Problem 1. Let \( h \) be a function such that if the vectors \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) differ in at most one coordinate (that is, for some \( k \), \( x_i = y_i \) for all \( i \neq k \)) then \( |h(x) - h(y)| \leq 1 \). Let \( X_1, \ldots, X_n \) be independent random variables. Then, with \( X = (X_1, \ldots, X_n) \), we have for \( a > 0 \) that
\[
\begin{align*}
(i) & \Pr\{h(X) - E[h(X)] \geq a\} \leq e^{-a^2/2n}, \\
(ii) & \Pr\{h(X) - E[h(X)] \leq -a\} \leq e^{-a^2/2n},
\end{align*}
\]

Problem 2. If \( T_1 \) and \( T_2 \) are stopping times with respect to the filtration \( \mathcal{F} \), show that \( T_1 + T_2 \) and \( \max\{T_1, T_2\} \) are stopping times also.

Problem 3. Let \( X_1, X_2, \ldots \) be a sequence of non-negative independent r.v.s and let \( N(t) = \max\{n : X_1 + X_2 + \cdots + X_n \leq t\} \). Show that \( N(t) + 1 \) is a stopping time with respect to a suitable filtration to be specified.

Problem 4. If \( X_i, i \geq 1 \) are i.i.d. with \( E[|X_1|] < \infty \) and if \( T \) is a stopping time for \( X_1, X_2, \ldots \) with \( E[T] < \infty \), then
\[
E[\sum_{i=1}^{T} X_i] = E[T]E[X_1].
\]

*Hint:* Use the OST: if \( \{Z_n\}, n \geq 1, \) is a martingale sequence, and \( T \): a stopping time with \( E[T] < \infty \), and there is a \( K < \infty \) such that
\[
E[|Z_n - Z_{n-1}| \mid Z_1, \ldots, Z_{n-1}] < K,
\]
then \( E[Z_T] = E[Z_1] \).

Problem 5. Suppose that \( T \) is a stopping time such that for some \( N \in \mathbb{N} \) and some \( \epsilon > 0 \), we have, for every \( n \in \mathbb{N} \):
\[
\Pr(T \leq n + N) > \epsilon, \quad \text{a.s.}
\]
Then \( E(T) < \infty \).

*Hint:* Prove by induction that for \( k = 1, 2, 3, \ldots \)
\[
\Pr(T > kN) \leq (1 - \epsilon)^k.
\]

Problem 6. Prove that if \( \mathcal{C} \) and \( \mathcal{D} \) are UI classes of r.v.s, and if we define
\[
\mathcal{C} + \mathcal{D} := \{X + Y : X \in \mathcal{C}, Y \in \mathcal{D}\},
\]
then \( \mathcal{C} + \mathcal{D} \) is UI.