Representations of Integers as the Sum of k Terms

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ABSTRACT

A set of natural numbers is called an *asymptotic basis* of order k if every number (sufficiently large) can be expressed as a sum of k distinct numbers from the set. In this paper we prove that, for every fixed k, there exists an asymptotic basis of order k such that the number of representations of n is $\Theta(\log n)$.

Key Words: Sidon sequences, representations of integers, random constructions

1. INTRODUCTION

Let $\mathcal{S}$ be a sequence of natural numbers, and let $r_k(n)$ denote the number of representations of n in the form

$$n = a_1 + a_2 + \cdots + a_k,\ 0 < a_1 < \cdots < a_k\ (a_i \in \mathcal{S})$$

$\mathcal{S}$ is said to be an *asymptotic basis* of order k if there exists a natural number n_0 such that $r_k(n) > 0$ for $n > n_0$. In 1932 Sidon raised the following question: Does there exist an asymptotic basis of order 2 such that, for every $\epsilon > 0$, $\lim_{n\to\infty} r_2(n)/n^\epsilon = 0$? In 1956 Erdös [2] proved that there exist positive constants c_1 and c_2 such that

$$c_1 \log n \le r_2(n) \le c_2 \log n \text{ for large } n\,,$$

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which more than answered the question. For a thorough exposition of the above result and several other results of similar nature, the reader is strongly recommended to refer to [5].

In this paper we prove a more general version of Sidon's original question using probabilistic techniques, old and new. Although the following general version was believed to be true, no complete proof appears in the literature.

Theorem 1. *For every fixed k there exists an asymptotic basis of order k such that* $r_k(n) = \Theta(\log n)$.

We shall actually conclude that in a suitable probability (measure) space almost all sequences are of *the right kind.*

2. PROBABILISTIC TOOLS

Disjointness Lemma

Let $A_1, A_2, \ldots$ be events in a probability space. The following elementary lemma is often useful in proving the probability of occurrence of "large" subsets of mutually independent events is "small." We refer to this as the *disjointness lemma* in view of the nature of its application in this paper.

Lemma 1 (Disjointness Lemma). *If* $\Sigma_i \Pr[A_i] \le \mu$, *then*

$$\sum_{\substack{\{A_1, \ldots, A_l\} \\ \text{mutually} \\ \text{independent}}} \Pr[A_1 \wedge A_2 \wedge \cdots \wedge A_l] \le \mu^l / l!$$

Proof. Let Σ' denote the summation in the statement of the lemma. Then by mutual independence of the probabilities,

$$\begin{aligned}
\sum{}' \Pr[A_1 \wedge \cdots \wedge A_l] &= \sum{}' \Pr[A_1] \cdots \Pr[A_l] \\
&\le \sum_{\text{all}\{A_1, \ldots, A_l\}} \Pr[A_1] \cdots \Pr[A_l] \\
&= \frac{1}{l!} \sum_{(A_1, \ldots, A_l)} \Pr[A_1] \cdots \Pr[A_l] \\
&= \frac{1}{l!} \Big(\sum_i \Pr[A_i] \Big)^l \\
&= \frac{1}{l!} \mu^l .
\end{aligned}$$ ∎

Erdös–Rado Δ-system

Let n and l be positive integers, $l \ge 3$. We say that l sets form a Δ-*system* (of size l), if they have pairwise the same intersection. Let $\mathcal{F}$ be a family of sets, each of size n. Then

Lemma 2 (Δ-System Lemma). *$\mathscr{F}$ contains a Δ-system of size l whenever*

$$|\mathscr{F}| \geq (l-1)^n N!$$

The proof and a more general version of the theorem (for infinite set systems) can be found in [3].

Correlation Inequality

In a probability space, we know that the probability of a (finite) conjunction of events is exactly the product of the individual probabilities, provided that the events are *mutually independent*. In a similar spirit, the following *correlation inequality* allows us to estimate the probability of a conjunction by the product of the probabilities, provided that the events have *low correlation*! We need some notation to state the lemma precisely.

Let Ω be a finite set of elements, and S be a random subset of Ω given by $\Pr[x \in S] = p_x$, where we assume these events to be mutually independent over $x \in \Omega$. Let A_i denote the event $X_i \subset S$, where $X_1, \ldots, X_n$ are subsets of the universal set Ω. Write $i \sim j$ if $i \neq j$ and $X_i \cap X_j \neq \phi$. Further let all $\Pr[A_i] \leq 1/2$, then

Lemma 3 (Correlation Inequality).

$$\prod \Pr[\bar{A}_i] \leq \Pr\Big[\bigwedge \bar{A}_i\Big] \leq \Big[\prod \Pr[\bar{A}_i]\Big]\, e^{2\sum_{i\sim j}\Pr[A_i \cap A_j]}.$$

The proof requires only elementary probability theory and appears in [1]. The original proof of the result was given by Svante Janson and appears in [6].

Borel–Cantelli Lemma

The last tool we make use of in this paper is one of great importance in probability theory. We merely state the well known Borel–Cantelli lemma in the following; the proof can be found in [5] and also in any standard book on probability theory.

Lemma 4 (Borel–Cantelli). *Let $\{A_i\}$ be a sequence of events in a probability space. If*

$$\sum_{j=1}^{\infty} \Pr(A_j) < \infty,$$

then with probability 1, *at most a finite number of the events A_j can occur.*

3. PROOF OF THE MAIN THEOREM

Let k be fixed. Our aim is to show that there exists an asymptotic basis of order k and that the number of representations is roughly $\log n$ for n *large*. Consider a sequence $\mathscr{S}$ of natural numbers where the probability $p(z)$ of z occuring in $\mathscr{S}$ is given by

$$p(z) = C\,\frac{(\log z)^{1/k}}{z^{(k-1)/k}} \text{ for } z > z_0$$
$$= 0 \text{ otherwise,}$$

where z_0 is the smallest constant such that $p(z) \leq 1/2$. We shall choose C so that $C^k D_k > 3$, where

$$D_k = [2^{1/k} - 1][3^{1/k} - 2^{1/k}] \cdots [(k-1)^{1/k} - (k-2)^{1/k}]\left(\frac{k^{k-1}}{k-1}\right)^{(k-1)/k},$$

a *not-so-nice* positive absolute constant!

Let $r_k(n)$ be the random variable denoting the number of representations of n as a sum of k distinct numbers from $\mathscr{S}$. Then

Lemma 5.

$$\mu = E[r_k(n)] = \Theta(\log n)$$

Proof.

$$\mu = \sum_{\substack{x_1 + \cdots + x_k = n \\ 1 \leq x_1 < \cdots < x_k < n}} \Pr[x_1] \cdots \Pr[x_k]$$
$$= \sum\nolimits^* C\left(\frac{\log x_1}{x_1^{k-1}}\right)^{1/k} \cdots C\left(\frac{\log x_k}{x_k^{k-1}}\right)^{1/k}$$

where Σ^* is *short hand for* $\Sigma_{\substack{x_1 + \cdots + x_k = n \\ 1 \leq x_1 < \cdots < x_k < n}}$. We estimate the above sum by breaking it into two parts $(\mu_1 + \mu_2)$, depending on whether $x_1 > n/(\log n)$ or not. (It may be noted that "$\log n$" here may be replaced by any positive function that approaches infinity sufficiently slowly as $n \to \infty$.) We prove the lemma by showing that $\mu_1 = \Theta(\log n)$ and that $\mu_2 = o(\log n)$. For this purpose, let $\delta_n = 1/(\log n)$. Then

$$\mu_1 = C^k \sum_{n\delta_n < x_1 < n}^{*} \frac{[(\log x_1) \cdots (\log x_k)]^{1/k}}{(x_1 \cdots x_k)^{(k-1)/k}}$$
$$= C^k(1 + o(1))(\log n) \sum_{n\delta_n < x_1 < n}^{*} \frac{1}{(x_1 \cdots x_k)^{(k-1)/k}}$$
$$= C^k(1 + o(1))(\log n) \cdot (SUM) \quad \text{(say)},$$

where the numerator comes out as a common term, in view of the range of summation. It now suffices to show that $SUM = \Theta(1)$. Note that $n/k < x_k < n$, since x_i's form an increasing sequence. As $n/k < x_k$, we bound SUM from above by

$$SUM < \frac{1}{(n/k)^{(k-1)/k}} \sum_{n\delta_n < x_1 < n}^{*} \frac{1}{(x_1 \cdots x_{k-1})^{(k-1)/k}}.$$

We can bound this tricky summation from above by summation over all $(x_1, \ldots, x_{k-1})$, with $1 \le x_i \le n$. That is,

$$SUM < \frac{1}{(n/k)^{(k-1)/k}} \sum_{\substack{1 \le x_i \le n \\ i=1,\ldots,k-1}} \frac{1}{(x_1 \cdots x_{k-1})^{(k-1)/k}}$$

$$= \left(\frac{k}{n}\right)^{(k-1)/k} \left(\sum_{1 \le x_1 \le n} \frac{1}{(x_1)^{(k-1)/k}} \right)^{k-1} .$$

We now estimate this discrete sum by integrating over the same range:

$$SUM < \left(\frac{k}{n}\right)^{(k-1)/k} \left(\int_1^n \frac{1}{x_1^{(k-1)/k}}\, dx_1 + O(1) \right)^{k-1}$$

$$= \left(\frac{k}{n}\right)^{(k-1)/k} (k x_1^{1/k}|_1^n + O(1))^{k-1}$$

$$= \frac{k^{(k-1)(k+1)/k}}{n^{(k-1)/k}} [n^{(k-1)/k} + o(n^{(k-1)/k})]$$

$$= [1 + o(1)] k^{(k-1)(k+1)/k} .$$

In the other direction, set $\alpha = 1/k(k-1)$. As $x_k < n$, we bound SUM from below by

$$SUM > \frac{1}{n^{(k-1)/k}} \sum_{n\delta_n < x_1 < n}^{*} \frac{1}{(x_1 \cdots x_{k-1})^{(k-1)/k}} .$$

For a *lower bound* on this summation, our idea is to restrict the x_i's to *strips* of width αn, i.e., we bound the summation from below by summation over $(x_1 \cdots x_{k-1})$, with restrictions $n\delta_n < x_1 < \alpha n$, $\alpha n < x_2 < 2\alpha n, \ldots, (k-2)\alpha n < x_{k-1} < (k-1)\alpha n$. (Our choice of α ensures that $x_{k-1} < x_k$.) Thus

$$SUM > \frac{1}{n^{(k-1)/k}} \sum_{n\delta_n < x_1 < \alpha n} \frac{1}{x_1^{(k-1)/k}} \cdots \sum_{(k-2)\alpha n < x_{k-1} < (k-1)\alpha n} \frac{1}{x_{k-1}^{(k-1)/k}}$$

and once again, estimating by integrals,

$$= \frac{1}{n^{(k-1)/k}} \left(\int_{n\delta_n}^{\alpha n} \frac{1}{x_1^{(k-1)/k}} + O(1) \right) \cdots \left(\int_{(k-2)\alpha n}^{(k-1)\alpha n} \frac{1}{x_{k-1}^{(k-1)/k}} + O(1) \right)$$

$$= \frac{1}{n^{(k-1)/k}} (k x_1^{1/k}|_{n\delta_n}^{\alpha n} + O(1)) \cdots (k x_{k-1}^{1/k}|_{(k-2)\alpha n}^{(k-1)\alpha n} + O(1))$$

$$= [1 + o(1)] k^{k-1} \alpha^{(k-1)/k} [1 - (\alpha\delta_n)^{1/k}][2^{1/k} - 1] \cdots [(k-1)^{1/k} - (k-2)^{1/k}]$$

$$= D_k + o(1)$$

where

$$D_k = [2^{1/k} - 1][3^{1/k} - 2^{1/k}] \cdots [(k-1)^{1/k} - (k-2)^{1/k}]\left(\frac{k^{k-1}}{k-1}\right)^{(k-1)/k},$$

a positive absolute constant!

Consider now the second part:

$$\begin{aligned}\mu_2 &= C^k \sum_{1<x_1<n\delta_n}^{*} \frac{[(\log x_1)\cdots(\log x_k)]^{1/k}}{(x_1\cdots x_k)^{(k-1)/k}}\\ &< C^k(\log n) \sum_{1<x_1<n\delta_n}^{*} \frac{1}{(x_1\cdots x_k)^{(k-1)/k}}\\ &= C^k(\log n)\cdot(sum) \quad \text{(say)}.\end{aligned}$$

It now suffices to show that $sum = o(1)$, since we are trying to prove $\mu_2 = o(\log n)$. The calculation is similar to what we did for SUM and proceeds as follows:

$$\begin{aligned} sum &< \frac{1}{(n/k)^{(k-1)/k}} \sum_{1\le x_1\le n\delta_n}^{*} \frac{1}{(x_1\cdots x_{k-1})^{(k-1)/k}}\\ &< \frac{1}{(n/k)^{(k-1)/k}} \sum_{\substack{1\le x_1\le n\delta_n\\ 1\le x_i\le n\\ i=2,\ldots,k-1}} \frac{1}{(x_1\cdots x_{k-1})^{(k-1)/k}}\\ &= \left(\frac{k}{n}\right)^{(k-1)/k}\left(\sum_{1\le x_1\le n\delta_n}\frac{1}{(x_1)^{(k-1)/k}}\right)\left(\sum_{1\le x_2\le n}\frac{1}{(x_2)^{(k-1)/k}}\right)^{k-2}\\ &= \left(\frac{k}{n}\right)^{(k-1)/k}\left(\int_1^{n\delta_n}\frac{1}{x_1^{(k-1)/k}}\,dx_1 + O(1)\right)\left(\int_1^{n}\frac{1}{x_2^{(k-1)/k}}\,dx_2 + O(1)\right)^{k-2}\\ &= \left(\frac{k}{n}\right)^{(k-1)/k}\left(kx_1^{1/k}\Big|_1^{n\delta_n} + O(1)\right)\left(kx_2^{1/k}\Big|_1^{n} + O(1)\right)^{k-2}\\ &= \frac{k^{(k-1)(k-2)/k}}{n^{(k-1)/k}}\,[\delta_n^{1/k} n^{(k-1)/k} + o(\delta_n^{1/k} n^{(k-1)/k})]\\ sum &= O(\delta_n^{1/k}).\end{aligned}$$

This shows that

$$\begin{aligned}\mu_2 &= C^k(\log n)O(\delta_n^{1/k})\\ &= o(\log n).\end{aligned}$$

Thus we have established

$$[b_1 + o(1)]\log n < \mu_1 < [b_2 + o(1)]\log n \text{ and } \mu_2 = o(\log n)$$

and hence

$$[b_1 + o(1)]\log n < \mu < [b_2 + o(1)]\log n,$$

where $b_1 = D_k C^k$ and $b_2 = C^k k^{(k-1)(k+1)/k}$.

3.1. First Part of the Main Theorem

Let $S_1 = \{a_1, \ldots, a_k\}$ and $S_2 = \{b_1, \ldots, b_k\}$ be two different representations of n in basis $\mathcal{S}$; that is, $S_1 \neq S_2$, S_1, $S_2 \subset \mathcal{S}$ and

$$a_1 + \cdots + a_k = b_1 + \cdots + b_k = n\,.$$

We say S_i and S_2 are *disjoint* if they share no element in common. The following lemmas guarantee (with high probability) that a maximal collection of pairwise disjoint representations has size at most $O(\log n)$.

Lemma 6.

$$\sum_{\substack{\{S_1, \ldots, S_{6\mu}\} \\ \text{pairwise} \\ \text{disjoint}}} \Pr[S_1 \wedge \cdots \wedge S_{6\mu}] \leq \mu^{6\mu}/(6\mu)!\,.$$

Proof. Pairwise disjointness of the sets implies mutual independence of the associated events. The proof is hence a simple application of the *disjointness lemma* (Lemma 1) with $l = 6\mu$. ∎

This in turn bounds the probability of having more than 6μ pairwise disjoint representations of n as follows. Let $r_k^*(n)$ denote the size of a maximal collection of pairwise disjoint representations. Then

Lemma 7. *For* $C > (1/3D_k)^{1/k}$,

$$\Pr[r_k^*(n) > 6\mu] \leq n^{-2+o(1)}\,.$$

Proof. Using Lemma 6 and the bound $m! > m^m e^{-m}$

$$\begin{aligned}
\Pr[r_k^*(n) > 6\mu] &< \frac{1}{(6\mu)!}\,(\mu)^{6\mu} \\
&< \frac{1}{(6\mu/e)^{6\mu}}\,(\mu)^{6\mu} \\
&= \left(\frac{e}{6}\right)^{6\mu} \\
&< \left(\frac{e}{6}\right)^{[6b_1 + o(1)]\log n} \\
&= n^{-6b_1 + o(1)}\,.
\end{aligned}$$

To complete the proof of the lemma, we have to show that $b_1 \geq 1/3$. As $b_1 = D_k C^k$, we pick C large enough (so that $D_k C^k > 1/3$) to make $b_1 > 1/3$, yielding

$$\Pr[r_k^*(n) > 6\mu] \leq n^{-2+o(1)}\,.$$

∎

Lemma 7a. A.a. $\exists\, n_0$ s.t. $r_k^*(n) \le 6\mu$, for $n > n_0$.

Proof. Use Borel–Cantelli lemma! First, a few words about the *probability space* we are dealing with. Let Ω denote universal set of all subsequences of the sequence of natural numbers. Our *random* set $\mathcal{S}$ is the subsequence formed by choosing number z with probability $p(z)$. Thus the triple $(\Omega, \mathcal{S}, p(\cdot))$ is our *probability space*. (For a rigorous proof that this is a *valid* probability space, Ref. [5], pp. 141–142.)

Now let A_n denote the event $r_k^*(n) > 6\mu$. By Lemma 7, we have

$$\sum_{n=1}^{\infty} \Pr(A_n) = \sum_{n=1}^{\infty} n^{-2+o(1)} < \infty .$$

By the Borel–Cantelli lemma, this implies that with probability 1, at most a *finite* number of A_n occur. Hence the lemma. ∎

To prove the first part of the main theorem, we must extend Lemma 7a and show that $r_k(n)$ is $O(\log n)$. For this purpose we need a bound [of $O(1)$] on the number of representations of n as a sum of $k-1$ (distinct) numbers. (The puzzled reader is advised to skip the next few lemmas and read the proof of Theorem 2 for further motivation.) But we need to generalize some of our notation before we can proceed. Let $r_l(n)$ denote the number of representations of n as a sum of l distinct numbers from $\mathcal{S}$. And let $r_l^*(n)$ denote the size of a maximal collection of *pairwise disjoint* such representations. Further, let μ_l denote the expectation of $r_l(n)$ (note that μ_k is simply μ)

Lemma 8. *For* $2 \le l \le (k-1)$,

$$\mu_l \le n^{-1+l/k+o(1)} .$$

Proof. The line of calculation is similar to that for μ, only simpler. Let l be fixed. Let Σ^* denote $\Sigma_{\substack{x_1+\cdots+x_l=n \\ 1\le x_1<\cdots<x_l<n}}$. Then by definition,

$$\begin{aligned}
\mu_l &= \sum_{\substack{x_1+\cdots+x_l=n \\ 1\le x_1<\cdots<x_l<n}} \Pr[x_1]\cdots\Pr[x_l] \\
&= \sum\nolimits^* C\left(\frac{\log x_1}{x_1^{k-1}}\right)^{1/k} \cdots C\left(\frac{\log x_l}{x_l^{k-1}}\right)^{1/k} \\
&= n^{o(1)} \sum\nolimits^* \frac{1}{(x_1\cdots x_l)^{(k-1)/k}} \\
&= n^{o(1)} SUM_l .
\end{aligned}$$

As $n/l < x_l$,

$$SUM_l = n^{-(k-1)/k+o(1)} \sum\nolimits^* \frac{1}{(x_1\cdots x_{l-1})^{(k-1)/k}} .$$

And once again we estimate crudely by the summation over all $(x_1, \ldots, x_{l-1})$, with $1 \le x_i \le n$, $i = 1, 2, \ldots, l-1$:

$$SUM_l \le n^{-(k-1)/k+o(1)} \sum_{\substack{1 \le x_i \le n \\ i=1,\ldots,l-1}} \frac{1}{(x_1 \cdots x_{l-1})^{(k-1)/k}}$$
$$= n^{-(k-1)/k+o(1)} \Bigg(\sum_{1 \le x_1 \le n} \frac{1}{x_1^{(k-1)/k}} \Bigg)^{l-1}$$
$$= n^{-(k-1)/k+o(1)} (n^{1/k+o(1)})^{l-1}$$
$$= n^{-1+l/k+o(1)} .$$

Substituting this in the estimate for μ_l above, we get

$$\mu_l \le n^{o(1)}(n^{-1+l/k+o(1)})$$
$$= n^{-1+l/k+o(1)} .$$

■

Let S^l denote a representation of n as a sum of l distinct numbers. Then:

Lemma 9. *For* $2 \le l \le (k-1)$,

(i) $$\sum_{\substack{\{S_1^l, \ldots, S_{2k}^l\} \\ \text{pairwise} \\ \text{disjoint}}} \Pr[S_1^l \wedge \cdots \wedge S_{2k}^l] < \frac{\mu_l^{2k}}{(2k)!}$$

(ii) $\Pr[r_l^*(n) > 2k] < n^{-2+o(1)}$.

Proof. (i) Note that when S_i^l and S_j^l are disjoint, $\Pr[S_i^l]$ and $\Pr[S_j^l]$ are mutually independent probabilities. So the first part of the lemma follows immediately from the *disjointness lemma*.

(ii) From part (i) it follows that

$$\Pr[r_l^*(n) > 2k] < \frac{\mu_l^{2k}}{(2k)!}$$
$$< \frac{1}{(2k)!} (n^{-1+l/k+o(1)})^{2k}$$
$$= n^{-2k+2l+o(1)}$$

As $l \le k-1$, we can conclude

$$\Pr[r_l^*(n) > 2k] < n^{-2+o(1)} .$$

■

Once again, by the Borel–Cantelli lemma, the above assertion implies that

$$\text{a.a. for } 2 \le l \le (k-1) \quad \exists\, n_l \quad \text{s.t.} \quad n > n_l \Rightarrow r_l^*(n) < 2k .$$

But for any finite n_l, there are at most a finite [certainly $<\binom{n_l}{l}$] number of representations as a sum of l numbers. Therefore,

$$\text{a.a. for } 2 \le l \le (k-1) \quad \exists\, c_l \quad \text{s.t.} \quad \forall n, r_l^*(n) < c_l\,.$$

At this point we remind the reader that we set out to prove that $r_{k-1}(n)$ is at most a constant. And we make critical use of the Δ-*system lemma* (Lemma 2) to prove this.

Lemma 10.

$$\text{a.a.} \quad \exists\, c \quad \text{s.t.} \quad r_{k-1}(n) < c\,, \quad \forall n\,.$$

Proof. We know

$$\text{a.a.} \quad \text{for } 2 \le l \le (k-1) \quad \exists\, c_l \quad \text{s.t.} \quad \forall n\,, \quad r_l^*(n) < c_l\,. \qquad (*)$$

Set $c_{\max} = \max_l\{c_l\}$. We claim (whenever c_l's exist),

$$\forall n \; r_{k-1}(n) \le (c_{\max})^{k-1}(k-1)!$$

Suppose not, i.e., the claim is false for some $n = n'$. Then by the Δ-*system lemma*, there exists a Δ-system $\{S_1^{k-1}, \ldots, S_\Delta^{k-1}\}$ of size $\Delta = c_{\max} + 1$. Let the common intersection of the system be R. So $R = \{x_1, \ldots, x_r\}$, where $0 \le r \le k-2$. If $\Sigma_i x_i = m$, then removing the common intersection R from each set will yield $r_{k-1-r}^*(n' - m) \ge c_{\max} + 1$. This is impossible in view of $(*)$ and the definition of $c_{\max}$. This proves the lemma and, in fact, also shows that $c \le c_{\max}^{k-1}(k-1)!$ ∎

We are now ready to prove one side of the main theorem.

Theorem 2. *A.a.* $\exists\, c\, \exists\, n_0$ s.t. $r_k(n) < [6b_2ck + o(1)]\log n\,,\ n > n_0\,.$

Proof. Let $\hat{\mathbf{S}}$ be any maximal collection of pairwise disjoint representations of n as a sum of k distinct numbers. By our notation, $|\hat{\mathbf{S}}| = r_k^*(n)$. By Lemma 7a, we have

$$\text{A.a.} \quad \exists\, n_0 \quad \text{s.t.} \quad r_k^*(n) < 6\mu, \quad n > n_0\,.$$

This implies there are at most $(k) \times (6\mu)$ numbers in our collection $\hat{\mathbf{S}}$. [There are (almost always) at most 6μ sets in the collection, and each set has exactly k numbers.] As $\hat{\mathbf{S}}$ is maximal, any representation of n must use at least one number from the collection. However, the number of representations of n which use x is precisely $r_{k-1}(n-x)$. By Lemma 10 we know $r_{k-1}(n-x) < c$. Thus the total number of representations of n is at most $(6k\mu) \times c$. Hence the theorem follows by substituting the bound for μ.

3.2. Second part of the main theorem

In the first part, we have proved that $r_k(n) < C_2 \log n$, for large n. We now would like to prove that $r_k(n) > C_1 \log n$, for large n. We prove the stronger inequality, that $r_k^*(n) > C_1 \log n$, for large n. Before we can prove this, we need two more lemmas.

Let S_i and S_j be two valid representations of n as a sum of k distinct numbers from $\mathscr{S}$. With abuse of notation, we let $\Pr[S_i]$ denote the probability that S_i is a valid representation. We would like to show that these events have *low correlation* in the following precise sense.

Lemma 11.

$$\sum_{i \sim j} \Pr[S_i \wedge S_j] = o(1) .$$

Proof. Note that $i \sim j$ implies S_i and S_j share at least 1 number and at most $k-2$ numbers.

$$\sum_{i \sim j} \Pr[S_i \wedge S_j] = \sum_{l=1}^{k-2} \sum_{|S_i \cap S_j| = l} \Pr[S_i \wedge S_j] .$$

Consider S_i, S_j such that $|S_i \cap S_j| = l$. Say,

$$S_i = (z_1, \ldots, z_l, x_1, x_2, \ldots, x_{k-l}) \text{ and } S_j = (z_1, \ldots, z_l, y_1, y_2, \ldots, y_{k-l}) .$$

Let $\Sigma_i z_i$ be m. Then $\Sigma_i x_i = \Sigma_i y_i = n - m$. So

$$\begin{aligned}
\sum_{|S_i \cap S_j| = l} \Pr[S_i \wedge S_j] &= \sum_m \sum_{\substack{z_1 + \cdots + z_l = m \\ x_1 + \cdots + x_{k-l} = n-m \\ y_1 + \cdots + y_{k-l} = n-m}} (\Pr[z_1] \cdots \Pr[z_l])(\Pr[x_1] \cdots \Pr[x_{k-l}]) \\
&\qquad \times (\Pr[y_1] \cdots \Pr[y_{k-l}]) \\
&= \sum_m \Big(\sum_{z_1 + \cdots + z_l = m} \Pr[z_1] \cdots \Pr[z_l] \Big) \\
&\qquad \times \Big(\sum_{x_1 + \cdots + x_{k-l} = n-m} \Pr[x_1] \cdots \Pr[x_{k-l}] \Big)^2 \\
&= \sum_m \mu_l(m) [\mu_{k-l}(n-m)]^2 .
\end{aligned}$$

Fortunately, we already made (in Lemma 8) the estimates $\mu_l(n) < n^{-1 + l/k + o(1)}$, for $1 \le l \le k-1$. Fix $\epsilon < l/2k$. Then we can pick m_0 such that

$$\mu_l(m) < m^{-1 + l/k + o(1)} , \text{ for } m > m_0 .$$

Since m_0 is a constant, $\mu_l(m) < M$ (some constant), for $m \le m_0$. We estimate the above summation in four parts $(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)$: (i) $m \le m_0$, (ii) $m_0 < m \le n/2$, (iii) $n/2 < m \le n - m_0$, and (iv) $n - m_0 < m$.

Two of the following four estimates are easy; we do the easy ones first:

$$\begin{aligned}\Delta_1 &= \sum_{m \le m_0} \mu_l(m)[\mu_{k-l}(n-m)]^2 \\ &< (n^{-1+(k-l)/k+o(1)})^2 \sum_{m \le m_0} M \\ &= n^{-2+2(k-l)/k+o(1)} = o(1)\,.\end{aligned}$$

The other extreme $(m > n - m_0)$ can be estimated similarly:

$$\begin{aligned}\Delta_4 &= \sum_{m > n-m_0} \mu_l(m)[\mu_{k-l}(n-m)]^2 \\ &< (n^{-1+l/k+o(1)}) \sum_{m > n-m_0} M^2 \\ &= n^{-1+l/k+o(1)} = o(1)\,.\end{aligned}$$

Second and third parts of the sum:

$$\begin{aligned}\Delta_2 &= \sum_{m_0 < m \le n/2} \mu_l(m)[\mu_{k-l}(n-m)]^2 \\ &< (n^{-1+(k-l)/k+o(1)})^2 \sum_{m_0 < m \le n/2} m^{-1+l/k+\epsilon} \\ &= n^{-2l/k+o(1)} \sum_{m_0 < m \le n/2} m^{-1+l/k+\epsilon}\end{aligned}$$

It now suffices to estimate the sum by an integral over the full range $0 \le m \le n$:

$$\begin{aligned}\Delta_2 &< n^{-2l/k+o(1)}\left(\int_{m=0}^{n} m^{-1+l/k+\epsilon} + O(1)\right) \\ &= n^{-2l/k+o(1)}(m^{l/k+\epsilon}|_0^n + O(1)) \\ &= n^{-l/k+\epsilon+o(1)} = o(1)\,.\end{aligned}$$

Similarly,

$$\begin{aligned}\Delta_3 &= \sum_{n/2 < m \le n-m_0} \mu_l(m)[\mu_{k-l}(n-m)]^2 \\ &< (n^{-1+l/k+o(1)}) \sum_{n/2 < m \le n-m_0} [(n-m)^{-1+(k-l)/k+\epsilon}]^2 \\ &= (n^{-1+l/k+o(1)}) \sum_{n/2 < m \le n-m_0} (n-m)^{-2l/k+2\epsilon}\end{aligned}$$

Estimating by integration over the full range, st:

$$\Delta_3 < n^{-1+l/k+o(1)}\left(\int_{m=0}^{n} (n-m)^{-2l/k+2\epsilon} + O(1)\right)$$
$$< n^{-1+l/k+o(1)}(-(n-m)^{1-2l/k+2\epsilon}|_0^n + O(1))$$
$$= n^{-l/k+2\epsilon+o(1)} = o(1)\,.$$

Thus,

$$\sum_{|S_i\cap S_j|=l} \Pr[S_i \wedge S_j] = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = o(1)$$

and hence the claim,

$$\sum_{i\sim j}\Pr[S_i \wedge S_j] = \sum_{l=1}^{k-2} o(1) = o(1)\,. \qquad \blacksquare$$

Let $s = \{S_1, \ldots, S_t\}$ be a collection of pairwise disjoint representations of n. We would like to show that, when $|s| < C_1 \log n$, deleting sets S_i from $\mathscr{S}$ does not hurt $r_k(n)$ much. For this purpose, set $r_k(n; s) = |\{\{x_1, \ldots, x_k\} : x_1 + \cdots + x_k = n; x_i$ distinct and $x_i \in \mathscr{S}\backslash s\}|$ and define $\mu' = E[r_k(n; s)]$.

Lemma 12.

$$|s| < C_1 \log n \Rightarrow \mu' = \Theta(\log n)\,.$$

Proof. By definition,

$\leq m \leq n$:

$$\mu' = \sum_{\substack{x_1+\cdots+x_k=n\\ 1<x_1<\cdots<x_k<n\\ x_1,\ldots,x_k\notin\cup_i S_i}} \Pr[x_1]\cdots\Pr[x_k]$$
$$> \sum^{**}_{x_1,\ldots,x_k\notin\cup_i S_i} \Pr[x_1]\cdots\Pr[x_k]$$

where Σ^{**} denotes the special summation (of Lemma 5) with the x_i's restricted to *strips* of width αn; i.e., the summation is over $x_1, \ldots, x_k$ such that

i. $x_1 + \cdots + x_k = n$.
ii. $n\delta_n < x_1 < \alpha n,\ \alpha n < x_2 < 2\alpha n, \ldots, (k-2)\alpha n < x_{k-1} < (k-1)\alpha n$.

$$\mu' \geq \sum^{**} \Pr[x_1]\cdots\Pr[x_k] - \sum_j \sum^{**}_{x_j\in\cup_i S_i} \Pr[x_1]\cdots\Pr[x_k]\,,$$
$$> [b_1 + o(1)]\log n - \sum_j \sum^{**}_{x_j\in\cup_i S_i} \Pr[x_1]\cdots\Pr[x_k]\,,$$

where the above estimate follows from the estimate (from below) for μ in Lemma 5. We would now like to show that for $j = 1, \ldots, k$

$$\sum^{**}_{x_j\in\cup_i S_i} \Pr[x_1]\cdots\Pr[x_k] = o(\log n)\,.$$

Without loss of generality, we show that for x_k(i.e., $j = k$). As $S_1, \ldots, S_{C_1 \log n}$ are pairwise disjoint

$$\sum_{x_k \in \cup_i S_i}^{**} \Pr[x_1] \cdots \Pr[x_k] = (C_1 \log n) \sum_{x_k \in S_1}^{**} \Pr[x_1] \cdots \Pr[x_k]$$
$$< (C_1 \log n) \mu_{k-1}(n - x_k)$$

But our choice of the range for Σ^{**} was such that $x_k < n - (k-1)\alpha n$. Thus

$$\sum_{x_k \in \cup_i S_i}^{**} \Pr[x_1] \cdots \Pr[x_k] = (C_1 \log n)(n^{-1+(k-1)/k+o(1)}) = o(\log n) .$$

[In the general case, the range for x_j ensures that $(n - x_j) > (1 - j\alpha)n$, allowing us the estimate $\mu_{k-1}(n - x_j) < n^{-1+(k-1)/k+o(1)}$.] Thus

$$\sum_j \sum_{x_j \in \cup_i S_i}^{**} \Pr[x_1] \cdots \Pr[x_k] = o(\log n)$$

and hence

$$\mu' > [b_1 + o(1)] \log n - o(\log n) .$$

On the other hand, we have, trivially, $\mu' < \mu < [b_2 + o(1)] \log n$, completing the proof of the lemma. ■

We are now set to complete the second part of the main theorem.

Theorem 3. *A.a.* $\exists\, n_1$ *s.t.* $r_k(n) > C_1 \log n, \quad n > n_1$.

Proof. We would first want to show the probability that $r_k^*(n) \le C_1 \log n$ is *small*, for all n. Then we could complete the proof by using the Borel–Cantelli lemma. Recall that s denotes a collection of pairwise disjoint representations of n. We want to estimate

$$\Pr[\exists \mathrm{s} : |\mathrm{s}| \le C_1 \log n, \quad \mathrm{s} \text{ maximal}]$$

$$= \sum_{i=0}^{C_1 \log n} \Pr[\exists \mathrm{s} : |\mathrm{s}| = i, \quad \mathrm{s} \text{ maximal}]$$

$$= \sum_{i=0}^{C_1 \log n} \sum_{\substack{S_1, \ldots, S_i \\ \text{pairwise} \\ \text{disjoint}}} (\Pr[S_1 \wedge \cdots \wedge S_i] \times \Pr[\{S_1, \ldots, S_i\} \text{ is maximal}])$$

$$= \sum_{i=0}^{C_1 \log n} \Bigg(\sum_{\substack{S_1, \ldots, S_i \\ \text{pairwise} \\ \text{disjoint}}} \Pr[S_1 \wedge \cdots \wedge S_i] \Bigg) \times \Pr[\{S_1, \ldots, S_i\} \text{ is maximal}] \qquad (**)$$

We already know how to estimate the first factor. By the *disjointness lemma*,

$$\sum_{\substack{S_1, \ldots, S_i \\ \text{pairwise} \\ \text{disjoint}}} \Pr[S_1 \wedge \cdots \wedge S_i] \leq \mu^i / i!$$

We estimate the second factor by the *correlation inequality*! Let B_j be the event that a set $S_j' \subset \mathcal{S} \backslash s$ of k distinct numbers is a representation of n. Then,

$$\Pr[\{S_1, \ldots, S_i\} \text{ is maximal}] = \Pr[\cap_j \bar{B}_j]$$

By the correlation inequality (Lemma 3),

$$\begin{aligned} \Pr[\cap_j \bar{B}_j] &\leq \left[\prod_j \Pr[\bar{B}_j]\right] e^{2\Sigma_{j\sim j'} \Pr[B_j \wedge B_{j'}]} \\ &= \left[\prod_j (1 - \Pr[B_j])\right] e^{2\Sigma_{j\sim j'} \Pr[B_j \wedge B_{j'}]} \\ &\leq e^{-\Sigma_j \Pr[B_j]} e^{2\Sigma_{j\sim j'} \Pr[B_j \wedge B_{j'}]} . \end{aligned}$$

With foresight, we have computed the two exponents in Lemmas 11 and 12 preceding this theorem! $r_k(n; s)$ is precisely the number of the probabilistic events B_j. Thus by Lemma 12,

$$\sum_j \Pr[B_j] = E[r_k(n; s)] = \mu' > [b_1 - o(1)] \log n .$$

By Lemma 11, the correlation

$$2 \sum_{j\sim j'} \Pr[B_j \wedge B_{j'}] \leq 2 \sum_{i\sim j} \Pr[S_i \wedge S_j] = o(1) .$$

Together these facts imply

$$\Pr[\{S_1, \ldots, S_i\} \text{ is maximal}] < e^{-[b_1 - o(1)] \log n} .$$

Plugging this in $(**)$ yields

$$\begin{aligned} \Pr[\exists s\colon |s| \leq C_1 \log n, s \text{ maximal}] &< \sum_{i=0}^{C_1 \log n} \left(\frac{\mu^i}{i!}\right) e^{-[b_1 - o(1)] \log n} \\ &= e^{-[b_1 - o(1)] \log n} \sum_{i=0}^{C_1 \log n} \frac{\mu^i}{i!} \\ &< e^{-[b_1 - o(1)] \log n} \left(\frac{e}{C_1}\right)^{C_1 \mu} , \text{ provided that} \\ & \quad 0 < C_1 < 1 , \\ &< e^{-[b_1 - o(1)] \log n} \left(\frac{e}{C_1}\right)^{C_1 [b_2 + o(1)] \log n} \end{aligned}$$

We choose C_1 small enough so that $(e/C_1)^{C_1[b_2+o(1)\log n} < e^{\log n}$. This is possible, since $(e/C_1)^{C_1} \to 1$ as $C_1 \to 0^+$.

$$\Pr[\exists s\colon |s| \le C_1 \log n, \text{s maximal}] < e^{-[b_1 - o(1)] \log n}\, e^{\log n}$$
$$= n^{-b_1 + 1 + o(1)}$$

Recall that $b_1 = D_k C^k$. So by choosing $C > (3/D_k)^{1/k}$, we can make $b_1 > 3$. [Note that this choice makes the condition $C > (1/3D_k)^{1/k}$ in Lemma 7 superfluous.] Thus,

$$\Pr[\exists s\colon |s| \le C_1 \log n,\ \text{s maximal}] < n^{-2+o(1)}\,.$$

But this implies

$$\sum_{n=1}^{\infty} \Pr[\exists s\colon |s| \le C_1 \log n,\ \text{s maximal}] = \sum_{n=1}^{\infty} n^{-2+o(1)} < \infty\,,$$

allowing us to conclude, by the Borel–Cantelli lemma,

$$\text{a.a.} \quad \exists\, n_1 \quad \text{s.t.} \quad r_k(n) > C_1 \log n\,, \quad \text{for } n > n_1\,.$$

Theorems 2 and 3 together imply the main theorem (Theorem 1).

Alternative Proof of Theorem 3. We have realized Theorem 3 can be proved with much less effort with a very recent result of Svante Janson. (In particular, Lemma 12 becomes unnecessary.) Janson's result [7] is a generalization of his original correlation inequality and can be described as follows. Let $A_1, \ldots, A_n$ be the events as defined in the correlation inequality lemma. Further let N denote the (random) number of events in the family $\{A_i\}$ which occur. Define

$$\mu = E[N] = \sum_i \Pr[A_i] \text{ and } \delta = \frac{1}{\mu} \sum_{i \sim j} \Pr[A_i \wedge A_j]$$

Theorem (S. Janson). If $0 \le \epsilon \le 1$, then

$$\Pr[N \le (1-\epsilon)\mu] \le e^{-[1/2(1+\delta)]\epsilon^2 \mu}\,.$$

(Note that correlation inequality is the special case when $\epsilon = 1$.)

To prove Theorem 3 we first identify A_i to be the event that a valid representation S_i of n is present in our random set $\mathcal{S}$. Then N is precisely $r_k(n)$. By Janson's theorem it would follow, for $0 \le C_1 \le 1$

$$\Pr[r_k(n) \le C_1 \mu] \le e^{-[1/2(1+\delta)](1-C_1)^2 \mu}$$

where $\delta = (1/\mu) \sum_{i \sim j} \Pr[A_i \wedge A_j] = o(1)$, by Lemma 11. We also know, $\mu > [b_1 + o(1)]\log n$. Thus

$$\Pr[r_k(n) \le C_1 \mu] \le e^{-\{1/2[1+o(1)]\}(1-C_1)^2 [b_1 + o(1)]\log n}\,.$$

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Once again, we observe that C_1 can be chosen arbitrarily small, and b_1 can be made greater than 4, so that

$$\Pr[r_k(n) \le C_1\mu] \le n^{-2+o(1)} .$$

Theorem 3 now follows by simply applying the Borel–Cantelli lemma.

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