1. Prove the simple case of Erdős-Ko-Rado theorem: if \( k = n/2 \), then the number of \( k \)-sets in any intersecting family is at most \( \frac{1}{2} \binom{n}{k} \).

2. Let \( X = \{1, 2, \ldots, 7\} \), and let \( \mathcal{B} \) be the following family of seven subsets of \( X \):

\[
\mathcal{B} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}.
\]

Then let \( \mathcal{F} = \{A \subset X : A \text{ contains } B \text{ for some } B \in \mathcal{B}\} \). Show that \( \mathcal{F} \) is intersecting and \( |\mathcal{F}| = 2^7 - 1 = 64 \).

3. Let \( G = (V, E) \) be a graph on \( n \) vertices and let \( t(G) \) be the number of triangles in it. Show that

\[
t(G) \geq \frac{|E|}{3n} (4 \cdot |E| - n^2).
\]

Sketch: For an edge \( e = \{x, y\} \), let \( t(e) \) be the number of triangles containing \( e \). Let \( B = V \setminus \{x, y\} \). Among the vertices in \( B \) there are precisely \( t(e) \) vertices which are adjacent to both \( x \) and \( y \). Every other vertex in \( B \) is adjacent to at most one of these two vertices. We thus obtain \( d(x) + d(y) - t(e) \leq n \). Summing over all edges \( e = \{x, y\} \) we obtain

\[
\sum_{e \in E} (d(x) + d(y)) - \sum_{e \in E} t(e) \leq n \cdot |E|.
\]

Apply the Cauchy-Schwarz inequality to estimate the first sum.

Comment: This implies that a graph on an even number \( n \) of vertices with \( n^2/4 + 1 \) edges not only contains one triangle (as it must be by Mantel’s theorem), but more than \( n/3 \).

Optional Problems.

4. Let \( G = (V, E) \) be a graph. Let \( d(x) \) denote the degree of \( x \), for each \( x \in V \). Explain why the following holds:

\[
\sum_{x \in V} d(x)^2 = \sum_{\{x, y\} \in E} (d(x) + d(y)).
\]

5. A family of subsets of \( X \) is said to be 2-colorable, if it is possible to assign one of two colors (RED and BLUE, say) to the elements of \( X \) so that no set in the family is monochromatic – meaning no set should be all RED or all BLUE. Is the family \( \mathcal{B} \) in Problem 2 above 2-colorable?