have $\tilde{t}, \tilde{s} < T_\delta$ if $\gamma$ and $\epsilon$ are sufficiently small. Therefore using the definition of viscosity solution we obtain

$$\frac{\tilde{t} - \tilde{s}}{\gamma} - \delta \mu \epsilon K_\mu(T - \tilde{t}) \|\bar{x}\|^2 - 2\delta \mu e K_\mu(T - \tilde{t}) \langle A \tilde{x}, \bar{x} \rangle - \frac{1}{\epsilon} \langle A \tilde{x}, A^{-\alpha}(\bar{x} - \bar{y}) \rangle$$

$$- \frac{1}{\epsilon} b(x, x, A^{-\alpha}(\bar{x} - \bar{y})) + F(\tilde{t}, \tilde{x}, \frac{1}{\epsilon} A^{-\alpha}(\bar{x} - \bar{y}) + 2\delta \mu e K_\mu(T - \tilde{t}) \tilde{x} \geq \frac{\mu}{T^2} \quad (4.59)$$

and

$$\frac{\tilde{t} - \tilde{s}}{\gamma} + \delta \mu \epsilon K_\mu(T - \tilde{s}) \|\bar{y}\|^2 + 2\delta \mu e K_\mu(T - \tilde{s}) \langle A \bar{y}, \bar{y} \rangle - \frac{1}{\epsilon} \langle A \bar{y}, A^{-\alpha}(\bar{x} - \bar{y}) \rangle$$

$$- \frac{1}{\epsilon} b(\bar{y}, \bar{y}, A^{-\alpha}(\bar{x} - \bar{y})) + F(\tilde{s}, \tilde{y}, \frac{1}{\epsilon} A^{-\alpha}(\bar{x} - \bar{y}) - 2\delta \mu e K_\mu(T - \tilde{s}) \bar{y} \leq -\frac{\mu}{T^2}. \quad (4.60)$$

Combining these two inequalities it follows that

$$K_\mu \delta \left( e^{K_\mu(T - \tilde{t})} \|\bar{x}\|^2_0 + e^{K_\mu(T - \tilde{s})} \|\bar{y}\|^2_0 \right) + 2\delta \left( e^{K_\mu(T - \tilde{t})} \|\bar{x}\|^2_0 + e^{K_\mu(T - \tilde{s})} \|\bar{y}\|^2_0 \right)$$

$$+ \frac{1}{\epsilon} \|\bar{x} - \bar{y}\|^2_{1-\alpha} + \frac{1}{\epsilon} b(x, x, A^{-\alpha}(\bar{x} - \bar{y})) - \frac{1}{\epsilon} b(\bar{y}, \bar{y}, A^{-\alpha}(\bar{x} - \bar{y})) + F(\tilde{s}, \bar{y}, \frac{1}{\epsilon} A^{-\alpha}(\bar{x} - \bar{y}) - 2\delta \mu e K_\mu(T - \tilde{s}) \bar{y} - F(\tilde{t}, \bar{x}, \frac{1}{\epsilon} A^{-\alpha}(\bar{x} - \bar{y}) + 2\delta \mu e K_\mu(T - \tilde{t}) \bar{x} \leq -2\frac{\mu}{T^2}. \quad (4.61)$$

We now estimate

$$|F(\tilde{t}, \bar{x}, \frac{1}{\epsilon} A^{-\alpha}(\bar{x} - \bar{y}) + 2\delta \mu e K_\mu(T - \tilde{t}) \bar{x} - F(\tilde{t}, \bar{x}, \frac{1}{\epsilon} A^{-\alpha}(\bar{x} - \bar{y}))|$$

$$+ |F(\tilde{s}, \bar{y}, \frac{1}{\epsilon} A^{-\alpha}(\bar{x} - \bar{y}) - 2\delta \mu e K_\mu(T - \tilde{s}) \bar{y} - F(\tilde{s}, \bar{y}, \frac{1}{\epsilon} A^{-\alpha}(\bar{x} - \bar{y}))|$$

$$\leq \omega((1 + \|\bar{x}\|_0)2\delta \mu e K_\mu(T - \tilde{t}) \|\bar{x}\|_1) + \omega((1 + \|\bar{y}\|_0)2\delta \mu e K_\mu(T - \tilde{s}) \|\bar{y}\|_1) \leq \frac{\mu}{T^2} + C_\mu((1 + \|\bar{x}\|_0)2\delta \mu e K_\mu(T - \tilde{t}) \|\bar{x}\|_1) + C_\mu((1 + \|\bar{y}\|_0)2\delta \mu e K_\mu(T - \tilde{s}) \|\bar{y}\|_1)$$

$$\leq \frac{\mu}{T^2} + 2\delta C_\mu^2 (2 + e^{K_\mu(T - \tilde{t})} \|\bar{x}\|^2_0 + e^{K_\mu(T - \tilde{s})} \|\bar{y}\|^2_0) + \delta \left( e^{K_\mu(T - \tilde{t})} \|\bar{x}\|^2_0 + e^{K_\mu(T - \tilde{s})} \|\bar{y}\|^2_0 \right).$$

Using this and the fact that $K_\mu = 2C_\mu^2$ in (4.61) we therefore obtain

$$\delta(\|\bar{x}\|^2_1 + \|\bar{y}\|^2_1) + \frac{1}{\epsilon} \|\bar{x} - \bar{y}\|^2_{1-\alpha}$$

$$+ \frac{1}{\epsilon} b(x, x, A^{-\alpha}(\bar{x} - \bar{y})) - \frac{1}{\epsilon} b(\bar{y}, \bar{y}, A^{-\alpha}(\bar{x} - \bar{y}))$$

$$+ F(\tilde{s}, \bar{y}, \frac{1}{\epsilon} A^{-\alpha}(\bar{x} - \bar{y}) - F(\tilde{t}, \bar{x}, \frac{1}{\epsilon} A^{-\alpha}(\bar{x} - \bar{y})) \leq -\frac{\mu}{T^2} + \sigma_1(\delta, \mu) \quad (4.62)$$
for some local modulus $\sigma_1$. The rest of the proof follows that of the stationary case. We first let $\gamma \to 0$ and use (4.39) to eliminate the dependence on $\bar{t}$ and $\bar{s}$ above, and then estimate all the terms as in the proof of Theorem 4.3. After doing this we obtain
\[
\left( \frac{1}{2} - \sigma_2(\epsilon, \delta) \right) \frac{||\bar{x} - \bar{y}||_{\alpha}^2}{\epsilon} - D_{\mu,\delta}||\bar{x} - \bar{y}||_{1-\alpha} \leq -\frac{\mu}{2T^2} + \sigma_3(\gamma, \epsilon, \delta; \mu)
\]
for some constant $D_{\mu,\delta}$, a local modulus $\sigma_2$, and a function $\sigma_3$ such that for every fixed $\mu$
\[
\lim \sup_{\delta \to 0} \lim \sup_{\epsilon \to 0} \lim \sup_{\gamma \to 0} \sigma_3(\gamma, \epsilon, \delta; \mu) = 0.
\]
This inequality, upon employing (4.54), yields a contradiction if we let $\gamma \to 0$, $\epsilon \to 0$, and then $\delta \to 0$.

5 Optimal control problem for the Navier-Stokes equation

In this section we study an optimal control problem for the Navier-Stokes equation. We only consider the finite horizon problem. Similar results can be obtained for the infinite horizon problem by the same methods. We recall that in the finite horizon case for a fixed $T > 0$ and $0 \leq t < T$ we try to minimize the cost functional
\[
J(t, x; a(\cdot)) = \int_t^T l(s, X(s; t, x, a), a(s)) ds + g(X(T; t, x, a)),
\]
where $X(\cdot; t, x, a)$ is the solution of (1.1). We assume the following.

Hypothesis 5.1 There exist a constant $0 < \beta < \frac{1}{2}$, an absolute constant $C > 0$, and families of moduli $\sigma_R$ and constants $L_R$ for $R > 0$ such that

(i) $l$ is continuous on $[0, T] \times V_\beta \times \Theta$ and for $t, s \in [0, T], x, y \in V_\beta$, and $a \in \Theta$ we have
\[
|l(t, x, a) - l(s, y, a)| \leq \sigma_R(|t - s|) + L_R||x - y||_\beta \text{ if } ||x||_0, ||y||_0 \leq R;
\]
\[
|l(s, x, a)| \leq C(1 + ||x||_1);
\]

(ii) $g : H \to \mathbb{R}$ is such that
\[
|g(x) - g(y)| \leq L_R||x - y||_{\beta, 1} \text{ if } ||x||_0, ||y||_0 \leq R,
\]
\[
|g(x)| \leq C(1 + ||x||_0);
\]

(iii) $f : [0, T] \times \Theta \to H$ is bounded, continuous, and $f(\cdot, a)$ is uniformly continuous, uniformly for $a \in \Theta$. 

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Let us briefly discuss the applicability of our control problem. Note that if we take \( \Theta \) as a separable Banach space then the following example fits Hypothesis 5.1:

\[
f(t, a) = \kappa(||a||_{\Theta})Na,
\]

where \( \kappa(\cdot) \in C^{\infty,0}(\mathbb{R}_+) \) here acts like an engineering constraint on the control actuation and the control operator \( N \in L(\Theta; H) \) is a linear continuous operator representing spatial localization of the control. Here \( C^{\infty,0}(\mathbb{R}_+) \) denotes smooth functions which identically vanish outside a finite interval \([0, R]\). For conducting fluids (even slightly conducting fluids such as salt water) such controllers can be realized through the action of Lorentz force \([27, \text{Ch. 10}], [8, \text{Ch. IV}]\). Since the Lorentz force is given by \( j \times B \) where \( j \) is the current and \( B \) is the magnetic field, spatial localization of such distributed control force can be realized by choosing a suitable shape and location for the magnets. This is implemented for example in a boundary layer stabilization experiment in \([21]\). A cost functional of the form

\[
J(t, x; a(\cdot)) = \int_t^T ||A^{\beta/2}(X(s) - X_d(s))||_0 ds + \int_t^T \lambda(||a(s)||_{\Theta})||a(s)||_{\Theta}^2 ds + ||A^{\beta/2}X(T)||_0
\]

with \( \lambda(\cdot) \in C^{\infty,0}(\mathbb{R}_+) \) being a discount factor and \( X_d(t) \) either a desired field or a known smooth field satisfies Hypothesis 5.1. We note that if the control is bounded (control set is bounded) then the discount factors \( \kappa(\cdot) \) and \( \lambda(\cdot) \) are not needed. Although it is weaker than the enstrophy norm \( ||A^{1/2}X(s)||_0 \), the above integrand \( ||A^{\beta/2}(X(s) - X_d(s))||_0 \) still amounts to minimizing spatial turbulence structures contained in the deviation field \( X(t) - X_d(t) \) and hence is useful in practice. In data assimilation problems in meteorology \([11]\) and oceanography \([40]\) a major step is to estimate the unknown distributed forces due to environmental effects which cannot be completely incorporated into the model due to the complexity of the actual problem. In these applications \( X_d \) would represent the extrapolated measurements and the unknown time dependent forces are regarded as controls with possible spatial localization and hence fit within the above abstract formulation. The integrand we chose above is in fact stronger than the \( L^2 \)-norm integrands used in the literature on data assimilation of geophysics.

We will prove that the value function for the problem,

\[
V(t, x) = \inf_{a(\cdot) \in U} J(t, x; a(\cdot)). \tag{5.68}
\]

is the unique viscosity solution of the HJB equation

\[
\begin{cases}
  u_t - \langle Ax + B(x, x), Du \rangle + \inf_{a \in \Theta} \{ (f(t, a), Du) + l(t, x, a) \} = 0 \quad \text{in } (0, T) \times H, \\
  u(T, x) = g(x) \quad \text{in } H.
\end{cases} \tag{5.69}
\]

The lemma below is easy and we omit the proof.

**Lemma 5.2** Under the assumptions of Hypothesis 5.1 the Hamiltonian \( F \) given by

\[
F(t, x, p) = \inf_{a \in \Theta} \{ (f(t, a), p) + l(t, x, a) \}
\]

is continuous on \([0, T] \times V_\beta \times V\) and it satisfies (4.37), (4.38), (4.39).
5.1 Properties of the value function

The continuity properties of the value function are described by the following result.

**Proposition 5.3** Let Hypothesis 5.1 hold. Let \( \alpha = 1 - \beta \). Then for every \( R > 0 \) there exists a constant \( C_R \) such that the value function \( V \) defined by (5.68) satisfies

\[
|V(t_1, x) - V(t_2, y)| \leq C_R \left( |t_1 - t_2|^\alpha + \|x - y\|_\alpha \right)
\]  

(5.70)

for \( t_1, t_2 \in [0, T] \) and \( \|x\|_0, \|y\|_0 \leq R \). Moreover

\[
|V(t, x)| \leq C(1 + \|x\|_0).
\]

(5.71)

**Proof.** We will first prove the local Lipschitz continuity of \( V(t, \cdot) \). Let

\[
M = M_R = \left( R^2 + TK^2 \right)^{\frac{1}{2}},
\]

where \( K \) is such that \( \sup_{t \in [0,T], \mathbf{a} \in \Theta} \|f(t, \mathbf{a})\|_1 \leq K \). It then follows from (2.13), (2.15) and (2.16) that for \( \|x\|_0, \|y\|_0 \leq R \) we have

\[
|V(t, x) - V(t, y)| \leq \sup_{\mathbf{a}(\cdot) \in \mathcal{U}} |J(t, x; \mathbf{a}(\cdot)) - J(t, y; \mathbf{a}(\cdot))|
\]

\[
\leq \sup_{\mathbf{a}(\cdot) \in \mathcal{U}} \left( \int_t^T |l(s, X(s), \mathbf{a}(s)) - l(s, Y(s), \mathbf{a}(s))| ds + |g(X(T)) - g(Y(T))| \right)
\]

\[
\leq L_M \sup_{\mathbf{a}(\cdot) \in \mathcal{U}} \left( \int_t^T \|X(s) - Y(s)\|_\beta ds + \|X(T) - Y(T)\|_\alpha \right)
\]

\[
\leq C_R \|x - y\|_\alpha.
\]

Estimate (5.71) is obvious from (2.13), (5.65), and (5.67). It remains to prove the local Hölder continuity of \( V \) in \( t \). Let \( 0 \leq t_1 \leq t_2 \leq T \). Then

\[
|V(t_1, x) - V(t_2, x)| \leq \sup_{\mathbf{a}(\cdot) \in \mathcal{U}} |J(t_1, x; \mathbf{a}(\cdot)) - J(t_2, y; \mathbf{a}(\cdot))|
\]

\[
\leq \sup_{\mathbf{a}(\cdot) \in \mathcal{U}} \int_{t_1}^{t_2} |l(s, X(s; t_1, x, \mathbf{a}(\cdot)), \mathbf{a}(s))| ds
\]

\[
+ \sup_{\mathbf{a}(\cdot) \in \mathcal{U}} \int_{t_2}^{T} |l(s, X(s; t_1, x, \mathbf{a}(\cdot)), \mathbf{a}(s)) - l(s, X(s; t_2, x, \mathbf{a}(\cdot)), \mathbf{a}(s))| ds
\]

\[
+ \sup_{\mathbf{a}(\cdot) \in \mathcal{U}} |g(X(T; t_1, x, \mathbf{a}(\cdot))) - g(X(T; t_2, x, \mathbf{a}(\cdot)))|
\]

so that, writing \( X_1(s) \) for \( X(s; t_1, x, \mathbf{a}(\cdot)) \) and \( X_2(s) \) for \( X(s; t_2, x, \mathbf{a}(\cdot)) \), and using Hypothesis 5.1 and (2.13),

\[
|V(t_1, x) - V(t_2, x)| \leq \sup_{\mathbf{a}(\cdot) \in \mathcal{U}} \int_{t_1}^{t_2} |l(s, X_1(s), \mathbf{a}(s))| ds
\]
\[ + C(||x||_0) \sup_{a(\cdot) \in U} \left( \int_{t_2}^{T} ||X_1(s) - X_2(s)||_\beta ds + ||X_1(T) - X_2(T)||_{-\alpha} \right). \]

We now observe that for every \( a(\cdot) \in U \)
\[
\int_{t_1}^{t_2} |l(s, X_1(s), a(s))| ds \leq C \int_{t_1}^{t_2} (1 + ||X_1(s)||_1) ds
\]
\[
\leq C (t_2 - t_1)^{1/2} \left( \int_{t_1}^{t_2} (1 + ||X_1(s)||^2_1) ds \right)^{1/2} \leq C(||x||_0) (t_2 - t_1)^{1/2},
\]
and since
\[ X_1(s) = X(s; t_1, x, a(\cdot)) = X(s; t_2, X(t_2; t_1, x, a(\cdot)), a(\cdot)), \]
we have by (2.15), (2.16) and (2.17)
\[
\sup_{a(\cdot) \in U} \left( \int_{t_2}^{T} ||X_1(s) - X_2(s)||^2_\beta ds + ||X_1(T) - X_2(T)||^2_{-\alpha} \right) \leq C(||x||_0)(t_2 - t_1)^{1/2}.
\]
Combining the last two inequalities we therefore obtain
\[
|\mathcal{V}(t_1, x) - \mathcal{V}(t_2, x)| \leq C(||x||_0)(t_2 - t_1)^{1/2}.
\]

\[ \blacksquare \]

**Remark 5.4** It is clear from the proof that to obtain Proposition 5.3 we do not have to assume that \( f \) has values in \( H \). It is enough to assume that the values of \( f \) are bounded in \( V_{-1} \).

\[ \blacksquare \]

### 5.2 Existence of solutions of HJB equations

Let us begin with Bellman principle of optimality for Navier-Stokes equation [33, 32].

**Proposition 5.5** For every \( 0 \leq t \leq \tau \leq T \) and \( x \in H \) we have
\[
\mathcal{V}(t, x) = \inf_{a(\cdot) \in U} \left\{ \int_t^\tau l(s, X(s; t, x, a(\cdot)), a(s)) ds + \mathcal{V}(\tau, X(\tau; t, x, a(\cdot))) \right\}.
\]

**Theorem 5.6** Assume that Hypothesis 5.1 is true. Then the value function \( \mathcal{V} \) is the unique viscosity solution of the HJB equation (5.69) that satisfies (5.70) and (5.71).

**Proof.** The uniqueness part follows from Theorem 4.2 and Lemma 5.2. Here we will only prove that \( \mathcal{V} \) is a viscosity solution and in fact only that the value function is a viscosity supersolution. The subsolution part is very similar and in fact easier. We set \( B(x) = B_x(x, x) \) throughout this proof. Let \( \psi(t, x) = \varphi(t, x) + \delta(t)||x||_0^2 \) be a test function. Let \( \mathcal{V} + (\varphi + \delta ||\cdot||_0^2) \) have a local minimum at \((t_0, x_0) \in (0, T) \times H\).
Step 1. We prove that \( x_0 \in \mathcal{V} \). For every \((t, x) \in (0, T) \times H\)

\[
\mathcal{V}(t, x) - \mathcal{V}(t_0, x_0) \geq -\varphi(t, x) + \varphi(t_0, x_0) - \delta(t) \|x\|_0^2 + \delta(t_0) \|x_0\|_0^2. \tag{5.72}
\]

By the dynamic programming principle for every \( \varepsilon > 0 \) there exists \( a_\varepsilon(\cdot) \in \mathcal{U} \) such that, writing \( X_\varepsilon(s) \) for \( X(s; t_0, x_0, a_\varepsilon(\cdot)) \), we have

\[
\mathcal{V}(t_0, x_0) + \varepsilon^2 > \int_{t_0}^{t_0+\varepsilon} l(s, X_\varepsilon(s), a_\varepsilon(s)) \, ds + \mathcal{V}(t_0 + \varepsilon, X_\varepsilon(t_0 + \varepsilon)).
\]

Then, by (5.72),

\[
\varepsilon^2 - \int_{t_0}^{t_0+\varepsilon} l(s, X_\varepsilon(s), a_\varepsilon(s)) \, ds \geq \mathcal{V}(t_0 + \varepsilon, X_\varepsilon(t_0 + \varepsilon)) - \mathcal{V}(t_0, x_0)
\]

\[
\geq -\varphi(t_0 + \varepsilon, X_\varepsilon(t_0 + \varepsilon)) + \varphi(t_0, x_0) - \delta(t_0 + \varepsilon) \|X_\varepsilon(t_0 + \varepsilon)\|_0^2 + \delta(t_0) \|x_0\|_0^2.
\]

Using the chain rule and the orthogonality relation (2.9) in the above inequality, and then dividing both sides by \( \varepsilon \) we obtain

\[
\varepsilon - \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} l(s, X_\varepsilon(s), a_\varepsilon(s)) \, ds
\]

\[
\geq - \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \varphi_t(s, X_\varepsilon(s)) \, ds + \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \langle A X_\varepsilon(s), D \varphi(s, X_\varepsilon(s)) \rangle \, ds
\]

\[
+ \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \langle B(X_\varepsilon(s)), D \varphi(s, X_\varepsilon(s)) \rangle \, ds
\]

\[
- \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \langle f(s, a_\varepsilon(s)), D \varphi(s, X_\varepsilon(s)) \rangle \, ds + \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \delta(s) \langle A X_\varepsilon(s), X_\varepsilon(s) \rangle \, ds
\]

\[
- \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \delta(s) \langle f(s, a_\varepsilon(s)), X_\varepsilon(s) \rangle \, ds - \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \delta'(s) \|X_\varepsilon(s)\|_0^2 \, ds. \tag{5.73}
\]

Now, setting \( \lambda = \inf_{t \in [t_0, t_0+\varepsilon]} \delta(t) \) for some fixed \( \varepsilon_0 > 0 \), we have for \( \varepsilon < \varepsilon_0 \)

\[
\frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \delta(s) \langle A X_\varepsilon(s), X_\varepsilon(s) \rangle \, ds \geq \frac{\lambda}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \|X_\varepsilon(s)\|_0^1 \, ds.
\]

Therefore it follows that

\[
\frac{\lambda}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \|X_\varepsilon(s)\|_0^1 \, ds \leq \varepsilon - \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} l(s, X_\varepsilon(s), a_\varepsilon(s)) \, ds \tag{5.74}
\]

\[
+ \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \varphi_t(s, X_\varepsilon(s)) \, ds - \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \langle A X_\varepsilon(s), D \varphi(s, X_\varepsilon(s)) \rangle \, ds
\]

\[
- \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \langle B(X_\varepsilon(s)), D \varphi(s, X_\varepsilon(s)) \rangle \, ds
\]

\[
+ \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \langle f(s, a_\varepsilon(s)), D \varphi(s, X_\varepsilon(s)) \rangle \, ds
\]

\[
+ \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \delta'(s) \|X_\varepsilon(s)\|_0^2 \, ds + \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \delta(s) \langle f(s, a_\varepsilon(s)), X_\varepsilon(s) \rangle \, ds.
\]
The assumptions on \( l \) yield
\[
\left| \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} l(s, X_\varepsilon(s), a_\varepsilon(s)) \, ds \right| \leq C \left[ 1 + \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \| X_\varepsilon(s) \|_1 \, ds \right]
\leq C + \frac{\lambda}{8} \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \| X_\varepsilon(s) \|_1^2.
\] (5.75)

Moreover the assumptions placed on \( \varphi \) and (2.18) yield that
\[
\| A^{1/2} D\varphi(s, X_\varepsilon(s)) \|_0 \leq C(\| x_0 \|_0) \] (5.76)
for \( \varepsilon < \varepsilon_0 \) for some fixed \( \varepsilon_0 > 0 \). Therefore
\[
\langle AX_\varepsilon(s), D\varphi(s, X_\varepsilon(s)) \rangle = \| A^{1/2} X_\varepsilon(s) \|_0 \| A^{1/2} D\varphi(s, X_\varepsilon(s)) \|_0 \leq C(\| x_0 \|_0) \| X_\varepsilon(s) \|_1 \leq C + \frac{\lambda}{8} \| X_\varepsilon(s) \|_2^2.
\]

Regarding the third term we have, using (2.12), (2.13) and (5.76),
\[
\| \langle B X_\varepsilon(s), D\varphi(s, X_\varepsilon(s)) \rangle \| = \| b(X_\varepsilon(s), X_\varepsilon(s), D\varphi(s, X_\varepsilon(s))) \| \leq C \| X_\varepsilon(s) \|_0 \| X_\varepsilon(s) \|_1 \| D\varphi(s, X_\varepsilon(s)) \|_1 \leq C(\| x_0 \|_0) \| X_\varepsilon(s) \|_1 \leq C + \frac{\lambda}{8} \| X_\varepsilon(s) \|_2^2.
\]

Finally
\[
\| \langle f(s, a_\varepsilon(s)), X_\varepsilon(s) \rangle \|, \| \langle f(s, a_\varepsilon(s)), D\varphi(s, X_\varepsilon(s)) \rangle \| \leq C(\| x_0 \|_0).
\] (5.77)

Therefore, using the inequalities (5.75)–(5.77) above and the fact that \( \varphi_\varepsilon(s, X_\varepsilon(s)) \) and \( \delta'(s) \) are bounded independently of \( \varepsilon \) on \( [t_0, t_0+\varepsilon_0] \) for some \( \varepsilon_0 > 0 \), we obtain from (5.74)
\[
\frac{\lambda}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \| X_\varepsilon(s) \|_1^2 \, ds \leq C
\] (5.78)
for some constant \( C \) independent of \( \varepsilon \). We now take \( \varepsilon = 1/n \) and set \( X_n(s) = X(s; t_0, x_0, a_{1/n}(\cdot)) \). Inequality (5.78) gives then
\[
n \int_{t_0}^{t_0+1/n} \| X_n(s) \|_1^2 \, ds \leq C
\]
so that, along a sequence \( t_n \in (t_0, t_0 + 1/n) \),
\[
\| X_n(t_n) \|_1^2 \leq C,
\]
and thus along a subsequence, still denoted by \( t_n \), we have
\[
X_n(t_n) \rightharpoonup x
\]
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weakly in $V$ for some $\bar{x} \in V$. This also clearly implies weak convergence in $H$. However, by (2.18), $X_n(t_n) \to x_0$ strongly (and weakly) in $H$. Therefore, by the uniqueness of the weak limit in $H$ it follows that $x_0 = \bar{x} \in V$.

**Step 2.** The supersolution inequality. We go back to inequality (5.74). By (2.13) (or (2.14)) we have $\|X_\varepsilon(s)\| \leq C(\|x_0\|) = R$ for every $\varepsilon$ and $s \in [t_0, T]$. Therefore using (5.64) and (2.20) we obtain

$$\left| \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} [l(s, X_\varepsilon(s), a_\varepsilon(s))] ds - l(t_0, x_0, a_\varepsilon(s))] ds \right|$$

$$\leq \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \left( \sigma_R(|s-t_0|) + L_R \|X_\varepsilon(s) - x_0\| \right) ds \leq \omega_1(\varepsilon)$$

for some modulus $\omega_1$. We will be using $\omega_1$ to denote various moduli throughout the rest of the proof. Similarly, by the continuity of $\varphi_t$, we obtain

$$\left| \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \varphi_t(s, X_\varepsilon(s)) ds - \varphi_t(t_0, x_0) \right| \leq \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \omega_{t_0, x_0}(|s-t_0| + \|X_\varepsilon(s) - x_0\|) ds \leq \omega_1(\varepsilon).$$

Moreover

$$\left| \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \left[ \langle A \varphi_t(s, X_\varepsilon(s)) - \varphi_t(t_0, x_0) \rangle \right] ds \right|$$

$$\leq \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \left| \langle A, (X_\varepsilon(s) - x_0), D \varphi(s, X_\varepsilon(s)) \rangle \right| ds$$

$$+ \int_{t_0}^{t_0+\varepsilon} \left| \langle A x_0, D \varphi(s, X_\varepsilon(s)) \rangle - D \varphi(t_0, x_0) \right| ds \right|.$$

It now follows from (5.76) and (2.20) that

$$\|A (X_\varepsilon(s) - x_0), D \varphi(s, X_\varepsilon(s))\| = \|A^{1/2} (X_\varepsilon(s) - x_0), A^{1/2} D \varphi(s, X_\varepsilon(s))\|$$

$$\leq \|X_\varepsilon(s) - x_0\| \|A^{1/2} D \varphi(s, X_\varepsilon(s))\|$$

$$\leq C(\|x_0\|) \|X_\varepsilon(s) - x_0\| \leq \omega_1(\varepsilon).$$

By the continuity of $A^{1/2} D \varphi$ we also get

$$\|A x_0, D \varphi(s, X_\varepsilon(s)) - D \varphi(t_0, x_0)\|$$

$$\leq \|A^{1/2} x_0, A^{1/2} [D \varphi(s, X_\varepsilon(s)) - D \varphi(t_0, x_0)]\|$$

$$\leq \|x_0\| \omega_{\varphi,t_0,x_0}(|s-t_0| + \|X_\varepsilon(s) - x_0\|) \leq \omega_1(\varepsilon).$$

Next we have

$$\left| \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} [(B (X_\varepsilon(s)), D \varphi(s, X_\varepsilon(s)) - (B (x_0), D \varphi(t_0, x_0))] ds \right|$$

$$= \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} |b(X_\varepsilon(s), X_\varepsilon(s), D \varphi(s, X_\varepsilon(s))) - b(x_0, x_0, D \varphi(t_0, x_0))| ds$$

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Also

Finally the estimates of the terms containing $\delta$. We have

$$
\leq \frac{1}{\varepsilon} \left[ \int_{t_0}^{t_0+\varepsilon} |b(X_{\varepsilon}(s) - x_0, X_{\varepsilon}(s), D\phi(s, X_{\varepsilon}(s)))| + |b(x_0, X_{\varepsilon}(s) - x_0, D\phi(s, X_{\varepsilon}(s)))| \, ds \\
+ \int_{t_0}^{t_0+\varepsilon} |b(x_0, x_0, D\phi(s, X_{\varepsilon}(s)) - D\phi(t_0, x_0))| \, ds \right].
$$

Now observe that by (2.11)

$$
|b(X_{\varepsilon}(s) - x_0, X_{\varepsilon}(s), D\phi(s, X_{\varepsilon}(s)))| \leq C \|X_{\varepsilon}(s) - x_0\|_1 \|X_{\varepsilon}(s)\|_1 \|D\phi(s, X_{\varepsilon}(s))\|_1,
$$

and, by (2.12)

$$
|b(x_0, X_{\varepsilon}(s) - x_0, D\phi(s, X_{\varepsilon}(s)))| \leq C \|x_0\|_0 \|x_0\|_1 \|D\phi(s, X_{\varepsilon}(s)) - D\phi(t_0, x_0)\|_1 \\
\leq C \|x_0\|_0 \|x_0\|_1 \omega_{\phi, t_0, x_0} (|s - t_0| + \|X_{\varepsilon}(s) - x_0\|_0).
$$

It then follows from (5.76) and (2.20) that

$$
\left| \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \left[ \langle B(X_{\varepsilon}(s)), D\phi(s, X_{\varepsilon}(s)) \rangle - \langle B(x_0), D\phi(t_0, x_0) \rangle \right] \, ds \right| \leq \omega_1(\varepsilon).
$$

Also

$$
\left| \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \left[ \langle f(s, a_\varepsilon(s)), D\phi(s, X_{\varepsilon}(s)) \rangle - \langle f(t_0, a_\varepsilon(s)), D\phi(t_0, x_0) \rangle \right] \, ds \right| \\
\leq \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} |\langle f(s, a_\varepsilon(s)), D\phi(s, X_{\varepsilon}(s)) - D\phi(t_0, x_0) \rangle| \, ds \\
+ \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} |\langle f(s, a_\varepsilon(s)) - f(t_0, a_\varepsilon(s)), D\phi(t_0, x_0) \rangle| \, ds \\
\leq \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \|f(s, a_\varepsilon(s))\|_0 \|D\phi(s, X_{\varepsilon}(s)) - D\phi(t_0, x_0)\|_0 \, ds \\
+ \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \omega_f(|s - t_0|) \|D\phi(t_0, x_0)\|_0 \, ds \\
\leq \omega_1(\varepsilon).
$$

Finally the estimates of the terms containing $\delta$. We have

$$
\left| \frac{2}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \left[ \delta(s) \langle A X_{\varepsilon}(s), X_{\varepsilon}(s) \rangle - \delta(t_0) \langle A x_0, x_0 \rangle \right] \, ds \right| \\
\leq \omega_1(\varepsilon) + \frac{2}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \delta(s) \left| \left| \left\langle A^\frac{1}{2} X_{\varepsilon}(s), A^\frac{1}{2} (X_{\varepsilon}(s) - x_0) \right\rangle \right| + \left| \left\langle A^\frac{1}{2} (X_{\varepsilon}(s) - x_0), A^\frac{1}{2} x_0 \right\rangle \right| \right| \, ds \\
\leq \omega_1(\varepsilon) + \frac{C}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \|X_{\varepsilon}(s)\|_1 \|X_{\varepsilon}(s) - x_0\|_1 + \|x_0\|_1 \|X_{\varepsilon}(s) - x_0\|_1 \, ds \leq \omega_1(\varepsilon)
$$
estimating as before. Similarly we obtain
\[ \left| \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \delta(s) \langle f(s, a_\varepsilon(s)), X_\varepsilon(s) \rangle ds - \delta(t_0) \langle f(t_0, a_\varepsilon(t_0)), x_0 \rangle \right| \leq \omega_1(\varepsilon) \]
and
\[ \left| \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \delta'(s) ||X_\varepsilon(s)||_0^2 ds - \delta'(t_0) ||x_0||_0^2 \right| \leq \omega_1(\varepsilon). \]
Using all these estimates in (5.74) yields
\[ \omega_1(\varepsilon) - \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} l(t_0, x_0, a_\varepsilon(s)) ds \]
\[ \geq -\varphi_\varepsilon(t_0, x_0) + \langle Ax_0, D\varphi(t_0, x_0) \rangle \]
\[ + \langle B(x_0), D\varphi(t_0, x_0) \rangle - \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} \langle f(t_0, a_\varepsilon(s)), D\varphi(t_0, x_0) \rangle ds \]
\[ - \delta(t_0) \langle f(t_0, a_\varepsilon(t_0)), x_0 \rangle + 2\delta(t_0) \langle Ax_0, x_0 \rangle - \delta'(t_0) ||x_0||_0^2 \]
so that
\[ -\varphi_\varepsilon(t_0, x_0) - \delta'(t_0) ||x_0||_0^2 + \langle Ax_0 + B(x_0), D\varphi(t, x_0) \rangle - \delta(t_0) \langle f(t_0, a_\varepsilon(t_0)), x_0 \rangle \]
\[ + \frac{1}{\varepsilon} \int_{t_0}^{t_0+\varepsilon} [(f(t_0, a_\varepsilon(s)), -D\varphi(t_0, x_0)) + l(t_0, x_0, a_\varepsilon(s))] ds \leq \omega_1(\varepsilon). \]
The claim now follows upon taking first the infimum over \( a \in \Theta \) inside the integral and then letting \( \varepsilon \to 0. \)

**5.3 Applications and possible further developments**

The characterization of the value function given by Theorem 5.6 can be considered as a starting point for the dynamic programming analysis of the optimal control problem considered in this paper. It opens up several possible directions for future research. We outline some of these issues below:

1. One such direction is obtaining verification theorems in the context of viscosity solutions. Verification theorems use the HJB equation to give sufficient (and/or necessary) conditions for optimality. They are natural and easy to prove if the value function is a smooth solution of the HJB equation (see e.g. [15]). In the nonsmooth case such optimality conditions have been proved in the finite dimensional case (see e.g. [5], [39, Section 5.3]). For the dynamic programming of Navier-Stokes equation, such a verification theorem is proved in [33], [13], [32, Ch. I, Theorem 1.3.12] with a weaker definition of solution. Verification theorems and optimal synthesis results are also proved for an abstract infinite dimensional problem in [39, Section 6.5] however these results are rather weak. Moreover the abstract problem considered there cannot be applied to problems
with unbounded nonlinearities. Completion of the verification theory for the control of Navier-Stokes equations would require some new ideas, in particular the introduction of appropriate generalized sub- and supergradients that would be consistent with our definition of viscosity solution. Optimality conditions provided by the theory can then be used to construct optimal feedback controls, at least theoretically.

2. Another important direction is the construction of \( \epsilon \)-optimal controls and trajectories, and numerical approximations. There exist several numerical methods appropriate for solving optimal control problems for infinite dimensional systems (in particular for problems governed by nonlinear PDE). One of such promising and recently popular methods is the POD (proper orthogonal decomposition) method (see for instance [24, 29, 30]). It is a reduced order method that is based on projecting the control system onto finite dimensional subspaces that are spanned by basis elements that contain some characteristics of the expected solution. It also produces finite dimensional approximations of optimal controls. We are not aware of any applications of these methods to HJB equations in Hilbert spaces. This field is wide open. However some results in this direction seem possible. One is a construction of an abstract approximation scheme that is connected to discrete Dynamic Programming. The existence and uniqueness results proved in this paper lay a groundwork for such an attempt. We refer the reader to [5] for an overview of such results in finite dimensions. A different approach to numerical approximation and construction of \( \epsilon \)-optimal controls may use the specific nature of our problem, more precisely the weak continuity of the viscosity solution. In light of this one can try to approximate (uniformly on bounded sets) the unique viscosity solution (and therefore the value function) of the infinite dimensional HJB equation by viscosity solutions of suitable finite dimensional HJB equations connected to finite dimensional control problems. Procedures of this type have been used to prove existence of solutions for other infinite dimensional problems [10, Part IV], [34]. One could then use known numerical methods for finite dimensional HJB equations and dynamic programming techniques to produce \( \epsilon \)-optimal controls and trajectories. Of course the case of Navier-Stokes equations is different from the problems considered in [10, 34] but the idea is worth pursuing.

Sequential quadratic programming is a very promising method where the nonlinearity in the Navier-Stokes equation is approximated by Newton’s method to obtain a series of linear problems [20, 28]. An approximation of this type applied to the nonlinear \( \mathbf{B}(\mathbf{x}, \mathbf{x}) \) term in the Hamilton-Jacobi equation would lead (for a quadratic cost) to a sequence of approximate problems which fall within the realm of linear control theory and can be solved effectively by well established methods. An interesting theoretical question would then be to establish the convergence of such a sequence of approximate solutions to the unique viscosity solution proved in this paper.

3. Optimal control with state constraint is an important component of flow control as it allows us to deal with the task of controlling interesting regions in the flow domain where we have concentrated turbulence activity, for example [14]. In the Hamilton-Jacobi framework this will correspond to an initial-boundary value problem (boundary value problem for the stationary case) which would be a substantial additional step that needs to be addressed in the future.

4. As we pointed out in the introduction, unique solution of the Hamilton-Jacobi
equation corresponding to the dynamic programming of the Navier-Stokes equations have been established for a neighborhood of the origin in [4] using different methods. The uniqueness of viscosity solution proved in this paper may be used to extend this local result to a global one.

5. Finally we would like to mention that we believe that our ideas and methods can be extended to handle the important case of control of three dimensional Navier-Stokes equations. The essential reason for the analysis to be limited to two dimensions was the lack of global solvability theorem and the related continuous dependence theorems for the Navier-Stokes equations in three dimensions. The form of the Hamilton-Jacobi equations however is independent of the spatial dimensions.

References


