(4) (a) Solve $(u_1)_{tt} = a^2 (u_1)_{xx}$

$u_1(0,x) = 0, \quad (u_1)_t(0,x) = \cos \pi x$

$u_{1x}(t,0) = u_{1x}(t,1) = 0$

$u = u_1 + u_2$ where u_1 solves (a) and u_2 solves (b)

Separation of variables

$\mu_n = (n\pi)^2 \quad X_n = \cos n\pi x \quad n = 0,1,2 \ldots$

$T_n = \tilde{A}_n \sin(an\pi t)$

Match the in. cond. to get

$u_1(x,t) = \sum_{n=0}^{\infty} \hat{A}_n \sin(an\pi t) \cos n\pi x$

$u_1 = \frac{1}{a\pi} \sin a\pi t \cos \pi x$

(b) Solve $(u_2)_{tt} = a^2 (u_2)_{xx} + \sin \pi x$

0 in. cond + b.c.

$u_2(x,t) = \sum_{n=0}^{\infty} u_n(t) \cos n\pi x$

$\sin \pi x = \sum_{n=0}^{\infty} A_n \cos n\pi x \qquad A_0 = \frac{2}{\pi}$

$A_n = 2 \int_0^1 \sin \pi x \cos n\pi x \, dx$

Now $\sum_{n=0}^{\infty} u_n''(t) \cos n\pi x = \sum_{n=0}^{\infty} -a^2 (n\pi)^2 u_n(t) \cos n\pi x + \sum_{n=0}^{\infty} A_n \cos n\pi x$

$\therefore \quad u_0''(t) = \frac{2}{\pi}, \quad u_0(0) = 0, \quad u_0'(0) = 0 \qquad u_0(t) = \frac{t^2}{\pi}$

$$\begin{cases} u_n''(t) = -(a n\pi)^2 u_n(t) + A_n \\ u_n(0) = u_n'(0) = 0 \end{cases}$$

$u_n(t) = \frac{-A_n \cos(n\pi a t) + A_n}{(n\pi a)^2}$

Use

$\sin \pi x \cos n\pi x = \frac{1}{2}\left[\sin(n+1)\pi x - \sin(n-1)\pi x\right]$

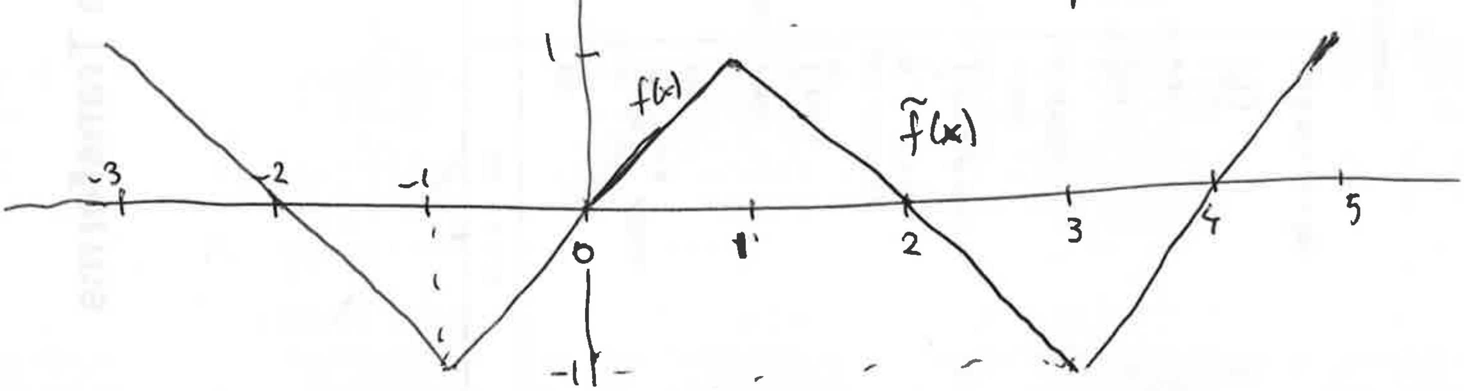
⑤

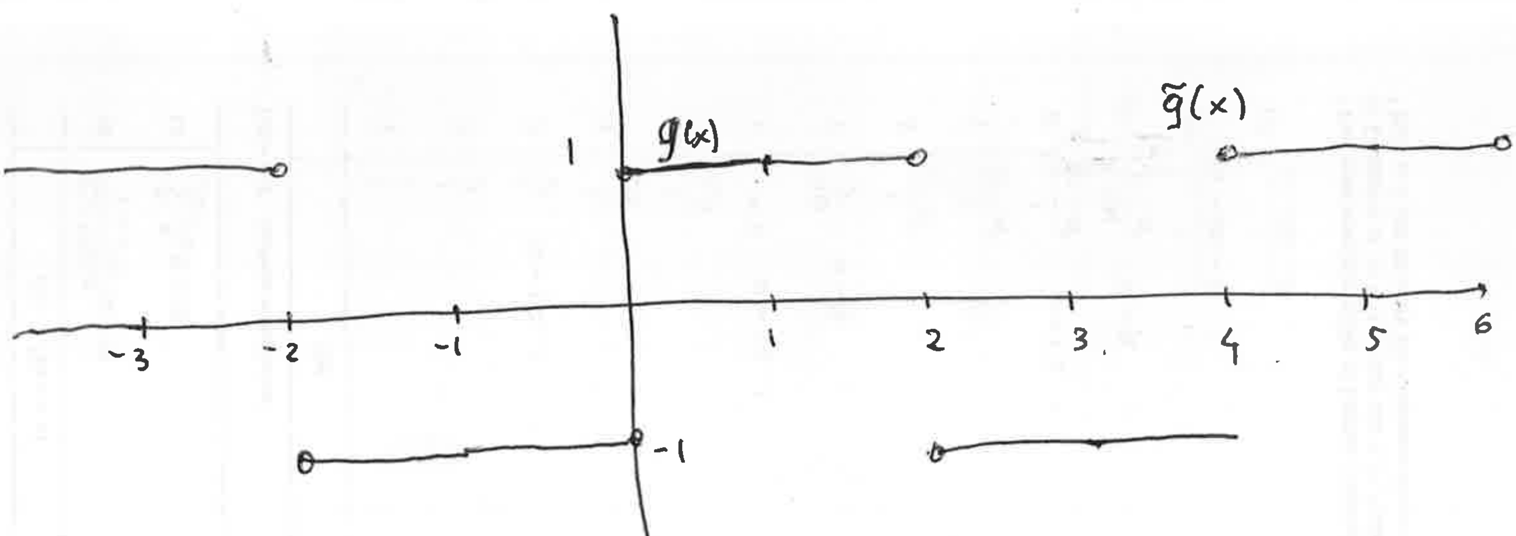
Use D'Alembert's solution formula

$$u(x,t) = \frac{1}{2}\left[\tilde{f}(x+2t) + \tilde{f}(x-2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} \tilde{g}(z)\,dz$$

where $\tilde{f}$ and $\tilde{g}$ are

$\tilde{f}$ - odd extension to $(-1,1)$, then even extension to $(-1,3)$, periodic with period 4





$\tilde{g}$ is obtained in the same way as f.

$$\therefore\ u(0.5, 3) = \frac{1}{2}\left[\tilde{f}(6.5) + \tilde{f}(-5.5)\right] + \frac{1}{4}\int_{-5.5}^{6.5} \tilde{g}(z)\,dz$$

$$= \frac{1}{2}\left[\tilde{f}(2.5) + \tilde{f}(2.5)\right] + 0 = -\frac{1}{2}$$

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$$-(s^2y - s)' - 2(sy)' - 2(sy-1) - (y)' - 2y = 0$$

$$-2sy - s^2y' + 1 - 2y - 2sy' - 2sy + 2 - y' - 2y = 0$$

$$-y'(s^2 + 2s + 1) + y(-4s - 4) = -3$$

$$-y'(s+1)^2 - y4(s+1) = -3$$

$$y' + \frac{4}{s+1}y = \frac{3}{(s+1)^2}$$

$$y'(s+1)^4 = \int 3(s+1)^2 = (s+1)^3 + C$$

$$y(s) = \frac{1}{s+1} + \frac{C}{(s+1)^4}$$

$$Y(t) = e^{-t} + Ct^3e^{-t}$$

$$y(s) = \mathcal{L}\{Y\}(s)$$