Problem 1 Let \((\Omega, \mathcal{F}, \mu)\) be a measure space. For measurable subsets \(A\) and \(B\) of \(\Omega\), let \(S(A, B)\) denote the symmetric difference of \(A\) and \(B\). Show that, if \(\mu(S(A, B)) = 0\), then, for every nonnegative measurable function \(f\),
\[
\int_A f \, d\mu = \int_B f \, d\mu.
\]

Solve the next two problems WITHOUT using any convergence theorems – monotone convergence, Fatou’s lemma, or Dominated convergence.

Problem 2. Let \((\Omega, \mathcal{F}, \mu)\) be a measure space and \(f\) and \(g\) nonnegative measurable functions. Show that, if \(g\) is simple, then, for all \(E \in \mathcal{F}\),
\[
\int_E (f + g) \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu.
\]

Problem 3. Let \((\Omega, \mathcal{F}, \mu)\) be a measure space with \(\mu(\Omega) < \infty\). Let \(f\) be a bounded nonnegative function and let \(s_n\) be the sequence of simple functions constructed in Theorem 6 (of the handout from Adams-Guillemin book). Show that for \(E \in \mathcal{F}\),
\[
\int_E s_n \, d\mu \rightarrow \int_E f \, d\mu.
\]

(Hint: The sequence \(s_n\) converges uniformly to \(f\). Moreover,
\[
\int_E (f - s_n) \, d\mu + \int_E s_n \, d\mu = \int_E f \, d\mu
\]
by Problem 2.)

Problem 4. For \(n = 1, 3, 5, \ldots\) let \(f_n\) be the characteristic function of the interval \((0, 1/2)\), and for \(n = 2, 4, 6, \ldots\) let \(f_n\) be the characteristic function of the interval \((1/2, 1)\). Compare \(\int \liminf f_n \, d\mu_L\) and \(\liminf \int f_n \, d\mu_L\), where \(\mu_L\) is the Lebesgue measure on \([0, 1]\).

Problem 5. Let \(F\) be a distribution function and \(r\) a positive integer. Show that the following are distribution functions:

1. \(F(x)^r\),
2. \(F(x) + (1 - F(x)) \log(1 - F(x))\),
3. \((F(x) - 1)e + \exp(1 - F(x))\).

Problem 6. Which of the following are density functions? Find \(c\) and the corresponding distribution function \(F\) for those that are.

1. \(f(x) = \begin{cases} cx^{-d}, & x > 1 \\ 0, & \text{otherwise} \end{cases}\)
2. \(f(x) = ce^x(1 + e^x)^{-2}, \quad x \in \mathbb{R}\).

Problem 7. Express the distribution functions of
\[
X^+ = \max\{0, X\}, \quad X^- = -\min\{0, X\}, \quad |X| = X^+ + X^-, \quad -X,
\]
in terms of the distribution function \(F\) of the random variable \(X\).