Unique Coloring of Planar Graphs

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Dedication

This Thesis is Dedicated to the Everlasting Trinity,

God of the Old and New Testaments.

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First and foremost I would like to thank the God of the Old and New Testaments, for "from Him and through Him and to Him are all things" (Romans 11:36). Indeed, without His continued kind providence, I would not exist let alone have enjoyed the many favorable circumstances that have made it possible for me to pursue to near-completion a Ph.D. in mathematics.

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A graph G is said to be uniquely $k-$vertex colorable if there is exactly one partition of the vertices of G into k independent sets, and uniquely edge $k-$colorable if there is exactly one partition of the edges of G into k matchings. This thesis explores unique coloring and positively resolves a 1977 conjecture of Fiorini, Wilson and independently Fisk, that a uniquely edge 3-colorable cubic planar graph with at least four vertices always contains a triangle. This is equivalent to the statement that every uniquely vertex 4-colorable planar graph has a vertex of degree three and implies that every such graph can be constructed from the complete graph on four vertices by repeatedly adding vertices of degree three. We give a computer-assisted proof of the conjecture. More precisely, using the techniques employed in the proof of the Four-Color Theorem we prove from first principles that every "internally 6-connected" planar triangulation has at least two 4-colorings. The Four-Color Theorem is a corollary.

Chapter 1

Introduction, Overview and Notation

1.1 Statement of the Principal Result

A graph is *cubic* if every vertex has degree three. Two functions f and g with identical domain and finite range $B = \{1, 2, \ldots, k\}$ are said to be *equivalent* if $\{f^{-1}(\{1\}), f^{-1}(\{2\}), \ldots, f^{-1}(\{k\})\} = \{g^(\{1\}), g^{-1}(\{2\}), \ldots, g^{-1}(\{k\})\}$. An *edge-coloring* of a graph is a function c from the edges of a graph to a set of colors having the property that if two edges share a common vertex as an endpoint, then c assigns them different colors. An *edge-k-coloring* is an edge-coloring in which k colors are used. A graph G is *uniquely edge-k-colorable* if there is an edge-k-coloring c such that every other edge-k-coloring of G is equivalent to c. It is equivalent to say that G is uniquely edge-k-colorable if there is exactly one partition of the edges of G into exactly k matchings. In 1977 Fiorini and Wilson ([1]) conjectured the following:

Conjecture 1.1.1 *Every uniquely edge-3-colorable cubic planar graph on at least 4 vertices contains a triangle.*

In the same year an equivalent form of this was posed as an unsolved problem by Fisk in [2]. Heretofore this problem will be referred to as the Fiorini-Wilson-Fisk Conjecture.

There have been many references to this problem in the literature, (see [1], [2], [3], [4], [5]. [6], [7]), and some partial results concerning the structure of a minimum counterexample were discovered as recently as 1995 [5]. These partial results as well as the technique used to prove the Four Color Theorem (see [8]) have been combined to produce a positive proof of the Fiorini-Wilson-Fisk Conjecture. This is the principal result of this thesis.

This result also gives a characterization of all uniquely edge-3-colorable planar graphs having at least 4 vertices. Specifically, every uniquely edge-3-colorable cubic planar graph with at least 4 vertices can be obtained from K_4, (the complete graph on 4 vertices) by repeatedly replacing a vertex w with neighbors x_1, x_2 and x_3 by a triangle with vertex set $\{w_1, w_2, w_3\}$ where w_i is joined to x_i by an edge for every $1 \leq i \leq 3$. The proof that this characterizes all uniquely edge-3-colorable graphs on more than 4 vertices follows easily from the truth of the Fiorini-Wilson-Fisk Conjecture. To see this, note first that K_4 is uniquely edge-3-colorable, and that the above operation applied to a uniquely edge-3-colorable cubic graph results in another uniquely edge-3-colorable cubic graph. Conversely, given a cubic planar uniquely edge-3-colorable graph, the Fiorini-Wilson-Fisk Conjecture implies the existence of a triangle whose contraction (the reverse of the operation above) results in a uniquely edge-3-colorable cubic planar graph with fewer vertices. By induction, this smaller graph can be constructed in the prescribed manner and thus the original graph can also be constructed in the prescribed manner.

1.2 Overview of how the proof proceeds

A *vertex-k-coloring* of a graph G is a function $c : V(G) \rightarrow \{1, \ldots, k\}$ such that if two vertices x, y are joined by an edge, then $c(x) \neq c(y)$. A graph is *uniquely vertex-k-colorable* if it has one exactly one vertex-k-coloring up to permutation of colors. In this section we state the main theorems to give the reader a broad overview of the method used to prove the Fiorini-Wilson-Fisk Conjecture. We first translate the Fiorini-Wilson-Fisk Conjecture to an equivalent statement concerning vertex-4-colorings.

Conjecture 1.2.1 *A uniquely vertex-4-colorable simple planar graph has a vertex of degree three.*

This conjecture stated in terms of vertex colorings appeared as early as 1977 as an open problem in a paper of Fisk. (Section I, problem 11 in [2]).

Theorem 1.2.1 *The above formulation is equivalent to the Fiorini-Wilson-Fisk Conjecture.*

A more exciting but equivalent version of conjecture 1.2.1 follows in conjecture 1.2.1.

Conjecture 1.2.2 *Every uniquely vertex-4-colorable simple planar graph G arises from a sequence of planar uniquely vertex-4-colorable graphs G_0,G_1,. . .,G_k where $G_0 = K_4$, $G = G_k$ and G_i is formed by taking some embedding of G_{i-1} in the plane and adding a vertex x into some triangular face of G_{i-1} and putting x adjacent to the three vertices incident to that triangular face.*

Theorem 1.2.2 *Conjecture 1.2.1 is equivalent to conjecture 1.2.1.*

A proof of this theorem, although essentially done in Section 1.1 in the context of the edge formulation of the Fiorini-Wilson-Fisk conjecture, is given in Section 2.8.3.

Instead of proving Conjecture 1.2.1 directly, we prove the following theorem which says that "highly connected" planar graphs (the sense of which will be defined in Chapter 3) always have at least two vertex-4-colorings. More precisely, we prove using the same techniques that were used to prove the Four Color Theorem:

Theorem 1.2.3 *Every internally six connected planar triangulation has at least two non-equivalent vertex-*4*-colorings.*

Theorem 1.2.3, combined with the following two theorems, the first of which will be proved in Chapter 3, has two important corollaries:

Theorem 1.2.4 *(Goldwasser and Zhang, [5]) Every minimal counterexample to the Fiorini-Wilson-Fisk Conjecture is internally six connected.*

Theorem 1.2.5 *(Birkhoff, 1913, [9]) Every minimal counterexample to the four color theorem is internally six connected.*

Corollary 1.2.1 *(The Vertex Fiorini-Wilson-Fisk Conjecture) Every simple uniquely vertex-*4*-colorable planar graph has a vertex of degree* 3.

Corollary 1.2.2 *(The Four Color Theorem) Every loopless planar graph admits a vertex-*4*-coloring.*

The proof of Theorem 1.2.3 is split into two components, proving reducibility and proving unavoidability. The ideas of reducibility and unavoidability presuppose the idea of a configuration, which is defined in Chapter 4. The definition of what it means for a configuration to *appear in* a triangulation will also be defined in Chapter 4. Let $\mathcal{K}$ be the set of configurations in Appendix A. The proof of Theorem 1.2.3 then amounts to proving the next two theorems.

Theorem 1.2.6 *(Reducibility) No configuration in $\mathcal{K}$ can appear in a minimum counterexample to Theorem 1.2.3.*

This we prove in Chapter 5 with the aid of a computer. In Chapter 7 and again with the aid of a computer, we prove:

Theorem 1.2.7 *(Unavoidability) For every internally 6-connected triangulation G, there is a configuration in $\mathcal{K}$ that appears in G.*

The recent proof of the Four Color Theorem in Robertson et. al. in [8], uses the same techniques of reducibility and unavoidability. Their proof shows that every internally six-connected triangulation has at least one vertex-4-coloring, whereas in this thesis it is shown that every such graph has at least two vertex-4-colorings.

1.3 Notation

We will use standard set theoretic notation. For the difference of two sets we write $A - B$, and mean this to be the set of all elements of A that do not belong to B. A multi-set will be a set in which individual elements can appear more than once. Unless explicitly stated, sets should be interpreted as regular sets rather than multi-sets.

A *graph* G is an ordered pair (V, E) where E is a multi-set of two element subsets of V. Thus we allow a graph to have multiple edges but forbid loops. Here V will be referred to as the set of *vertices* of G and E will be known as the set of *edges* of G. We will also denote V by $V(G)$ and E by $E(G)$. An edge $e \in E$ of the form $e = \{x, y\}$ is said to have *endpoints* x and y. If two or more edges have the same vertices as endpoints, they are said to be *parallel edges*. A graph with no parallel edges is called a *simple graph*. If H is a graph in which $V(H) \subset V(G)$ and $E(H) \subset E(G)$, then we say H is a *subgraph of* G. If every edge of G with both endpoints in $V(H)$ is also an edge of H, then we say that H is an *induced subgraph* of G. From this definition, it can be seen that for each subset $A \subset V(G)$ there is a unique induced subgraph of G whose vertex set is A, namely the graph H with $V(H) = A$ and with edge set consisting of every edge in G with both endpoints in A. We will call this subgraph the *subgraph of* G *induced by* A. The notation $G - A$ will denote the subgraph induced by the vertex set $V(G) - A$. If $A = \{v\}$ consists of the single vertex v, we will sometimes write $G - v$ in place of $G - \{v\}$. If $F \subset E(G)$, then the graph $G - F$ will refer to the subgraph of G with vertex set $V(G)$ and edge set $E(G) - F$.

A *walk* in a graph G is a sequence of the form $v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$, where $v_i \in V(G)$ for $i = 0, 1, \ldots, k$ and e_i is an edge of G having endpoints v_{i-1} and v_i. If $v_0 = v_k$ then W is called a *closed walk*. If $v_0 = v_k$, and every other pair of vertices W is pairwise distinct, then W is a *circuit* of G provided there is at least one edge in W. The vertices v_0, v_1, …, v_k of W are not necessarily distinct but if they are, W is also called a *path*. If W is a path and $x = v_0$, $y = v_k$, then W is also called a *x-y path* or a *path joining x to y*. The *length* of a path W is defined to be the number of edges in the path, that is, k. If for every pair of vertices $x, y \in V(G)$, there is an x-y

path in G, then G is said to be *connected.* If G is not connected, then we say that G is *disconnected.* A disconnected graph G can be partitioned into maximal connected subgraphs H_1, H_2, …, H_k. The graphs H_1, H_2, …, H_k, are known as *connected components*, or just *components* of G. If there is a vertex x in a connected graph G such that the graph $G - x$ is disconnected, then x is said to be a *cut-vertex of* G or just a *cut-vertex.* If G has no cut vertex, then G is said to be $2-$*connected.* More generally, if $G - X$ is connected for each $X \subset V(G)$ with $|X| < k$, then we say that G is k-connected. A *vertex coloring* of a graph is a function from the vertices of a graph to a set of colors such that if two vertices are joined by an edge they receive different colors. A unique vertex-k-coloring of a graph G is a vertex coloring c using exactly k distinct colors and having the property that every other vertex-coloring of G using exactly k colors can be obtained from c by permuting colors. An *independent set* of a graph G is a subset A of $V(G)$, such that no edge of G has both endpoints in A. Every vertex-k-coloring of G partitions $V(G)$ into k non-empty independent sets, and so a graph is uniquely vertex-k-colorable if and only if there is exactly one partition of the vertices of G into k non-empty independent sets.

An *edge-coloring* of a graph is a function c from the edges of a graph to a set of colors having the property that if two edges share a common vertex as an endpoint, then c assigns them different colors.

1.3.1 Planar Graphs and Topology

1.3.1.1 Underlying Topology

Let Σ be the sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\} \subset R^3$ considered as a topological space. For $X \subset \Sigma$, $\bar{X}$ will denote the closure of X in Σ. A *line* of Σ is a subspace $h([0, 1])$, where h is a homeomorphism of $[0, 1]$ onto $h([0, 1])$. The *endpoints* of a line $h[0, 1]$ are the points $x = h(0)$ and $y = h(1)$. A subspace $A \subset \Sigma$ is *arc-wise connected* if any pair of points $x, y \in A$ are the endpoints of some line, all of whose points are in A. If $h([0, 1])$ is a line and x_1, x_2 are two points such that $h(t_1) = x_1$ and $h(t_2) = x_2$ for some $t_1, t_2 \in [0, 1]$, then we will let $h[x_1, x_2]$ denote the line in Σ that consists of the set $h([min\{t_1, t_2\}, max\{t_1, t_2\}])$. A *curve* of Σ is a homeomorphism C mapping $\{(x, y) : x^2 + y^2 = 1\}$ onto a subset of Σ.

Let C be a curve in Σ. We will make frequent use of the Jordan Curve Theorem, which states that $\Sigma - C$ consists of two disjoint arc-wise connected sets. An *open disc* is a subset of Σ that is homeomorphic to $\{(x, y) \in R^2 : x^2 + y^2 < 1\}$ and a *closed disc* is a subset of Σ that is homeomorphic to $\{(x, y) \in R^2 : x^2 + y^2 \leq 1\}$. For $u \in \Sigma$ and $\epsilon > 0$, $D(u, \epsilon)$ will denote the open disc in Σ centered at u and having radius $\epsilon > 0$. The set $\overline{D(u, \epsilon)} - D(u, \epsilon)$ will be referred to as the *circle centered at u of radius ϵ* and will be denoted by $C(u, \epsilon)$. If $A \subset \Sigma$ then the *boundary of A*, denoted by $bd(A)$, is defined to be the set $\overline{A} - A$.

1.3.1.2 Planar Graphs

Intuitively, a graph G is *planar* if it can be "drawn" in the plane in such a way that no two edges "cross" each other. Having laid the necessary topological foundation, we follow Massey ([10]) in making this precise.

A *drawing* is a pair (U, V) where U is a closed subspace of Σ, $V \subset U$ is a finite set of points of Σ, and

(i) $U(G) - V(G)$ has only finitely many arc-wise connected components, called *edges*

(ii) for each edge e, the closure of each edge e is a line, and $\bar{e} - e$ consists of two points in $V(G)$, called *endpoints of* e.

A drawing (U, V) naturally gives rise to a *corresponding graph* G with vertex set $V(G) = V$, and edge set $E(G) = U - V(G)$. The set $\Sigma - U(G)$ is a set of arc-wise connected components which will be called *faces* of (U, V). A *drawing of a graph* G is a drawing whose corresponding graph is isomorphic to G. Not all graphs have drawings; if a graph G is isomorphic to a graph which has a drawing, then we say that G is *planar*. Property (ii) excludes graphs with loops. A vertex v and an edge e are *incident* if the edge e has v as an endpoint. A vertex v and a face f are *incident* if $v \in \bar{f}$, and an edge e and f are *incident* if $e \subset \bar{f}$. Two vertices are *adjacent* if they are the endpoints of some edge. Two edges are *adjacent* if they are incident to a common vertex. Two faces are *adjacent* if they are both incident to a common edge. The *degree* of a vertex v in G, denoted $d_G(v)$ or $d(v)$, equals the number of edges of G that are incident to v. If $(U(H), V(H))$ is another drawing in which $V(H) \subset V(G)$ and $E(H) \subset E(G)$, then H is a *subdrawing* of G. If every edge of G which has both endpoints in $V(H)$ is an edge of H, then H is an *induced subdrawing* of G.

A drawing is said to be *planar* if one of the faces is designated as infinite. A drawing is a *triangulation* if every face is incident to exactly three edges. A face which is incident to exactly 3 edges called a *triangle* or is said to be *triangular*. A planar drawing is a *near triangulation* if every face, except possibly the one designated as infinite, is a triangle.

Chapter 2

Survey of Results in Unique Coloring

There is a wealth of literature about the topic of unique coloring. We will split our discussion of it into various categories, discussing the origin of unique coloring, results about unique coloring in general, computational complexity of unique coloring, criticality and unique coloring, uniquely vertex-k-colorable graphs with no small cycles, unique vertex-3-coloring in the plane, unique edge coloring, and the history of the Fiorini-Wilson-Fisk Conjecture as well extensions of it and conjectures related to it.

2.1 Origin of Unique Coloring

The origin of unique coloring appears to have been, perhaps surprisingly, in the field of psychology. There the problem of a signed graph was introduced, together with a coloring of signed graphs, to model a problem in that field [11]. A signed graph S is a ordered pair (G, ϕ), where G is an undirected graph and ϕ is a function $\phi : E(G) \to \{-1, 1\}$. These signed graphs are used in psychology to model the idea of clusterings. From there the idea of colorings and unique colorings a signed graph, closely related to the normal notion of coloring a graph arose in a 1968 paper of Cartwright and Harary [12]. A coloring c of a signed graph is a function from the vertex set of G to $\{1, 2, \ldots, k\}$ having the property that if x and y are two adjacent vertices in G, then

1) If $\phi(\{x, y\}) = 1$ then $c(x) = c(y)$.

2) If $\phi(\{x, y\}) = -1$ then $c(x) \neq c(y)$.

As usual the set $\{c^{-1}(\{i\}) : i \in \{1, 2, \ldots, k\}\}$ defines a partition of the vertices of S into *color classes*. This paper of Cartwright and Harary, as well as a 1967 paper of Gleason and Cartwright [11], established conditions for a signed graph to have a coloring, and introduced the notion of a unique coloring of a signed graph. To wit, a signed graph S is uniquely colorable if there is exactly one partition of S into color classes. Both papers gave fairly simple criterion for a signed graph to be uniquely colorable. In addition, [12] introduced the notion of unique coloring of a "normal" (unsigned) graph G, which is the topic of interest in this thesis.

Under the usual notion of a coloring c of a graph G being a function from the set of vertices to a set of integers (colors) having the property that adjacent vertices receive a different assignment under c, Cartwright and Harary defined a graph G to be uniquely colorable if either G is complete or G has a unique partition of the vertices of G into $t < |V(G)|$ color classes. In this same paper, they showed that if G has a unique coloring with say t colors, then, in fact $t = \chi(G)$, where $\chi(G)$ is the chromatic number of G, that is, the smallest positive integer s for which there is a coloring of G using exactly s colors.

2.2 General Results About Unique Vertex Colorings

2.2.1 Necessary Conditions for a Graph to be Uniquely Colorable

To warm up our understandings of unique coloring we mention some easy necessary consequences of a graph being uniquely vertex-colorable. The first is that the number of colors used in a unique coloring is unique and equals the chromatic number of G.

Proposition 2.2.1 *(Cartwright, Harary) If G has a unique coloring with t colors then $t = \chi(G)$.*

Proof: We may assume that G is not the complete graph on $\chi(G)$ vertices. Clearly $\chi(G) \leq t \leq |V(G)|$, since a unique coloring is also a proper coloring. If $t > \chi(G)$ then $|V(G)| > \chi(G)$ and for any $\chi(G)$-coloring c of G, pick a set of vertices $\{x_1, x_2, \ldots, x_{\chi(G)}\}$ having the property that $c(x_i) = i$. There are at least $t - \chi(G)$ vertices in G other than $\{x_1, \ldots, x_{\chi(G)}\}$ and these can be assigned colors from $\{\chi(G) + 1, \chi(G) + 2, \ldots, t\}$, to get two distinct t-colorings of G.

By this proposition, we may say unambiguously that G is uniquely vertex colorable, and mean that G is uniquely vertex-$\chi(G)$-colorable.

Let G be a graph and let $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ be a unique vertex-k-coloring of G. For i, $j \in \{1, 2, \ldots, k\}$, define $G_{i,j}$ to be the subgraph of G induced by the vertices which c assigns the colors i or j. A very useful necessary condition for G to be uniquely vertex-k-colorable was noticed by Harary et. al. in the following theorem which appears in [13].

Theorem 2.2.1 *(Harary, Hedetniemi, Robinson, 1969) If* $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ *is a unique vertex-k-coloring of* G*, then for all* $i \neq j$, $i, j \in \{1, 2, \ldots, k\}$*, the graph* $G_{i,j}$ *is connected.*

Proof: If some $G_{i,j}$ had two or more components, then by interchanging the colors i and j in exactly one of these components, we would arrive at a valid coloring different than c.

Corollary 2.2.1 *Let* c *be a unique vertex-k-coloring of* G*, let* x *be a vertex in* $V(G)$ *and let* $i \in \{1, \ldots, k\}$*. If* $i \neq c(x)$ *then there is a vertex* $y \in V(G)$ *such that* x *is adjacent to* y *and* $c(y) = i$*. In particular, every vertex of* G *has degree at least* $k - 1$*.*

Proof: Let $v \in V(G)$, let c be a unique vertex-k-coloring of G and let i be a color different from $c(x)$. By Theorem 2.2.1, $G_{i,c(x)}$ is a connected graph, and in particular, x is not an isolated vertex in $G_{i,c(x)}$ because Proposition 2.2.1 insures that some vertex receives the color i. Since there are $k-1$ other colors besides $c(x)$, the minimum degree of G must be at least $k - 1$. This completes the proof of Corollary 2.2.1.

Corollary 2.2.2 *(Harary et al.) If* G *is a uniquely vertex-k-colorable, then* G *has at least* $(k-1)n - \binom{k}{2}$ *edges.*

Proof: Let V_i be the set of vertices colored i. Theorem 2.2.1 insures that for $1 \leq i < j \leq k$, the graph $G_{i,j}$ with vertex set $V_i \bigcup V_j$ is connected. Thus $|E(G_{i,j})| \geq$

$|V_i| + |V_j| - 1$. Summing this inequality over all pairs $i \neq j$, we have that $|E(G)| \geq \sum_{1 \leq i < j \leq k} |V_i| + |V_j| - 1 = (k-1)(|V_1 + V_2 + \ldots V_k|) - \binom{k}{2} = (k-1)n - \binom{k}{2}$, which is the desired result.

Corollary 2.2.3 *(Geller, Chartrand) If G is a uniquely vertex-4-colorable simple planar graph, then any drawing of the graph G is a triangulation. Moreover, for $i \neq j$ and $i, j \in \{1, 2, 3, 4\}$, each subgraph $G_{i,j}$ is a tree.*

Proof: By Euler's formula $|E(G)| \leq 3|V(G)| - 6$, and from Corollary 2.2.2, $|E(G)| \geq 3|V(G)| - 6$, so $|E(G)| = 3|V(G)| - 6$. This implies that any drawing of G must be a triangulation. It also implies that equality holds throughout in the proof of the Corollary 2.2.2, so $|E(G_{i,j})| = |V_i| + |V_j| - 1 = |V(G_{i,j})| - 1$. Since $G_{i,j}$ is connected, it follows that $G_{i,j}$ is a tree. This completes the proof of the corollary.

2.2.2 Assorted Results About Uniquely Colorable Graphs

A function $\phi : V(G) \to V(G')$ is said to be a *homomorphism of the graph G into the graph G'* if it preserves adjacency of vertices, that is, if $\{x, y\} \in E(G)$ implies $\{\phi(x), \phi(y)\} \in E(G')$. If it is true that for every pair of vertices x', $y' \in V(G')$, x' is adjacent to y' in G' if and only if there is a pair x, y of adjacent vertices in G such that $\phi(x) = x'$ and $\phi(y) = y'$, then ϕ is said to be a homomorphism of G *onto* G', and G' is said to be a *homomorphic image* of G. The following propositions appear in [13].

Proposition 2.2.2 *If G is uniquely vertex-k-colorable and H is a homomorphic image of G such that $\chi(H) = k$, then H is uniquely vertex-k-colorable .*

Proposition 2.2.3 *If G is uniquely vertex-k-colorable then G is $(k-1)$-connected.*

Proof: Let A be a set with $|A| \leq k-2$, let c be a unique vertex-k-coloring of G, and let x, $y \in V(G) - A$. There are two distinct colors i, $j \in \{1, \ldots, k\}$ such that no vertex of A has a vertex colored i or j by c. Therefore, $V(G_{i,j}) \bigcap A = \emptyset$. By Corollary 2.2.1, there are vertices u_x and u_y such that x is adjacent to u_x, y is adjacent to u_y and $c(u_x) = c(u_y) = i$. Since $G_{i,j}$ is connected, there is a path P in $G_{i,j}$ joining u_x to u_y and thus there is a path in $G - A$ joining x and y. Thus, $G - A$ is connected. This completes the proof of the proposition.

2.3 Complexity Results for Unique Coloring

The following proposition is obvious.

Proposition 2.3.1 *A graph is uniquely vertex-*1*-colorable if and only if it consists of isolated vertices. A graph is uniquely vertex-*2*-colorable if and only if it is a connected bipartite graph.*

Beyond this there is not much hope of finding a "good" characterization of arbitrary uniquely vertex-k-colorable graphs when $k \geq 3$ because of the following complexity results contained in or implied by the work of Dailey in 1981 [14].

Theorem 2.3.1 *The following decision problems are NP-Complete:*

1) Given a graph G and a vertex-k-coloring c of G, is there a vertex-k-coloring c' of G that is not equivalent to c?

2) Given an integer k and a graph G, does G have either 0 *or at least* 2 *vertex-k-colorings?*

The result of Dailey probably dooms any possibility of a polynomial time algorithm for problems 1) or 2) above. In [7], the authors pose the question of whether there is a polynomial time algorithm for deciding whether a given planar graph is uniquely vertex-3-colorable. This problem is still open as far as this author knows.

2.4 A Sufficient Condition for Determining Unique Vertex-k-Colorability

The following sufficient condition for a graph to be uniquely vertex-k-colorable was given by Bollobas in [15].

Theorem 2.4.1 *Let k be an integer greater than one, let G be a vertex-k-colorable graph on n vertices, and let $\delta(G)$ denote the minimum degree of G. If $\delta(G) > \frac{(3k-5)n}{(3k-2)}$ then G is uniquely vertex-k-colorable . Moreover, if G has a vertex-k-coloring in which $G_{i,j}$ is connected for every $1 \leq i < j \leq k$, and $\delta(G) > \frac{(k-2)n}{(k-1)}$, then G is uniquely vertex-k-colorable. These results are best possible.*

This was generalized by Dmitriev according to a review of [16]. As we can see, this condition will apply only to very dense graphs.

2.5 Critical Uniquely Colorable Graphs and Forbidden Subgraphs

A graph G is said to be *k-critical* if $\chi(G) = k$ but $\chi(G-A) \leq k-1$ for every nonempty subset $A \subset V(G)$. A graph is *k-edge-critical* if $\chi(G) = k$ and $\chi(G-e) \leq k-1$ for every $e \in E(G)$. Harary et. al. pointed out in [13] that the only graph which is both k-critical and uniquely vertex-k-colorable is K_k, the complete graph on k vertices. This follows because for any two nonadjacent vertices x and y in a k-critical graph G, there is a vertex-k-coloring of G in which x and y receive the same color and there is a different vertex-k-coloring in which they receive different colors. This shows a fundamental difference between critical graphs and uniquely vertex-k-colorable graphs.

We can also extend the above concepts of critical and edge-critical to unique coloring, and both of these extensions have been considered in the literature on unique coloring. We will say that a graph G is *critically-uniquely vertex-colorable* if it is uniquely vertex-colorable and no proper induced subgraph of G is uniquely vertex-colorable. Nešetřil studied this problem in 1972 in [17]. The following theorems appear in this paper.

Theorem 2.5.1 *If G is a critically uniquely vertex-k-colorable graph then either G is isomorphic to K_k or $\delta(G) \geq k$.*

Theorem 2.5.2 *If G is a critically uniquely vertex-3-colorable graph then G is 3-connected.*

Theorem 2.5.3 *The subgraph induced by a cutset of a critically uniquely vertex-colorable graph contains two non-adjacent vertices.*

Theorem 2.5.3 has a related counterpart for critical graphs which appears in [13].

Theorem 2.5.4 *(Harary, Hedetniemi, and Robinson, 1969) No cutset of a k-critical graph induces a uniquely vertex-$(k-1)$-colorable graph.*

Nešetřil also showed in the following theorem of [17] that the class of induced subgraphs of uniquely vertex colorable graphs is quite rich.

Theorem 2.5.5 *(Nešetřil, 1972) Let H be any graph and let c be any $\chi(H)$-coloring of H. There is a uniquely vertex-$\chi(G)$-colorable graph G such that H is an induced subgraph of G and the unique coloring of G restricted to $V(H)$ equals c.*

Investigations about edge-critical uniquely vertex-colorable graphs were carried out by Müller in [18], Aksionov in [19] and Steinberg and Mel'nikov in [20]. A uniquely vertex-k-colorable graph G is said to be *uniquely edge-critical* if for every edge $e \in E(G)$, the graph $G - e$ is not uniquely vertex-k-colorable. Corollary 2.2.2 shows that every uniquely vertex-k-colorable graph on exactly $(k-1)n - \binom{k}{2}$ edges is uniquely edge-critical. Aksionov conjectured in [19] that if G was an edge-critical uniquely vertex-3-colorable planar graph on n vertices then $|E(G)| = 2n-3$, the value of the above formula when $k = 3$. This was disproved the same year by Steinberg and Mel'nikov in [20]. They posed the following problem: Find an exact upper bound for the number of edges in a planar, edge-critical uniquely vertex-3-colorable graph. Note that when vertex-3-coloring is replaced by vertex-4-coloring, then Euler's formula implies that every edge critical uniquely vertex-4-colorable planar graph G on n vertices has exactly $3n - 6$ edges.

Results of Müller in [18] characterize all induced subgraphs of edge critical uniquely vertex-colorable graphs.

Theorem 2.5.6 *(Müller, 1979) A graph G is an induced subgraph of some uniquely edge-critical graph with chromatic number k if and only if $\chi(G) \leq k$, and for every* *edge* $e = \{x, y\} \in E(G)$, *there is a homomorphism* $f_e : G - e \to K_k$ *satisfying* $f_e(x) = f_e(y)$.

He also proved a similar characterization for induced subgraphs of edge-critical uniquely vertex colorable graphs with large girth.

Theorem 2.5.7 *(Müller, 1979) Let* k *and* r *be positive integers. The graph* G *is an induced subgraph of some uniquely edge-critical graph* G *having* $\chi(G) = k$ *and minimum circuit length at least* r *if and only if the minimum circuit length of* G *is at least* r, $\chi(G) \leq k$, *and for every edge* $e = \{x, y\} \in E(G)$, *there is a homomorphism* $f_e : G - e \to K_k$ *satisfying* $f_e(x) = f_e(y)$.

2.6 Uniquely Colorable Graphs with Large Girth

The complete graph on k vertices is uniquely vertex-k-colorable . For various reasons it becomes tempting to conjecture that every uniquely vertex-k-colorable graph has a subgraph isomorphic to K_k. In fact, the results of Chartrand and Geller in [21] and this thesis show that this is the case for any integer k and any uniquely vertex-k-colorable planar graph G. However in general it is not true as the following theorem of Harary et. al. in [13, 22] shows:

Theorem 2.6.1 *For every* $k \geq 3$, *there is a uniquely vertex-*k*-colorable graph with no subgraph isomorphic to* K_k.

A number of even stronger results that generalize a classic result of Erdős soon followed. In [17, 18, 23], it was shown that for every $k \geq 3$ and every positive integer

g there are uniquely vertex-k-colorable graphs with no circuits of length g or less. We cite one of these results, due to Müller, which appears in [18].

Theorem 2.6.2 *(Müller, 1979) Let g and k be positive integers and let H be any k-vertex-colorable graph which has no circuits of length less than g. Then there is a uniquely vertex-k-colorable graph G with no circuits of length less than g which has an induced subgraph isomorphic to H.*

Another desirable structure in a uniquely vertex-k-colorable graph G is a vertex of degree exactly $k-1$. When such a vertex exists, one can make induction arguments because $G-x$ is uniquely vertex-k-colorable. Of course, this can not be expected in general because if $G \neq K_k$ and G is uniquely vertex-k-colorable then it is possible to add edges to G and preserve the unique vertex-k-colorability of G. One might then conjecture that if G has a minimum ($= (k-1)n - \binom{k}{2}$ by Corollary 2.2.2 in Section 2.2) number of edges then G has a vertex of degree $k-1$. This is true when $k=4$ and G is a planar graph, and is the central result of this thesis. In general this fails according to an English summary of a paper written by Dmitriev in Russian [24].

Theorem 2.6.3 *(Dmitriev, 1982) Let G be a uniquely vertex-k-colorable graph on n vertices with exactly $(k-1)n - \binom{k}{2}$ edges. Then the minimum degree $\delta(G)$ of G satisfies $k-1 \leq \delta(G) \leq 2k-3$. Moreover, for every $\delta \in \{k-1, k, \ldots, 2k-3\}$, there is a uniquely colorable graph with exactly $(k-1)n - \binom{k}{2}$ edges and with minimum degree δ.*

A 1990 conjecture of Xu in [25] asks if a uniquely vertex-k-colorable graph G with the minimum number of edges always contains a K_k.

To this we add the weaker conjecture:

Conjecture 2.6.1 *Every uniquely vertex-k-colorable graph with exactly* $(k-1)n-\binom{k}{2}$ *edges has either a subgraph isomorphic to* K_k *or a vertex of degree at most* $k-1$.

2.7 Unique Edge Coloring

2.7.1 Basic Results About Unique Edge Coloring

We first state some fundamental facts about about unique edge-colorings and then discuss characterizations of uniquely edge-k-colorable simple graphs.

The ideas of Theorem 2.2.1 and Corollary 2.2.2 are used in proving the the following about unique-edge-coloring which appears in [4].

Proposition 2.7.1 *(Fiorini and Wilson, 1978) If* G *is a uniquely edge-k-colorable graph on* n *vertices and* e *edges then*

1. Each edge of G *is adjacent to edges of every other color.*

2. The subgraph $H_{\alpha,\beta}$ *of* G *induced by edges colored either* α *or* β *is either a path or a circuit.*

3. $\frac{1}{2}nk-\binom{k}{2}\leq e\leq\frac{1}{2}nk$ *and both these bounds can be attained.*

4. If G *is a regular* $k-valent$ *graph* $(k\geq 3)$ *and if* H *is a graph obtained from* G *by subdividing any edge of* G*, then* H *is* k*-critical.*

The following theorem obtained by Greenwell and Kronk in [26], and independently by Fiorini [3], shows that with one exception, uniquely edge-k-colorable graph G is of class one, that is $\chi'(G)=\Delta(G)=k$, where $\Delta(G)$ denotes the maximum degree in G.

Theorem 2.7.1 *(Greenwell and Kronk, Fiorini) If G is a uniquely edge-k-colorable simple graph then unless G is isomorphic to K_3, G is of class one, that is $\chi'(G) = \Delta(G) = k$.*

2.7.2 Characterizing Unique Edge-k-Coloring for $k \geq 4$

We now discuss characterizations of simple uniquely edge-k-colorable graphs. If $k = 1$ it follows that the only uniquely edge-k-colorable graph consists of a graph with isolated vertices and edges. If $k = 2$ the only uniquely edge-k-colorable graph is an even cycle or a path. For $k \geq 4$ Fiorini in [3] and Wilson in [27] made a conjecture which we denote by $U(k)$.

$U(k)$: The only connected uniquely-k-edge colorable graph is $K_{1,k}$, the k-star.

We will now outline a proof that for $k \geq 5$, that $U(4)$ implies $U(k)$.

Let $k \geq 5$ and suppose that G is a uniquely edge-k-colorable graph with a unique edge-k-coloring $c : E(G) \to \{1, \ldots, k\}$. We first show that each vertex in G has degree either k or 1. A vertex cannot have degree 0 because the graph is connected and it cannot have degree greater than k because if it did, it would not be edge-k-colorable. Suppose some vertex y has degree d, where $2 \leq d \leq k-1$ and without loss of generality let 1 and 2 be two colors such that there are edges incident to y which receive colors 1 and 2 and let 3 be a color such that there is no edge incident to y that receives color 3. Consider the subgraph of G induced by the edges of G which receive some color in $\{1, 2, 3, 4\}$. This subgraph must include vertex y and the two edges incident to y which are colored 1 and 2. It must be uniquely edge-4-colorable, because otherwise the original coloring would not be a unique edge-k-coloring. Assuming $U(4)$ to be true, this subgraph must be $K_{1,4}$. But $y \in V(K_{1,4})$ has degree at least two and at most three which is a contradiction. Thus every vertex has degree 1 or k in G. Using similar reasoning, it can be shown that there is exactly one vertex of degree k, and that for every color a in $\{1, \ldots, k\}$, there is exactly one edge colored a. This shows that G is isomorphic to $K_{1,k}$, as desired.

Andrew Thomason proved that $U(4)$ was true in [28], and therefore the only case that remains to be considered is the case $k = 3$. We will now discuss some of the research and results surrounding this case.

2.8 Unique edge-3-coloring and the Fiorini-Wilson-Fisk Conjecture

The state of unique edge-colorings of simple graphs is nice because there are concise characterizations of uniquely edge-k-colorable graphs for every integer $k \neq 3$. Moreover, the result of this thesis characterizes uniquely edge-3-colorable cubic planar graphs. Thus the only cases remaining for unique edge-3-coloring are when G is non-planar or G is not 3-regular. We shall see that many of the conjectures about uniquely edge-3-colorable graphs which are non-planar or not $3-$regular claim a fairly specific structure for these graphs.

2.8.1 The Fiorini-Wilson-Fisk Conjecture and its Precursors

We will discuss the development of the Fiorini-Wilson-Fisk Conjecture as well as other conjectures about unique edge-3-coloring.

One of the first conjectures about unique edge-3-colorability is that of Greenwell and Kronk in [26]:

Conjecture 2.8.1 *(Greenwell and Kronk, 1973) If G is a uniquely edge-3-colorable cubic graph, then G is planar and contains a triangle.*

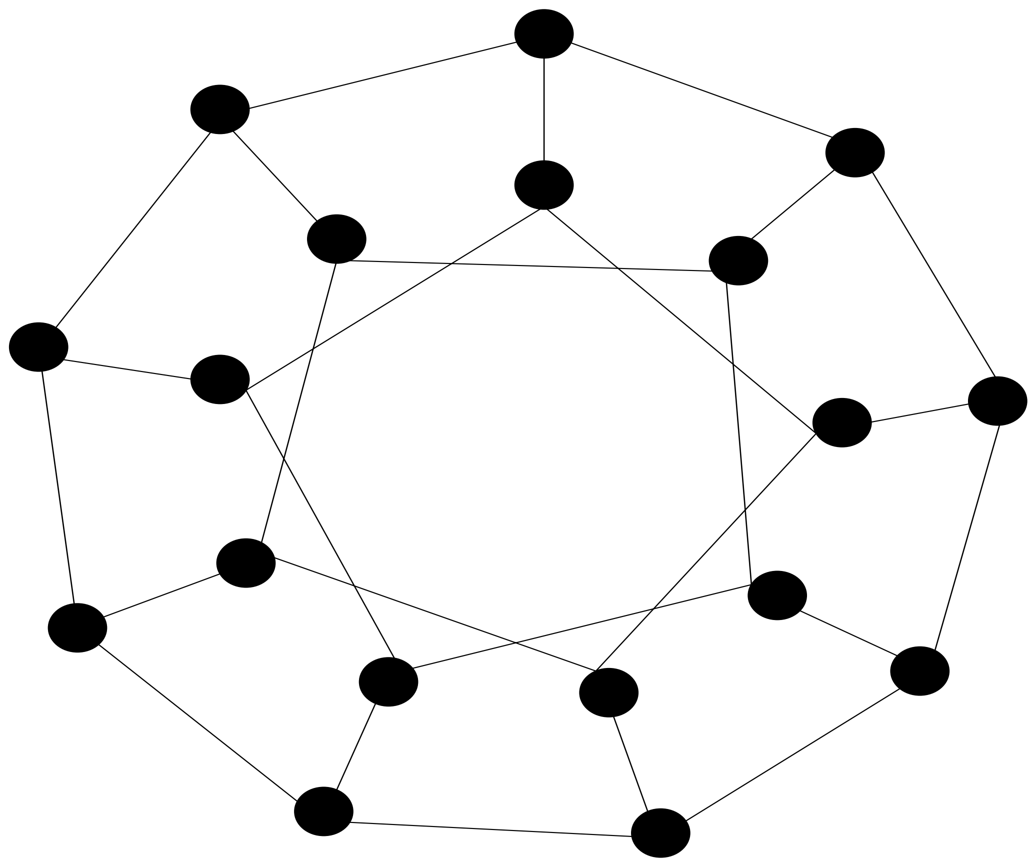

Figure 2.1: The Graph $P(9, 2)$

For this to be true, we would have to interpret the word "graph" in the above statement to mean simple graph, as the graph consisting of two vertices and three parallel edges is cubic, planar, has a unique edge-3-coloring and yet contains no triangle.

More fundamentally, this statement must be modified because of the graph $P(9,2)$, shown in Figure 2.1, which is non-planar, cubic and uniquely vertex-3-colorable.

The existence of $P(9,2)$ may well have led Fiorini and Wilson to propose in [1] what we have been referring to as the Fiorini-Wilson-Fisk Conjecture:

Conjecture 2.8.2 *Fiorini-Wilson-Fisk Conjecture: Every uniquely edge-*3*-colorable cubic planar graph on at least* 4 *vertices contains a triangle.*

The condition "on at least 4 vertices" is added to avoid the case of the two vertex cubic graph with three parallel edges. Notice that both planarity, and the condition that every vertex has degree exactly three are assumed. The vertex equivalent of the Fiorini-Wilson-Fisk Conjecture was stated in Conjecture 1.2.1 and appeared in [2] as an unsolved problem. A strengthening of the vertex version, due to Jensen [29], is given in Section 2.8.5.

2.8.2 Conjectures Which Relax Planarity or Regularity

Let G and H be graphs. If H is isomorphic to a graph obtained from from a subgraph of G by contracting edges, then G is said to have a *H-minor*. A graph G contains a *Petersen minor* if G has the well known Petersen graph as a minor. The first conjecture concerning non-planar uniquely edge-3-colorable cubic graphs is due to Fiorini and Wilson [4]:

Conjecture 2.8.3 *If G is a uniquely edge-*3*-colorable non-planar cubic graph then G has a triangle or G is isomorphic to $P(9,2)$.*

The graph $P(9,2)$ is a generalized Petersen graph, and it appears in Figure 2.1. There are some other conjectures about uniquely edge-3-colorable cubic graphs G which do not assume planarity. One due to Zhang [5] in 1995 states that

Conjecture 2.8.4 *If G is a uniquely edge-3-colorable simple cubic graph, then G contains a Petersen minor or a triangle.*

A *snark* is a two-edge connected cubic graph that is not edge-3-colorable. Again in 1995, Zhang also conjectured in [5] that

Conjecture 2.8.5 *If G is a uniquely edge-3-colorable, simple, cubic graph, then G contains either a snark as a minor or a triangle.*

In view of the truth of the Fiorini-Wilson-Fisk Conjecture, it would suffice to prove these conjectures for non-planar graphs only. Goldwasser and Zhang have also provided some additional help towards proving the second conjecture in the following theorem [5].

Theorem 2.8.1 *(Goldwasser & Zhang, 1995) If G is a cyclically 4-edge connected uniquely-edge-3-colorable graph and if G has a cyclic 4-edge cut, then G contains a snark as a minor.*

As for the assumption of a uniquely edge-3-colorable graph being sub-cubic, there is the following conjecture of Fiorini in 1973 [3] and Fiorini and Wilson in 1978 [4] which was also brought up by Kriessell [30].

Conjecture 2.8.6 *Every uniquely edge-3-colorable planar graph that is not isomorphic to $K_{1,3}$ contains a triangle.*

2.8.3 Structure of Uniquely Edge-3-Colorable Cubic Planar Graphs

We will say that a planar graph G is a *Fiorini-Wilson-Fisk graph* (or a *FWF-graph* for short) if there is a sequence G_0, G_1, …, G_p, such that G_0 is isomorphic to K_4, G_i arises from G_{i-1} by the operation defined in Section 1.2 of replacing a degree 3 vertex with a triangle, and $G_p = G$.

We will also say that a graph G is a *vertex-Fiorini-Wilson-Fisk* graph if there is a sequence of graphs G_0, G_1, …, G_p embedded in the plane such that G_0 is isomorphic to K_4, $G = G_p$ and G_i arises from G_{i-1} by adding a vertex v and joining v to exactly three vertices in a common facial triangle of G_{i-1}.

We now provide the promised proof of Theorem 1.2.2.

Theorem 2.8.2 *Conjecture 1.2.1 is equivalent to the statement that every simple uniquely vertex-4-colorable planar graph is a vertex Fiorini-Wilson-Fisk graph.*

Proof: Let G be a uniquely vertex-4-colorable planar graph. First assume that every uniquely vertex-4-colorable planar graph is a vertex Fiorini-Wilson-Fisk graph. It then immediately follows that G has a vertex of degree three.

Now assume that every uniquely vertex-4-colorable planar graph has a vertex of degree three. We prove that G is a vertex Fiorini-Wilson-Fisk graph by induction on $|V(G)|$. Because of the assumption, it follows that G has a vertex v of degree three. Also, it must be the case that $G - \{v\}$ is uniquely vertex-4-colorable and planar or else G would not be. By the induction hypothesis, $G - \{v\}$ must be a vertex Fiorini-Wilson-Fisk graph and it then follows that G itself is also a vertex Fiorini-Wilson-Fisk graph. This completes the proof of Theorem 2.8.2.

Although the Fiorini-Wilson-Fisk Conjecture states the existence of a triangle in a uniquely edge-3-colorable cubic planar graph, it actually implies more as the following result of Goldwasser and Zhang in [6] shows.

Theorem 2.8.3 *(Goldwasser and Zhang) A Fiorini-Wilson-Fisk graph on at G with at least 6 vertices contains at least two triangles, and all of its triangles are disjoint. If G has at least 8 vertices and exactly two triangles, then each triangle shares an edge with a 4-circuit which is disjoint from the other triangle.*

For convenience, we state the vertex version of this theorem.

Theorem 2.8.4 *A vertex Fiorini-Wilson-Fisk graph G on at least 5 vertices contains at least two degree 3 vertices and all of the degree 3 vertices of G are non-adjacent. If G has at least 6 vertices and exactly two degree three vertices v_1 and v_2, then there are two vertices u_1 and u_2 having degree 4 in G, and such that u_i is adjacent to v_i and not adjacent to v_{3-i} for $i = 1, 2$.*

2.8.4 Cantoni's Conjecture, A Converse of The Fiorini-Wilson-Fisk Conjecture

If the edge-coloring $c : E(G) \to \{1, \ldots, k\}$ is a unique edge-k-coloring, and $i, j \in \{1, \ldots, k\}$ are distinct colors, then by Theorem 2.2.1 applied to the line graph of G, the subgraph of G that consists of edges colored i or j and their endpoints must be a hamiltonian circuit. It follows that a uniquely edge-3-colorable graph has at least 3 hamiltonian circuits. Moreover, since a cubic graph has an even number of vertices, it can have at most 3 hamiltonian circuits because a fourth hamiltonian circuit defines a two coloring of the edges in the hamiltonian circuit which easily extends to another edge-3-coloring of G.

The converse of this was conjectured Greenwell and Kronk in 1973 [26] and by Fiorini and Wilson [1] in 1977, namely,

Conjecture 2.8.7 *Every cubic graph which has exactly* 3 *hamiltonian circuits is uniquely edge*-3-*colorable.*

Thomason disproved this conjecture in 1982 by showing that the family of generalized Petersen graphs $P(6k+3,2)$ $(k \geq 2)$ have exactly 3 hamiltonian cycles but more than one edge-3-coloring [31]. These graphs are non-planar and if the hypothesis of planarity is added, then the revised conjecture is still open. A related conjecture of Cantoni [32] is that any cubic graph with exactly three hamiltonian circuits contains a triangle. The truth of the Fiorini-Wilson-Fisk Conjecture implies that the conjecture that a cubic planar graph with exactly 3 hamiltonian circuits is uniquely edge-3-colorable is equivalent to the Cantoni Conjecture as we shall prove in Theorem 2.8.5.

Theorem 2.8.5 *Let G be a cubic planar graph with exactly* 3 *hamiltonian cycles. The following two statements are equivalent.*

1) (Cantoni's Conjecture) G contains a triangle.

2) G is uniquely edge-3-*colorable.*

Proof: Let G be a cubic planar graph with exactly three hamiltonian cycles. Consider the operation performed on a cubic graph in which a triangle is contracted to a vertex. This operation, and its inverse were mentioned in Section 1.2 and Section 2.8.3 and it is pointed out in [33] that they both preserve the number of hamiltonian cycles. Using this we prove the claimed equivalence.

First assume the Cantoni Conjecture is true and proceed by induction on $|V(G)|$. From the Cantoni Conjecture, deduce that G has a triangle since G has exactly 3 hamiltonian circuits. Perform the above operation by contracting this triangle to get a smaller cubic graph G', which also has exactly 3 hamiltonian circuits and so by induction is uniquely edge-3-colorable. This, in turn, implies that G is uniquely edge-3-colorable, as desired.

Now assume that a cubic planar graph with exactly 3 hamiltonian circuits is uniquely edge-3-colorable and let us assume that the Cantoni Conjecture holds for all cubic planar graphs on less than n vertices. If G is a cubic planar graph on n vertices with exactly 3 hamiltonian circuits, then G is uniquely edge-3-colorable. By the Fiorini-Wilson-Fisk Conjecture, G has a triangle, and so the proof is complete.

There is another connection of unique edge-3-colorability and the Cantoni Conjecture due to Zhang in [33]. A $(1,2)$-*eulerian weight* w of a $2-$connected graph is a function $w : E(G) \rightarrow \{1,2\}$ such that the total weight of each edge cut is even. A *faithful cover* of w is a family C of circuits such that each edge e is contained in precisely $w(e)$ circuits of C. If w is a $(1,2)$-eulerian weight of a cubic graph G then a faithful cover C of w is *hamiltonian* if C is a set of two hamiltonian circuits. A $(1,2)$-eulerian weight w of G is *hamiltonian* if every faithful cover of w is hamiltonian. With these definitions behind us we can state the next result of Zhang which gives some connection between the Cantoni Conjecture and uniquely edge-3-colorable planar graphs:

Theorem 2.8.6 *(Zhang, 1995) Let G be a cubic graph admitting a hamiltonian weight w. Then the following statements are equivalent:*

1. *G is uniquely edge-3-colorable.*
2. *G has precisely three hamiltonian circuits.*
3. *The hamiltonian weight has precisely one faithful cover.*
4. *The set $\{e : w(e) = 1\}$ is a hamiltonian circuit.*

2.8.5 Strengthenings implied by the Fiorini-Wilson-Fisk Conjecture

A nowhere zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ flow is a function $\phi : E(G) \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $\phi(e) \neq 0$ for every edge $e \in E(G)$ and for every vertex $v \in G$, $\sum_{e\in\delta(v)} \phi(e) = 0$. Here, $\delta(v)$ is the set of all edges which have vertex v as an endpoint. Given a nowhere zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ flow f, we say that f is a *unique flow* if every other nowhere zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ flow f' can be obtained from f by an automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The following is an observation of Robin Thomas [34].

Theorem 2.8.7 *If a connected graph G has a unique nowhere zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ flow f then G is cubic. If G is also planar, then G is a Fiorini-Wilson-Fisk graph, and so contains a triangle.*

Proof: The definition of $\mathbb{Z}_2 \times \mathbb{Z}_2$ shows that $2r(1,0) = 2r(0,1) = 2r(1,1) = (0,0)$ for every positive integer r. For a vertex x let a_x,b_x, $c_x \in \{0,1\}$ be the mod 2 parity of the number of edges incident to x that are assigned $(1,0)$, $(0,1)$, $(1,1)$ respectively. Now $(0,0) = \sum_{e\in\delta(x)} f(e) = a_x(1,0)+b_x(0,1)+c_x(1,1) = (a_x+c_x, b_x+c_x)$, and this implies either $a_x = b_x = c_x = 1$ or $a_x = b_x = c_x = 0$. Either way, any subgraph $L_{\alpha,\beta}$ induced by edges of G which are assigned two distinct elements $\alpha, \beta \in \{(1,0),(0,1),(1,1)\}$ has even degree at each vertex and so is a spanning eulerian subgraph. If $L_{\alpha,\beta}$ is not a simple circuit then one can exchange α, β along the edges of a simple proper subcircuit of L to get a different nowhere zero flow. This contradicts the fact that f is a unique flow. Thus L is a simple circuit for any distinct $\alpha, \beta \in \{(1,0),(0,1),(1,1)\}$, which implies that G is cubic.

Since there is a one to one correspondence between nowhere zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ flows and edge-3-colorings of a cubic graph, if the graph is planar, it must be a Fiorini-Wilson-Fisk graph, and, in particular, contains a triangle. This completes the proof of Theorem 2.8.7.

Let G and H be graphs. Recall that if H is isomorphic to a graph obtained from G by contracting and deleting edges, then G is said to have a *H-minor*.

Theorem 2.8.8 *(Jensen) A simple graph G with no K_5-minor is uniquely vertex-4-colorable if and only if there is a sequence G_0, G_1, ..., G_p, such that G_0 is isomorphic to K_4, G_i is obtained from G_{i-1} by adding a vertex v to G_{i-1} and joining v to exactly three vertices which induce a triangle in G_{i-1}.*

Before proving this, we will need a definition and a fundamental characterization of graphs with no K_5-minor. This characterization is originally due to Wagner, and the formulation we present is found in R. Diestel's textbook Graph Theory [35].

If G is a graph and G_1, G_2 and S are induced subgraphs of G having the properties that $G = G_1 \bigcup G_2$, $S = G_1 \bigcap G_2$, then G is said to be *obtained from G_1 and G_2 by pasting along S* [35]. The *Wagner graph*, denoted by V_8, is shown in figure 2.2.

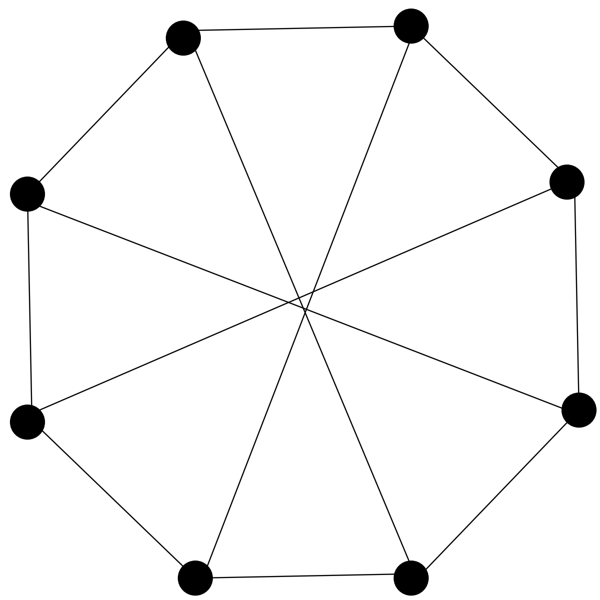

Figure 2.2: The Wagner Graph V_8

Theorem 2.8.9 *(Wagner, 1937) Let G be an edge maximal graph with no K_5−minor. If $|V(G)| \geq 4$, then G can be constructed recursively by pasting plane triangulations and V_8's along K_3's and K_2's.*

We now prove Jensen's extension by induction on $|V(G)|$. Let G be a uniquely vertex-4-colorable graph with no K_5-minor. Let G' be an edge maximal spanning super-graph of G with no K_5-minor. It can be shown using Wagner's Theorem that $|E(G')| \leq 3|V(G')| - 6$. By Corollary 2.2.2, $|E(G)| \geq 3|V(G)| - 6$ and thus $G = G'$.

By applying Wagner's Theorem to G, we may find two graphs G_1 and G_2, such that G is obtained from G_1 and G_2 by pasting along an edge or a triangle, G_1 is itself an edge-maximal graph with no K_5 minor, and G_2 is either a planar triangulation or is isomorphic to V_8, the Wagner graph.

Now $\chi(G_1), \chi(G_2) \leq 4$ since they are subgraphs of G. Using this fact, we note that for $i = 1, 2$, any vertex-4-coloring of G_i can be extended to a vertex-4-coloring of G since G_1 and G_2 are pasted along either an edge or a triangle. Therefore, each of G_1 and G_2 must be uniquely vertex-4-colorable since the graph G is uniquely vertex-4-colorable.

This implies that G_2 is not isomorphic to V_8, since $\chi(V_8) = 3$. Therefore, G_2 is a plane triangulation and by the assumed truth of the Fiorini-Wilson-Fisk Conjecture, must be a planar vertex Fiorini-Wilson-Fisk graph. If $|V(G_2)| = 4$, then G_2 is isomorphic to K_4 and we would be done after applying the induction hypothesis to G_1. If $|V(G_2)| \geq 5$, then Theorem 2.8.4 implies the existence of a vertex in G_2 which has degree 3 in G and whose neighbors induce a triangle in G. Jensen's extension thus follows.

2.9 Summary and Conclusion

Two things may be concluded in the light of this survey on the results of Unique Colorability. The first is the drastic difference in the difficulty between unique edge colorability and unique vertex colorability. On the one hand, there exist very nice and precise characterizations for uniquely-edge-colorable graphs for almost all conceivable graphs. Even in the cases which are not settled yet, namely graphs which have maximum degree three and which are either non-planar or sub-cubic, many of the conjectures propose a very specific structure.

On the other hand, uniquely vertex-colorable graphs seem to form a rich and varied class, and usually elude any nice structural characterization. Related to this is the fact that identifying them will probably be computationally difficult [14]. Intuitively, problems that are NP-Hard will not yield a "concise" characterization except in the case of the unlikely event that NP=coNP.

One result illustrating the richness of the class of uniquely vertex colorable planar graphs is that it is possible, given any graph H and any $\chi(H)$ coloring c of H, to find a uniquely-$\chi(H)$-colorable graph G whose unique coloring, when restricted to $V(H)$, equals c [17]. Another similar theorem says that the super-graph G can be taken to have arbitrarily large minimum circuit length and to be edge-critical, provided only that H has sufficiently high minimum circuit length, $\chi(H) \leq \chi(G)$, and a certain condition about the set of homomorphisms on subgraphs of H holds [18]. This result and others imply the existence of uniquely vertex-k-colorable graphs having arbitrarily large minimum circuit length [23, 36]. This implies, in particular, that there are uniquely vertex-k-colorable graphs which have no subgraph isomorphic to K_k. Another structure that might be expected in an edge-critical uniquely vertex-k-colorable graph is a vertex of degree $k-1$. This, however, cannot be expected in general, and in fact there exist edge critical uniquely vertex-k-colorable graphs which have minimum degree as large as $2k-3$ [24].

In light of these results, the resolution of the Fiorini-Wilson-Fisk Conjecture shows that unique vertex-4-coloring in the plane is exceptional. Its truth shows that it is possible to decide in polynomial time whether or not a planar graph is uniquely vertex-4-colorable. It also is the case that a uniquely vertex-4-colorable planar graph cannot contain C_4 as an induced subgraph. It also shows that uniquely vertex-4-colorable planar graphs must contain a K_4 as a subgraph and must also contain vertices of degree 3.

What then is the source of this difference in behavior between unique vertex-4-coloring in the plane and unique vertex-coloring in general? Two sources which come to mind are 1) planarity and 2) the fact that unique vertex-4-coloring is equivalent to unique edge-3-coloring. To better understand the effect of planarity, it may be good to look into unique vertex coloring on more general surfaces. The writer has some suspicion that the embedding on a surface tends to produce nice substructures, like degree $k-1$ vertices and subgraphs isomorphic to K_k's. If nothing else, it imposes conditions on the minimum degree. Perhaps, on the other hand, the real source of the nice characteristics of unique vertex-4-coloring of planar graphs is that it is really a problem about unique-edge-coloring. To investigate this, it might be interesting to look at unique vertex-colorability of graphs that are nearly line graphs, like claw-free graphs, since unique coloring is well understood for line graphs via unique edge coloring. This may make the unique vertex coloring problem significantly easier.

Chapter 3

Structure of Minimum Counterexample to the Fiorini-Wilson-Fisk Conjecture

3.1 Definitions and Notation

We start with some definitions to help us understand the discoveries that have been made about the structure of counterexamples to the Fiorini-Wilson-Fisk Conjecture. A *minimum counterexample* is a graph G such that

(1) G is planar,

(2) G is not a vertex Fiorini-Wilson-Fisk graph,

(3) G has at most one vertex-4-coloring, up to equivalence of colorings, and

(4) subject to conditions (1), (2) and (3), $|V(G)|$ is minimum.

A set of edges $F \subset E$ of a graph is an *edge cut* for a graph G if the graph $G - F$ has at least two components. A graph is $k-$*edge-connected* if every edge cut $F \subset E$ of G satisfies $|F| \geq k$. An edge cut F of a graph G is *cyclic* if at least two components of $G - F$ have circuits. A graph is *cyclically* k-*edge-connected* if every cyclic edge cut F satisfies $|F| \geq k$. A cyclic edge-cut is *trivial* if one of the components of $G - F$ is precisely a circuit containing $|F|$ edges. Recall that a graph is k-*connected* if after the removal of any set having at most $k - 1$ vertices, the graph is still connected. A graph G is *internally* 6-*connected* if it is 5-connected, and if for any set A of size 5 having the property that $G - A$ is disconnected, it must be the case that one of the components of $G - A$ consists of a single vertex.

Given a circuit C in a drawing (U, V) of a planar graph G, the Jordan Curve Theorem insures that C (treated as a curve in the topological sense), partitions the sphere Σ into two connected (topologically) regions which are homeomorphic to a disc. A circuit C in a drawing of a graph G is said to be a *separating circuit* of the drawing of G if when C, when considered as simple topological closed curve, has the property that the two arc-wise connected components of $\Sigma - C$ both contain vertices of $V(G)$. When the drawing of G is clear from the context, we will also say that C is a *separating circuit of* G.

Work on the Fiorini-Wilson-Fisk Conjecture has produced a number of results about what a counterexample to the conjecture must look like. It is not too difficult to prove that any counterexample to the vertex version of the Fiorini-Wilson-Fisk Conjecture must be a triangulation of the plane, and that a minimum counterexample must not have any separating triangles. Hind showed in his 1988 Ph.D. thesis that a minimum counterexample to the edge version of the Fiorini-Wilson-Fisk Conjecture must have girth 5, which implies that vertices in a vertex counterexample must have degree at least 5 [37]. Goldwasser and Zhang in [6] strengthened this to prove that a minimum counterexample to the edge version of the Fiorini-Wilson conjecture must be cyclically 5-edge-connected. They then showed in [5] that *every* counterexample to the edge version of the Fiorini-Wilson-Fisk Conjecture must be cyclically 5-edge-connected. They showed, moreover, that in a minimum counterexample, every cyclic edge cut F, with $|F| = 5$ must be trivial. By considering the dual graph, this can be seen to imply that a minimum counterexample as we have defined it in Section 3.1 is internally 6-connected. Boehme, Stiebitz and Voigt [38] independently proved that a minimum counterexample to the vertex version of the Fiorini-Wilson-Fisk Conjecture is 5-connected.

We now give a proof of this result of Goldwasser and Zhang that says that a minimum counterexample must be internally six-connected. First, we prove some lemmas which show that the graph must be 5−connected.

Lemma 3.1.1 *Any counterexample G to the Fiorini-Wilson-Fisk Conjecture must have a drawing that is a triangulation. Moreover, if G is a minimum counterexample, then no drawing of G which is a triangulation has a separating triangle.*

Proof: First let G be any counterexample to the Fiorini-Wilson-Fisk Conjecture. The fact that some drawing of G is a triangulation of the plane follows from Corollary 2.2.3.

Now let G be a minimum counterexample, as in the definition found in Section 3.1. Consider a drawing (U, V) of G which is a triangulation and which has a separating triangle C. Let G' be the drawing which consists of the subdrawing of (U, V) induced by all of the vertices that are in one of the arc-wise connected components of $\Sigma - C$ along with the all of the vertices of C. We see that G' is uniquely 4-vertex colorable or else G would have two distinct vertex-4-colorings. Since G is assumed to be a minimum counterexample, then G' is a vertex Fiorini-Wilson-Fisk graph, and so Theorem 2.8.4 implies that G' has at least two non-adjacent degree three vertices. Thus, one of these two degree 3 vertices must not be a vertex in C, and therefore must also be a degree three vertex in the original graph G, which is a contradiction. This completes the proof of the lemma.

3.2 Excluding Separating $4-$Circuits

If c is a proper vertex-4-coloring of a graph G which uses colors $\{1,2,3,4\}$, then we will denote by $G(i,j,c)$ the subgraph of G induced by vertices colored i or j. If the coloring c is understood from the context, then we abbreviate $G(i,j,c)$ by $G(i,j)$. Throughout this section and Section 3.3, we will be assuming that G is a minimum counterexample, and that C is a separating circuit in some drawing of (U,V) of G. We will denote by G_I be the subdrawing of (U,V) induced by all of the vertices in one of the arc-wise connected components of $\Sigma - C$ along with $V(C)$, and we will denote by G_O the subdrawing of (U,V) induced by $\{x_1,x_2,x_3,x_4\}$ and all of the vertices in the other arc-wise connected component of $\Sigma - C$. We note that G_I and G_O each have a unique face bounded by C that is not a triangle, and so we can consider each of G_I and G_O to be a near triangulation by declaring this unique face in each to be the infinite face. We note that since G is a minimum counterexample, then G_I, G_O and graphs which are formed from G_I or G_O by either identifying vertices or adding only edges and which are also planar will be either vertex Fiorini-Wilson-Fisk graphs or will have at least two vertex-4-colorings. In either event, they will have at least one vertex-4-coloring.

Lemma 3.2.1 *Let G be a minimum counterexample. No drawing of G has a separating four circuit.*

Proof: Suppose by way of contradiction that G has a drawing (U,V) with a separating four-circuit C having vertices x_1,x_2,x_3,x_4 with x_i adjacent to x_{i+1} for $i=1,2,3$ and x_4 adjacent to x_1. Note that x_1 is not adjacent to x_3 nor is x_2 adjacent to x_4, because Lemma 3.1.1 guarantees the absence of separating triangles.

Let $\mathcal{C}_I$ ($\mathcal{C}_O$) denote the set of restrictions of four-colorings of G_I (G_O) to the four circuit $\{x_1, x_2, x_3, x_4\}$. We will represent $4-$colorings of C by four element strings $a_1a_2a_3a_4$, where a_i is the color of x_i for $1 \leq i \leq 4$. Two colorings c and c' of $V(C)$ are deemed *equivalent* if there is a permutation π of $\{1, 2, 3, 4\}$ such that $c(x_i) = \pi(c'(x_i))$ for every $1 \leq i \leq 4$. Let $\mathcal{C}_I$ ($\mathcal{C}_O$) denote those colorings in $\{1234, 1213, 1232, 1212\}$ that are equivalent to the restriction to $\{x_1, x_2, x_3, x_4\}$ of a four-coloring of G_I (G_O). We denote by $\mathcal{C}_{1234}$ those colorings of C which are equivalent to the coloring which assigns $c(x_1) = 1$, $c(x_2) = 2$, $c(x_3) = 3$, and $c(x_4) = 4$, and make analogous definitions for $\mathcal{C}_{1213}$, $\mathcal{C}_{1232}$, and $\mathcal{C}_{1212}$. Claim:

(i) $\{1212, 1213\} \bigcap \mathcal{C}_I \neq \emptyset$

(ii) $\{1212, 1232\} \bigcap \mathcal{C}_I \neq \emptyset$

(iii) $\{1232, 1234\} \bigcap \mathcal{C}_I \neq \emptyset$

(iv) $\{1232, 1234\} \bigcap \mathcal{C}_I \neq \emptyset$

Proof of Claim. To prove (i), form a graph from G_I by identifying x_1 and x_3. This graph is planar and is loop-less, because x_1 is not adjacent to x_3 in G. Since G is a minimum counterexample, this graph is either a vertex Fiorini-Wilson-Fisk graph or it has two vertex-4-colorings. Either way, it has at least one vertex-4-coloring. This $4-$coloring naturally corresponds to a 4-coloring of G_I in which x_1 and x_3 receive the same color. The colorings which are equivalent to colorings in $\{1234, 1213, 1232, 1212\}$ and that give x_1 and x_3 the same color are equivalent to the colorings 1212 and 1213. This shows that (i) holds. By similar reasoning applied to x_2 and x_4 we see that (ii) holds. To prove (iii), add the edge $\{x_1, x_3\}$ in the infinite face of G_I and use the similar reasoning as above to produce a vertex-4-coloring of G_I in which x_1 and x_3 receive different colors. Since all colorings for which x_1 and x_3 are colored differently are equivalent to 1232 and 1234, then (iii) holds. By similar reasoning applied to x_2 and x_4, (iv) holds. This completes the proof of the claim. The same proof could be applied to $\mathcal{C}_O$ so the claim holds with $\mathcal{C}_O$ replaced by $\mathcal{C}_I$.

By (i) and (iii) of the above claim, $|\mathcal{C}_I| \geq 2$ and $|\mathcal{C}_O| \geq 2$. If $|\mathcal{C}_I| = 2$ then (i),(ii),(iii) and (iv) imply that $\mathcal{C}_I = \{1234, 1212\}$ or $\mathcal{C}_I = \{1213, 1232\}$. Similarly, if $|\mathcal{C}_O| = 2$ then $\mathcal{C}_O = \{1234, 1212\}$ or $\mathcal{C}_O = \{1213, 1232\}$.

We now show that we may assume $|\mathcal{C}_I \bigcap \mathcal{C}_O| \geq 1$. First note that unless $|\mathcal{C}_I| = 2$ and $|\mathcal{C}_O| = 2$ we would have $|\mathcal{C}_I \bigcap \mathcal{C}_O| \geq 1$ immediately. Thus we may assume that $\mathcal{C}_I = \{1234, 1212\}$ or $\mathcal{C}_I = \{1213, 1232\}$ and $\mathcal{C}_O = \{1234, 1212\}$ or $\mathcal{C}_O = \{1213, 1232\}$. We may therefore assume without loss of generality that $\mathcal{C}_I = \{1234, 1212\}$ and $\mathcal{C}_O = \{1213, 1232\}$ or we would be done. Let c be a vertex-4-coloring of G_I with the property that $c(x_1) = 1$, $c(x_2) = 2$, $c(x_3) = 1$ and $c(x_4) = 3$. If the vertices x_1 and x_3 are in the same component of the subgraph of $G_I(1, 4, c)$, then the vertices x_2 and x_4 can not be in the same component of the subgraph of $G_I(2, 3, c)$ because G_I is a drawing. Therefore, it is possible to interchange the colors 2 and 3 in the component of $G_I(2, 3, c)$ which contains the vertex x_4. This will result in a coloring of G_I which has restriction to C denoted by 1212 which is impossible because $\mathcal{C}_I = \{1213, 1232\}$. It follows from this contradiction that the vertices x_1 and x_3 are in different components of $G(1, 4, c)$. Therefore, we may interchange the colors 1 and 4 in that component of $G(1, 4, c)$ which contains the vertex x_3 to get a coloring of G_I whose restriction to C is given by 1243 and which can be made equivalent to the coloring 1234. This also contradicts $\mathcal{C}_I = \{1213, 1232\}$. Thus, we may assume that $|\mathcal{C}_I \bigcap \mathcal{C}_O| \geq 1$.

Now we complete the proof of the lemma. Since $|\mathcal{C}_I \bigcap \mathcal{C}_O| \geq 1$, if $|\mathcal{C}_I| = 2$ and $|\mathcal{C}_O| = 2$ the work above shows that we must have $|\mathcal{C}_I \bigcap \mathcal{C}_O| = 2$ since $|\mathcal{C}_I| = 2$ implies that $\mathcal{C}_I = \{1234, 1212\}$ or $\mathcal{C}_I = \{1213, 1232\}$ with a similar statement for $\mathcal{C}_O$. Also, if $|\mathcal{C}_I| \geq 3$ and $|\mathcal{C}_O| \geq 3$, then $|\mathcal{C}_I \bigcap \mathcal{C}_O| \geq 2$ and we would be done. So we may assume by the symmetry of $\mathcal{C}_I$ and $\mathcal{C}_O$ that $|\mathcal{C}_I| = 2$ and $|\mathcal{C}_O| = 3$. We will first show that $\mathcal{C}_I \neq \{1234, 1212\}$. So assume that $\mathcal{C}_I = \{1234, 1212\}$. Consider a four-coloring c of G_I that has restriction to $\{x_1, x_2, x_3, x_4\}$ equivalent to 1212. Without loss of generality, we may assume that $c(x_1) = c(x_3) = 1$ and $c(x_2) = c(x_4) = 2$. If there is a path in $G_I(1,2)$ joining x_1 and x_3, then the planarity of G_I implies that there is no x_2-x_4 path in G_I whose vertices are colored 2 or 4. But then by interchanging colors in the component of the subgraph of $G_I(2,4)$ containing the vertex x_4, and permuting the colors 3 and 4, we see that $1213 \in \mathcal{C}_I$ which is a contradiction. Thus, there is no x_1-x_3 path in G_I whose vertices are colored 1 and 3. Therefore, by interchanging the colors 1 and 3 in the component of $G_I(1,3,c)$ containing x_3, we see that $1232 \in \mathcal{C}_I$ which again is a contradiction. This shows that $\mathcal{C}_I \neq \{1234, 1212\}$.

Assume then that $\mathcal{C}_I = \{1213, 1232\}$. Consider a $4-$coloring of G_I that has restriction to $\{x_1, x_2, x_3, x_4\}$ equivalent to 1213. By using reasoning similar to the previous paragraph, we can dispose of this case. This completes the proof of the lemma.

3.3 Separating $5-$Circuits

Because of Lemma 3.1.1 and Lemma 3.2.1, it suffices to show that if C is a separating 5-circuit in a minimum counterexample, then either the interior or the exterior of C has exactly one vertex. Let the vertices of such a separating circuit C of length 5 be x_1, x_2, x_3, x_4, x_5. Note that by Lemma 3.1.1 and Lemma 3.2.1, C is chord-less. We will introduce the concept of a *minority vertex* whose definition depends on whether C receives 3 or 4 colors in a proper 4-coloring. If c is a proper 4-coloring of C, in which C receives exactly 3 different colors, (which we assume are 1,2, and 3), then one of the colors $\{1, 2, 3\}$ appears exactly once and the other two appear exactly twice. The vertex which receives the color which appears exactly once is called the *minority vertex*. If C receives four colors, say $\{1, 2, 3, 4\}$, then exactly one of the colors $\{1, 2, 3, 4\}$ appears twice. The unique vertex in $V(C)$ which is adjacent to the two vertices receiving this color will be the minority vertex in this case. By this definition, every 4-coloring of C is uniquely specified by stating which vertex in C is the minority vertex and whether C receives 3 or 4 colors. Thus we can denote a class of equivalent colorings of C by an ordered pair (i, α) where $i \in \{1, 2, 3, 4, 5\}$ and $\alpha \in \{3, 4\}$ if x_i is the minority vertex in a 4-coloring of C belonging to this class which uses α colors. We will denote by $\mathcal{C}_I$ (respectively $\mathcal{C}_O$) the set of ordered pairs of the form above which represent coloring classes of restrictions to C of 4-colorings of the graph G_I (respectively G_O).

Lemma 3.3.1 *Let $i \in \{1, \ldots, 5\}$.*

a) If $(i, 3) \in \mathcal{C}_I$, then $\{(i-1, 3), (i-2, 4)\} \bigcap \mathcal{C}_I \neq \emptyset$ and $\{(i+1, 3), (i+2, 4)\} \bigcap \mathcal{C}_I \neq \emptyset$.

b) If $(i, 4) \in \mathcal{C}_I$, then $\{(i+3, 3), (i+2, 4)\} \bigcap \mathcal{C}_I \neq \emptyset$, and $\{(i+2, 3), (i+3, 4)\} \bigcap \mathcal{C}_I \neq \emptyset$.

Here the first components of all ordered pairs should be interpreted modulo 5. The same statements hold for C_O as well.

Proof: Let c be a coloring of G_I that has restriction to x_1, x_2, x_3, x_4, x_5 which is represented by $(i, 3)$. Without loss of generality we may assume (after possibly re-labeling) that $i = 1$ and that $x_1 = 1$ and that $c(x_1) = 1$, $c(x_2) = c(x_4) = 2$ and $c(x_3) = c(x_5) = 3$. If x_2 and x_4 are not in the same component of $G_I(2, 4, c)$, then the colors 2 and 4 can be exchanged in the component containing x_4 to get a coloring of G_I whose restriction to x_1, x_2, x_3, x_4, x_5 is represented by $(4, 4)$. If x_2 and x_4 are in the same component of $G_I(2, 4, c)$, then x_1 and x_3 are not in the same component of $G_I(1, 3, c)$, because if they were, the path in this component would intersect the path joining x_2 and x_4 in $G_I(2, 4)$. Therefore the colors in the component of $G_I(1, 3, c)$ containing x_3 can be exchanged to get a coloring of G_I whose restriction to x_1, x_2, x_3, x_4, x_5 is represented by $(5, 3)$. This proves the assertion that if $(i, 3) \in \mathcal{C}_I$ then $\{(i-1, 3), (i-2, 4)\} \bigcap \mathcal{C}_I \neq \emptyset$.

Under the same assumption about c, if x_1 and x_4 are not in the same component of $G_I(1, 2, c)$, then it can be shown that $(2, 3) \in \mathcal{C}_i$. If x_1 and x_4 are in the same component of $G(1, 2, c)$, then x_3 and x_5 must be in different components of $G_I(3, 4, c)$, and by an appropriate interchange of the colors 3 and 4, it can be shown that $(3, 4) \in \mathcal{C}_i$. This establishes that $\{(i+1, 3), (i+2, 4)\} \bigcap \mathcal{C}_I \neq \emptyset$.

The proof of b) follows in the same way. Also, the same arguments can be applied to G_O to reach the same conclusions about G_O. Therefore, the proof is complete.

The following notation will also be helpful: for $i = 1, 2, 3, 4, 5$, let $\mathcal{T}_i = \{(i, 4), (j, 3), (k, 3)\}$ and $\mathcal{Q}_i = \{(i, 3),\ (i, 4),\ (j, 4),\ (k, 4)\}$, where $j, k \in \{1, 2, 3, 4, 5\}$ and $j \equiv i + 2 (\text{mod } 5)$ and $k \equiv i + 3 (\text{mod } 5)$. Arithmetic will be modulo 5, where the context demands.

Lemma 3.3.2 *For every $i \in \{1,2,3,4,5\}$,*

1) $\mathcal{T}_i \bigcap \mathcal{C}_I \neq \emptyset$.

2) $\mathcal{Q}_i \bigcap \mathcal{C}_I \neq \emptyset$.

Similar statements hold for the set $\mathcal{C}_O$ in place of $\mathcal{C}_I$.

Proof: Let $i \in \{1,2,3,4,5\}$. We first prove 1). Because C is chord-less, we can identify the vertices x_{i-1} and x_{i+1} in the graph G_I to get a graph G'_I. This identification is possible because C bounds the infinite face of G_I. Because G is a minimum counterexample, G'_I has a vertex-4-coloring which naturally defines a vertex-4-coloring c of the graph G_I by "un-identifying" x_{i-1} and x_{i+1}. In G'_I, the single vertex obtained by identifying x_{i-1} and x_{i+1} is in a triangle with the vertices x_{i+2} and x_{i+3}. Therefore, the coloring c of G_I that arises from this coloring of G'_I must assign the same color to x_{i-1} and x_{i+1}, and must assign two distinct colors other than $c(x_{i-1})$ to the vertices x_{i+2} and x_{i+3}. If c uses exactly 3 colors then we may assume $c(x_{i-1}) = c(x_{i+1}) = 1$, $c(x_{i+2}) = 2$,$c(x_{i+3}) = 3$ and $c(x_i) \in \{2,3\}$. Thus we see that either x_{i+2} or x_{i+3} is the minority vertex. If c gives C 4 colors, then x_i is the minority vertex. This completes the proof that $\{(i,4),((i+2),3),((i+3),3)\} \bigcap \mathcal{C}_I \neq \emptyset$.

To prove 2), form the graph G'_I from G_I by adding the edges $\{x_i, x_{i+2}\}$ and $\{x_i, x_{i+3}\}$ in the infinite face of G_I that is bounded by C. By the minimality of G, G'_I has a vertex-4-coloring whose restriction to C uses at least 3 colors. If the restriction to G'_I uses exactly three colors, then x_i is a minority vertex in this coloring. If it uses 4 colors then a little analysis will show that either x_i or x_{i+2} or x_{i+3} is a minority vertex. This proves 2).

Statements 1) and 2) clearly apply to G_O as well as G_I, since we used nothing about G_I except that it had infinite face bounded by C. This completes the proof of

Lemma 3.3.2.

The next proof is adapted from Robertson et. al.

Lemma 3.3.3 *(Birkhoff [9]) Let* $\mathcal{S} = \{(1,3),(2,3),(3,3),(4,3),(5,3)\}$ *and suppose that* $\mathcal{C}_I \bigcap \mathcal{S} \neq \emptyset$*. The set* $\mathcal{C}_I$ *contains one of* $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5, \mathcal{S}\}$*. The same holds for* $\mathcal{C}_O$*.*

Proof: If $\mathcal{S} \subset \mathcal{C}_I$, we are done. Therefore, without loss of generality assume that $(1,3) \in \mathcal{C}_I$ and $(2,3) \notin \mathcal{C}_I$. By Lemma 3.3.1, $\{(2,3),(3,4)\} \bigcap \mathcal{C}_I \neq \emptyset$, so $(3,4) \in \mathcal{C}_I$. If $(5,3) \in \mathcal{C}_i$, then $\mathcal{T}_3 \subset \mathcal{C}_I$, so we may assume $(5,3) \notin \mathcal{C}_I$. Again from Lemma 3.3.1, $\{(4,4),(5,3)\} \bigcap \mathcal{C}_I \neq \emptyset$, so $(4,4) \in \mathcal{C}_I$. Now $(4,4) \in \mathcal{C}_I$ and Lemma 3.3.1, together imply that $\{(1,4),(2,3)\} \bigcap \mathcal{C}_I \neq \emptyset$, so $(1,4) \in \mathcal{C}_I$. Thus $\mathcal{Q}_1 = \{(1,3),(1,4),(3,4),(4,4)\} \subset \mathcal{C}_I$, as desired. The same reasoning can be applied to $\mathcal{C}_O$ as well. This completes the proof of the lemma.

Theorem 3.3.1 *(Goldwasser & Zhang) A minimum counterexample is internally six connected.*

Proof: Because G has a drawing which is a triangulation, it suffices in terms of this notation to show that either

1) $|V(G_I) - \{x_1, x_2, x_3, x_4, x_5\}| = 1$ or that

2) $|V(G_O) - \{x_1, x_2, x_3, x_4, x_5\}| = 1$.

So assume otherwise. Because G is assumed to be a minimum counterexample, it follows that $|\mathcal{C}_I \bigcap \mathcal{C}_O| = 1$. Form a graph G'_I by adding a vertex v to the infinite face of G_I and adding the 5 edges $\{\{v, x_i\} : 1 \leq i \leq 5\}$ to $E(G_I)$. By the minimality of

G and the assumption that neither 1) nor 2) above hold, G'_I has a vertex-4-coloring, and so $\mathcal{C}_I \bigcap \mathcal{S} \neq \emptyset$. Similarly, $\mathcal{C}_O \bigcap \mathcal{S} \neq \emptyset$. Therefore, by Lemma 3.3.3, both $\mathcal{C}_I$ and $\mathcal{C}_O$ contain some element in $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5, \mathcal{S}\}$.

Suppose first that $\mathcal{S} \not\subset \mathcal{C}_I$, and $\mathcal{S} \not\subset \mathcal{C}_O$. Therefore there are sets $\mathcal{R}_I \subset \mathcal{C}_I$ and $\mathcal{R}_O \subset \mathcal{C}_O$ such that $R_I, R_O \in \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5, \mathcal{S}\}$. By Lemma 3.3.2, $|\mathcal{C}_I \bigcap \mathcal{R}_O| \geq 1$, and $|\mathcal{C}_O \bigcap \mathcal{R}_I| \geq 1$. This implies that if $|\mathcal{R}_I \bigcap \mathcal{R}_O| \in \{0, 2, 3, 4\}$ then $|\mathcal{C}_I \bigcap \mathcal{C}_O| \geq 2$. Therefore, $|\mathcal{R}_I \bigcap \mathcal{R}_O| = 1$. It is possible by relabeling to assume that $\mathcal{R}_O \in \{\mathcal{T}_1, \mathcal{Q}_1\}$.

Assume first that $\mathcal{R}_O = \mathcal{T}_1$ and $\mathcal{R}_I \in \{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5\}$. Since $|\mathcal{R}_O \bigcap \mathcal{R}_I| = 1$, it must be that $\mathcal{R}_I = \mathcal{Q}_1$ and that $\mathcal{C}_I \bigcap \mathcal{C}_O = \{(1, 4)\}$. By Lemma 3.3.2, $\{(4, 4), (1, 3), (2, 3)\} \bigcap \mathcal{C}_O \neq \emptyset$. Since $\{(4, 4), (1, 3)\} \subset \mathcal{R}_I \subset \mathcal{C}_I$, it must be that $(2, 3) \in \mathcal{C}_O$. Likewise, $\{(3, 4), (5, 3), (1, 3)\} \bigcap \mathcal{C}_O \neq \emptyset$ implies $(5, 3) \in \mathcal{C}_O$. This, Lemma 3.3.1 and the fact that $(1, 3) \in \mathcal{C}_I - \mathcal{C}_O$ imply that $(2, 4) \in \mathcal{C}_O$. Thus $\mathcal{T}_2 = \{(2, 4), (4, 3), (5, 3)\} \subset \mathcal{C}_O$. However, Lemma 3.3.2 applied to $\mathcal{C}_I$ with $i = 2$ establishes the fact that $\mathcal{C}_I \bigcap \mathcal{T}_2 \neq \emptyset$. This contradicts $|\mathcal{C}_I \bigcap \mathcal{C}_O| = 1$, and so completes the proof when $\mathcal{R}_O \in \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5\}$ and $\mathcal{R}_I \in \{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5\}$.

We now consider the case that $\mathcal{R}_O = \mathcal{T}_1 = \{(1, 4), (3, 3), (4, 3)\}$ and $\mathcal{R}_I \in \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5\}$. Since $|\mathcal{R}_O \bigcap \mathcal{R}_I| = 1$, it must be that $\mathcal{R}_I \in \{\mathcal{T}_2, \mathcal{T}_5\}$. By symmetry, we may assume that $\mathcal{R}_I = \mathcal{T}_2 = \{(2, 4), (4, 3), (5, 3)\}$, and so $\mathcal{C}_I \bigcap \mathcal{C}_O = \{(4, 3)\}$. We split the proof of this case up according to whether $(3, 4) \in \mathcal{C}_I$ or $(3, 4) \notin \mathcal{C}_I$.

Assume first that $(3, 4) \in \mathcal{C}_I$. Lemma 3.3.2 with $i = 3$ implies that $\{(3, 4), (5, 3), (1, 3)\} \bigcap \mathcal{C}_O \neq \emptyset$, and since $\{(3, 4), (5, 3)\} \subset \mathcal{C}_I$, it must be that $(1, 3) \in \mathcal{C}_O$. Lemma 3.3.1 and the fact that $(3, 4) \in \mathcal{C}_I$ show that $\{(1, 3), (5, 4)\} \bigcap \mathcal{C}_I \neq \emptyset$ which forces $(5, 4) \in \mathcal{C}_I$ because $(1, 3) \in \mathcal{C}_O$ and $\mathcal{C}_I \bigcap \mathcal{C}_O = \{(4, 3)\}$. We now have established that

$\mathcal{Q}_5 = \{(5,3),(5,4),(2,4),(3,4)\} \subset \mathcal{C}_I$ which reduces to the previous case.

Assume then that $(3,4) \notin \mathcal{C}_I$. By Lemma 3.3.2, $\{(3,3),(3,4),(5,4),(1,4)\} \bigcap \mathcal{C}_I \neq \emptyset$. Since $\{(3,3),(1,4)\} \subset \mathcal{C}_O$, this forces $(5,4) \in \mathcal{C}_I$. This fact and Lemma 3.3.1 show that $\{(2,3),(3,4)\} \bigcap \mathcal{C}_I \neq \emptyset$ and so $(2,3) \in \mathcal{C}_I$. Now $(2,3) \in \mathcal{C}_I$ and Lemma 3.3.1 guarantee that $\mathcal{C}_I \bigcap \{(3,3),(4,4)\}$ is nonempty and since $(3,3) \in \mathcal{R}_O \subset \mathcal{C}_O$, we have that $(4,4) \in \mathcal{C}_I$. Thus far we have shown that $\mathcal{Q}_2 = \{(2,3),(2,4),(4,4),(5,4)\} \subset \mathcal{C}_I$. This, however, reduces to a previous case that we already proved.

The next case is when $\mathcal{R}_O = \mathcal{Q}_1$ and $\mathcal{R}_I \in \{\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5\}$. The fact that $|\mathcal{R}_O \bigcap \mathcal{R}_I| = 1$ allows us to conclude $\mathcal{R}_I \in \{\mathcal{Q}_2, \mathcal{Q}_5\}$, and we may by symmetry assume that $\mathcal{R}_I = \mathcal{Q}_2 = \{(2,3),(2,4),(4,4),(5,4)\}$. Therefore, $\mathcal{C}_I \bigcap \mathcal{C}_O = \{(4,4)\}$. Lemma 3.3.2 guarantees that $\{(2,4),(4,3),(5,3)\} \bigcap \mathcal{C}_O \neq \emptyset$, and it follows that $\{(4,3),(5,3)\} \subset \mathcal{C}_O$.

If $(5,3) \in \mathcal{C}_O$, then $\mathcal{T}_3 \subset \mathcal{C}_O$ which corresponds to the first case proved. Thus, we may assume that $(4,3) \in \mathcal{C}_O$. Lemma 3.3.1 then implies that $\{(3,3),(2,4)\} \bigcap \mathcal{C}_O \neq \emptyset$ and this in turn implies that $(3,3) \in \mathcal{C}_O$, since $(2,4) \in \mathcal{C}_I$. Thus, $\mathcal{T}_1 = \{(1,4),(3,3),(4,3)\} \subset \mathcal{C}_O$, and we again can fall back to the proof of case 1.

The case when $\mathcal{R}_O = \mathcal{Q}_1$ and $\mathcal{R}_I \in \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5\}$ is just the second case proved with the roles of G_I and G_O interchanged. There are no other cases and we are forced to conclude that at least one of $\mathcal{C}_I$ and $\mathcal{C}_O$ contains $\mathcal{S}$.

Suppose then that $\mathcal{S} \subset \mathcal{C}_O$. We know that $\mathcal{C}_I \bigcap \mathcal{S} \neq \emptyset$. Therefore, $|\mathcal{C}_I \bigcap \mathcal{S}| = 1$ or else G would not be uniquely vertex-4-colorable, and so there exists an integer $i \in \{1,2,3,4,5\}$ such that $\mathcal{C}_I \bigcap \mathcal{S} = \{(i,3)\} = \mathcal{C}_I \bigcap \mathcal{C}_O$. Also, there is exactly one (up to a permutation of colors) vertex-4-coloring of G_I that has restriction to x_1, x_2, x_3, x_4, x_5 represented by $(i,3)$, or else G would not be uniquely vertex-4-colorable. It follows that the graph G'_I defined at the beginning of this proof is uniquely vertex-4-colorable, and is therefore a smaller counterexample than G. Thus the assumption that $|V(G_I)-\{x_1, x_2, x_3, x_4, x_5\}| \geq 2$ and $|V(G_O) - \{x_1, x_2, x_3, x_4, x_5\}| \geq 2$ is false, and so one of $G_I - V(C)$ or $G_O - V(C)$ consists of a single vertex. This proves that G must be internally 6-connected.

Chapter 4

Configurations, Projections and Free Completions

4.1 Combinatorial Representations of Drawings

Recall that a line in the sphere Σ is a subspace $h([0,1])$ where h is a homeomorphism of $[0,1]$. A *straight line in* Σ is a line h such that $h([0,1])$ is a $h(0)-h(1)$ geodesic in Σ. If a line h can be partitioned into a finite number of straight lines then we say that h is a *piecewise polygonal line.* If (U,V) is a drawing in which all of the edges are piecewise polygonal lines, then we say that (U,V) is a *piecewise polygonal drawing.* Recall that $C(u,\epsilon)$ is the *circle*, centered at u and having radius ϵ. If $x,y \in C(u,\epsilon)$, then the *clockwise arc between* x *and* y *on* $C(u,\epsilon)$, which will be denoted $A(x,y,v,\epsilon)$ (or simply $A(x,y)$ if u and ϵ are understood from the context) is the subset of $C(u,\epsilon)$ between x and y in the clockwise direction.

In this section we develop a combinatorial description of drawings. Before we do this, one minor detail must be disposed of. A consequence of the definition of drawing is that an edge incident to a vertex v can spiral around v an infinite number of times. This and other types of refractory edges can make the analysis of drawings complicated. Happily, this problem can be circumvented without loss of generality through the following lemma, a proof of which appears in Bollobas [39].

Lemma 4.1.1 *Let G be a drawing of a graph. There is a drawing of G in which every edge of G is piecewise polygonal.*

The advantage gained by this assumption will be made clear by the next lemma and some definitions which follow. We will then prove two lemma's which lay the foundation for a combinatorial description of drawings. Each of these lemmas relies on the piecewise polygonal drawing of edges.

Lemma 4.1.2 *Let G be a piecewise polygonal drawing of a graph. There is an $\epsilon > 0$ such that if $v \in V(G)$ and $g_1, g_2, \ldots, g_{d_G(v)}$ are the edges of G that are incident to v then:*

1) $\overline{D(v,\epsilon)} \bigcap (V(G) \bigcup (E(G) - \{g_1, \ldots, g_{d_G(v)}\})) = \{v\}$.

2) $\overline{D(v,\epsilon)} \bigcap g_i$ *is a straight line for every* $1 \leq i \leq d_G(v)$.

The proof of this lemma will be omitted.

For a given piecewise polygonal drawing of G, we call any ϵ which satisfies the conditions 1) and 2) of Lemma 4.1.2 a *critical radius for G*. Henceforth, we will assume that all drawings of graphs are piecewise polygonal. Let G be a drawing, let ϵ be a critical radius for G and let $v \in V(G)$. Consider all points of intersection of the circle $C(v,\epsilon)$ with the edges $g_1, \ldots, g_{d_G(v)}$ that are incident to v. Because of property 1), each edge in $E(G)$ that is incident to v intersects $C(v,\epsilon)$ at least once. From property 2), each edge incident to v intersects $C(v,\epsilon)$ at most once. Denote by x_i the unique point in $C(v,\epsilon) \bigcap g_i$. We will say that x_j *follows* x_i *on* $C(v,\epsilon)$ (or if C is evident from the context simply that x_j *follows* x_i if $A(x_i,x_j) \bigcap \{x_1, \ldots, x_d(v)\} = \{x_i, x_j\}$. If $\{j_1, j_2, \ldots, j_{d(v)}\} = \{1, 2, \ldots, d(x)\}$ is such that $x_{j_{i+1}}$ follows x_{j_i} for every $1 \leq i \leq d(v)$ (where if $i = d(x)$, j_{i+1} is interpreted to be j_1), then $g_{j_1}, g_{j_2}, \ldots, g_{j_{d(x)}}$ is *a clockwise* *listing in G at C of edges incident to v.* It is clear that any cyclic shift of $j_1, \ldots, j_{d(x)}$ naturally gives rise to another clockwise listing at C of edges incident to v. Property 2) of Lemma 4.1.2 implies that any two critical radii will lead to the same clockwise listing in G of edges incident to v. It is also natural to say that a set of vertices $u_1, u_2, \ldots, u_{d_G(v)}$ is a *clockwise listing in G of neighbors of v* if $g_1, \ldots, g_{d_G(v)}$ is a clockwise listing of edges in G having the property that the endpoint in $V(G) - v$ of g_i is u_i for $1 \leq i \leq d_G(v)$. The next lemma gives the basic but fundamental connection of clockwise listings to drawings.

Lemma 4.1.3 *Suppose that G is a near-triangulation and that $v \in V(G)$. Suppose that $g_1, \ldots, g_{d_G(v)}$ is a clockwise listing in G of edges incident to v. For every $1 \leq i \leq d_G(v)$, there is a face r_i of G which is incident to both g_i and g_{i+1} (where if $i = d_G(v)$ we interpret g_{i+1} as g_1). Moreover, if $d_G(v) \geq 3$, and r_i is finite and triangular then r_i is the unique finite face of G that is incident to g_i and g_{i+1} for $1 \leq i \leq d_G(v)$.*

Proof: Let ϵ be a critical radius of G, and for $1 \leq i \leq d_G(v)$, let x_i denote the unique point of intersection of the edge g_i with $C(v, \epsilon)$. It follows that $A(x_i, x_{i+1}, v, \epsilon) = A(x_i, x_{i+1})$ satisfies $A(x_i, x_{i+1}) \bigcap \{x_1, \ldots, x_{d_G(v)}\} = \{x_i, x_{i+1}\}$, for every $1 \leq i \leq d_G(v)$ (with the usual convention that when $i = d_G(v)$, $i + 1$ is interpreted to be 1). From property 2) of Lemma 4.1.2, we may also conclude that $A(x_i, x_{i+1}) \bigcap G = \{x_i, x_{i+1}\}$ for $1 \leq i \leq d_G(v)$. Therefore $A(x_i, x_{i+1}) - \{x_i, x_{i+1}\}$ is a subset of r_i, and this implies that $g_i \subset \overline{r_i}$ and $g_{i+1} \subset \overline{r_i}$ for $1 \leq i \leq d_G(v)$. This completes the first part of the proof.

Now assume that $d_G(v) \geq 3$ and that r_i is a triangular face. Suppose that $r \neq r_i$ is another face of G that is incident to both g_i and g_{i+1}. By relabeling we may assume that $i = 1$. The face r_2 is incident to the edges g_2 and g_3. The edge g_2 is incident to at most two faces and therefore $r_2 \in \{r, r_1\}$. If $r_2 = r_1$, then r_1 is incident to the three edges g_1, g_2, g_3 which is impossible since these three edges are all incident to v and r_1 is a triangular face. Therefore $r_2 = r$. However r is incident to both of g_1 and g_2 and since $r = r_2$, r must also incident to g_3 and this implies that r is not triangular and hence is the infinite face of G.

This completes the proof of Lemma 4.1.3.

4.2 Configurations, Free Completions, and Projections

The next definition is taken from Robertson et. al. [8] A *configuration* K is a near triangulation $G = G(K)$ together with a function γ_K defined on the vertex set of $G(K)$ such that

(i) For every vertex $v \in V(G)$, $G(K) - v$ has at most two components, and if there are two, then $\gamma_K(v) = d_G(v) + 2$.

(ii) For every vertex $v \in V(G)$, if v is not incident with the infinite region, then $\gamma_K(v) = d_G(v)$, otherwise, $\gamma_K(v) > d_G(v)$ and in either case $\gamma_K(v) \geq 5$.

(iii) K has ring size ≥ 2, where the *ring-size* of K is defined to be $\sum(\gamma_K(v) - d_G(v) - 1)$, summed over all the vertices v of $G(K)$ incident with the infinite region such that $G(K) - v$ is connected.

It will be convenient to represent configurations pictorially, by simply drawing the near-triangulation in the plane such that all of the finite regions are triangles and where $\gamma(v)$ is represented by having differing shapes for the vertex v according to the table in figure 4.1.

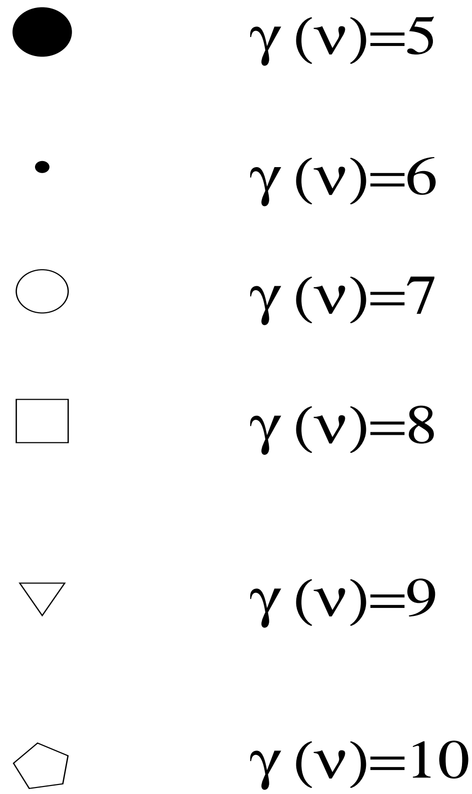

Figure 4.1: The Meaning Of Vertex Shapes

The set of configurations which we will be most concerned with are the 942 configurations that appear in Appendix A. The configuration in row y and column z of page x of Appendix A will be referred to by $\text{conf}(x, y, z)$. We will show in Chapter 5 that no configuration in Appendix A can be "found" in a minimum counterexample. Just what we mean by "found" is now made precise.

If T is a triangulation or a near triangulation, then a configuration K *appears* in T if $G(K)$ is an induced subdrawing of T, every finite face of G is a face of T and $\gamma_K(v) = d_T(v)$ for every vertex $v \in V(G(K))$.

Two configurations K and L are said to be *isomorphic* if there is a homeomorphism ϕ of the sphere Σ mapping $G(K)$ to $G(L)$ and a function $\psi : \{5, 6, \ldots\} \to \{5, 6, \ldots\}$ such that $\psi(\gamma_K(v)) = \gamma_L(\phi(v))$ for every $v \in V(G)$.

Let K be a configuration. The following definition appears in [8]. A *free completion S of K with ring R* is a near triangulation such that

(i) The infinite face of S is bounded by an induced circuit R called the *ring*.

(ii) The graph $G(K)$ is an induced subdrawing of S, $G(K) = S - R$, every finite region of $G(K)$ is a finite region of S, and the infinite region of $G(K)$ contains the infinite region of S.

(iii) $\gamma_K(v) = d_S(v)$ for every $v \in V(G(K))$.

We will show in Section 4.4 that this free completion is essentially unique.

Suppose a configuration K with underlying graph $G = G(K)$ appears in a triangulation or near-triangulation T. Let A be the set of vertices of $V(T)$ that are either in $V(G)$ or are incident to a vertex in $V(G)$. As mentioned above, a configuration could appear in a triangulation or near triangulation in various ways. Thus, while it is conceivable that a configuration K appears in a triangulation T in such a way that the subgraph of T that is induced by the vertices of A is a free completion of K, it is not the case that is has to be this way. One thing that can happen is that vertices on the ring of the free completion can be identified in T. Thankfully, it can be shown that the variety of different ways in which a configuration can appear in a triangulation is under control, in the sense that it will suffice to analyze only the free completion. A key part of how this is established is by establishing a close relationship between a free completion S of K, and any triangulation T in which K appears. We now make this more precise, using a definition from [8].

Suppose that T is a triangulation and that S is the free completion of a configuration K with ring R. Let $F(S)$ be the set of finite faces of S. A map ϕ from the set $V(S)\bigcup E(S)\bigcup F(S)$ into $V(T)\bigcup E(T)\bigcup F(T)$ is called a *projection of S into T* if

(i) ϕ maps $V(S)$ into $V(T)$, $E(S)$ into $E(T)$ and $F(S)$ into $F(T)$

(ii) for distinct $u, v \in V(S)$, $\phi(u) = \phi(v)$ only if u, v are both incident with the infinite region of S; for distinct $e, f \in E(S)$, $\phi(e) = \phi(f)$ only if e,f are both incident with the infinite region of S; and for distinct r,$s \in F(S)$, $\phi(r) \neq \phi(s)$, and

(iii) for x,$y \in V(S)\bigcup E(S)\bigcup F(S)$, if x,y are incident in S, then $\phi(x), \phi(y)$ are incident in T.

The next theorem gives some basic properties of free completions. The proof is omitted.

Lemma 4.2.1 *Let S be a free completion of a configuration K with ring R. Then the following are true:*

1) The configuration K appears in S.

2) Every edge $e \in E(S) - E(R)$ is incident to two finite faces of S and has at least one endpoint in $V(G)$.

3) For every $v \in V(R)$ there is an $x \in V(G)$ such that x is adjacent to v.

4) Every finite face of $F(S)$ is incident to some vertex $x \in V(G)$.

5) No $v \in V(G)$ is incident to the infinite face of S.

6) For every $v \in V(R)$, the clockwise listing in S of the edges incident to v is of the form $r_1, g_1, \ldots, g_p, r_2$, where $p > 0$, $r_1, r_2 \in E(R)$ and $g_1, \ldots, g_p \in E(S) - E(R)$.

7) No finite face of S is incident to two edges of $E(R)$.

4.3 The Existence of Projections

Let S be a a graph containing a vertex $v \in V(S)$. A *portion* of a clockwise listing in S of edges incident to v in S is a sequence of at least two edges $f_1, f_2, \ldots, f_p$ such that $f_1, \ldots, f_p, t_1, \ldots, t_k$ is a clockwise listing in S of edges incident to v.

Lemma 4.3.1 *Let G be a near triangulation that is an induced subdrawing of a near-triangulation S such that every finite face of G is a finite face of S. Suppose also that $d_S(v) \geq 5$ for $v \in V(G)$ and also that no edge of S which is incident to a vertex of $V(G)$ is incident with the infinite face of S. Let $v \in V(G(K))$, and let $g_1, g_2, \ldots, g_j$ be a portion of the clockwise listing in G of edges incident to v having the property that for every $1 \leq k < j$, the edges g_k and g_{k+1} are incident to a common triangular face of G. Then $g_1, \ldots, g_j$ is also a portion of the clockwise listing in S of edges incident to v.*

Proof: Let $s_1, s_2, \ldots, s_{d(v)}$ be a clockwise listing in S of edges incident to v in S. For a suitably small ϵ there is a circle $C(v, \epsilon)$ having the property that $\overline{D(v,\epsilon)} \bigcap V(S) = \{v\}$ and where $\overline{D(v,\epsilon)} \bigcap E(S) = \bigcup_{i=1}^{i=d_S(v)} s_i[v, x_i]$, where for each $1 \leq i \leq d_S(v)$, $s_i[v, x_i]$ consists of a straight line segment between the vertex v and the final point x_i of s_i on $C(v, \epsilon)$. For $x, y \in \Sigma \bigcap C(v, \epsilon)$, let $A(x, y)$ be the clockwise arc in $C(v, \epsilon)$ joining x to y.

By the choice of ϵ, $A(x_i, x_{i+1}) \bigcap (V(S) \bigcup E(S)) = \{x_i, x_{i+1}\}$ for every $1 \leq i \leq d_S(v)$ (where for $i = d_S(v)$ we interpret s_{i+1} as s_1).

Since G is a subdrawing of S, $E(G) \subset E(S)$ and so there is a function $f : \{1, \ldots, j\} \rightarrow \{1, \ldots, d_S(v)\}$, such that $g_i = s_{f(i)}$ for $i = 1, \ldots, j$. This also implies that the final point of g_i on $C(v, \epsilon)$ is $x_{f(i)}$ for $1 \leq i \leq j$. Let f_i be a finite face of G that is incident to both of the edges g_i and g_{i+1}. By Lemma 4.1.3, this finite face f_i of G that is incident to both g_i and g_{i+1} is unique. Since G appears in S, every finite face of G is a finite face of S and in particular f_i is a finite face of S for $1 \leq i < j$. Now the the open arc $A(x_{f(i)}, x_{f(i+1)}) - \{x_{f(i)}, x_{f(i+1)}\}$ of $C(v, \epsilon)$, is contained in f_i for every $1 \leq i < j$, and so $(A(x_{f(i)}, x_{f(i+1)}) - \{x_{f(i)}, x_{f(i+1)}\}) \bigcap (V(S) \bigcup E(S)) = \emptyset$ which implies that g_i follows g_{i+1} in the clockwise listing in S of edges incident to v. This establishes Lemma 4.3.1.

Lemma 4.3.2 *Let K be a configuration with underlying graph G, and let $v \in V(G)$. Let $g_1, \ldots, g_{d_G(v)}$ be a clockwise listing in G of edges incident to v. Let p be the number of indices i in $\{1, 2, \ldots, d_G(v)\}$ such that g_i and g_{i+1} are not incident to a common finite face of G. (Here, when $i = d_G(v)$ we understand that $g_{i+1} = g_1$).*

a) If v is not incident to the infinite face then $p = 0$

b) If v is incident to the infinite face of G and if v is not a cut-vertex of G then $p = 1$.

c) If v is incident to the infinite face of G and if v is a cut vertex of G, then $p = 2$.

Proof: We prove that $G - \{v\}$ has $max\{|L|, 1\}$ components, where L is the set of indices in $\{1, \ldots, d_G(v)\}$ such that g_i and g_{i+1} are not in a common finite face of G. By Lemma 4.1.3 , g_i and g_{i+1} must be incident to a common face r_i for $1 \leq i \leq d_G(v)$. Let $v_i \in V(G) - \{v\}$ be the endpoint of g_i for $1 \leq i \leq d_G(v)$.

First, suppose $|L| = 0$ or $|L| = 1$. After a suitable relabeling of $g_1, \ldots, g_{d_G(v)}$, we may assume that each r_i except possibly $r_{d_G(v)}$ is a finite triangular face so that there is an edge in G with endpoints v_i and v_{i+1} for every $1 \leq i < d_G(v)$. Therefore, the subgraph of G induced by the vertices $\{v_1, \ldots, v_{d_G(v)}\}$ contains a walk and therefore is connected. This implies that $G - \{v\}$ is also connected. This proves the case when $|L| = 0$ or $|L| = 1$.

Now suppose that $p = |L| \geq 2$, let ϵ be a critical radius and let $L = \{k_1, \ldots, k_p\}$, where we may write $1 \leq k_1 < k_2 < \ldots < k_p \leq d_G(v)$. Suppose that x_i is the unique point in $g_i \bigcap C(v, \epsilon)$ and let $1 \leq i \leq p-1$. Both of the faces r_{k_i} and $r_{k_{i+1}}$ are infinite faces of G, and since G is a near-triangulation, $r_{k_i} = r_{k_{i+1}}$. Therefore, and for any two points $x, y \in \Sigma$ where $x \in S(v, \epsilon, x_{k_i}, x_{k_i+1}) \bigcap r_{k_i}$ and $y \in S(v, \epsilon, x_{k_{i+1}}, x_{k_{i+1}+1}) \bigcap r_{k_{i+1}}$ there is a line $P_i \subset \Sigma - G$ with endpoints x and y. By letting ϵ approach 0, x and y approach v, and so we see that for $1 \leq i \leq p-1$ there is a closed curve C_i such that

1) $C_i \bigcap G = \{v\}$

2) Every edge in $\{g_{k_i+1}, \ldots, g_{k_{i+1}}\}$ is a subset of the topological interior of the curve C_i, and every edge in $\{g_1, \ldots, g_{d_G(v)}\} - \{g_{k_i+1}, \ldots, g_{k_{i+1}}\}$ is a subset of the topological exterior of the curve C_i.

Let Q be a v_{k_i}-v_{k_i+1} path in G. Condition 2 and the Jordan Curve Theorem imply that (treating Q as a subset of Σ rather than a subgraph of G) $Q \bigcap C_i \neq \emptyset$. By condition 1), this implies that $Q \bigcap C_i = \{v\}$. This implies that v_{k_i} and v_{k_i+1} are in different components of $G - \{v\}$. Since this holds for $1 \leq i \leq p-1$, $G - \{v\}$ will have p components, and this establishes that $G - \{v\}$ has $|L|$ components.

If v is a vertex in G that is not incident to the infinite face of G then $|L| = 0$ and the result claimed in the lemma holds. Also, if v is a non-cut vertex of G that is incident to the infinite face, then $|L| \geq 1$ from the very fact that v is incident to the infinite face, and $|L| < 2$ since v is not a cut-vertex of G. Finally, if v is a cut-vertex of G then $|L| \geq 2$ and property (i) of configurations implies that $|L| \leq 2$. This completes the proof of Lemma 4.3.2.

Lemma 4.3.3 *Let K be a configuration with underlying drawing G which appears in S where S is a either a free completion of K or a triangulation, and let $v \in V(G)$ be a non-cut vertex of G. Let $g_1, \ldots, g_{d_G(v)}$ be a clockwise listing in G of edges incident to v having the property that if v is incident to the infinite face of G, then the edges g_1 and $g_{d_G(v)}$ are also incident to the infinite face of G. Then a clockwise listing in S of edges incident to v is $g_1, g_2, \ldots, g_{d_G(v)}, s_1, \ldots, s_p$, where $d_G(v) + p = \gamma_K(v)$, $p = 0$ if and only if v is not incident to the infinite face of G, and $s_1, \ldots, s_p$ are edges in $E(S) - E(G)$ having their endpoints in $\{v\} \bigcup (V(S) - V(G))$*

Proof: From Lemma 4.3.2, there is either 0 or 1 indices in $\{1, \ldots, d_G(v)\}$ such that g_i and g_{i+1} are incident to the infinite face of G, (where when $i = d_G(v)$ we interpret g_{i+1} as g_1).

By Lemma 4.3.1, $g_1, g_2, \ldots, g_{d_G(v)}$ is a portion of the clockwise listing in S of edges incident to v.

If there are 0 indices, then Lemma 4.3.1 implies that $g_{d_G(v)-1}, g_{d_G(v)}, g_1$ is also a portion of the clockwise listing in S of edges incident to v. This implies that $g_1, g_2, \ldots, g_{d_G(v)}$ is the clockwise listing in S of edges incident to v. In this case $p = 0$, and thus v is not incident to the infinite face of G, so property (ii) of configurations insures that $\gamma_K(v) = d_G(v)$.

Now suppose that v is incident to the infinite face. Property (ii) of configurations implies that $d_G(v) < \gamma_K(v)$ and since K appears in S, $d_S(v) = \gamma_K(v)$. Therefore, there are $p = \gamma_K(v) - d_G(v)$ edges $s_1, \ldots, s_p \in E(S) - E(G)$ incident to v in addition to the edges $g_1, \ldots, g_{d_G(v)}$. If for $1 \leq i \leq p$, the vertex $v_i \in V(S) - \{v\}$ is an endpoint of s_i, then $v_i \notin V(G)$, because if it was, then the fact that G is an induced subdrawing of S would imply that $s_i \in E(G)$. We may assume without loss of generality that $s_1, s_2, \ldots, s_p$ is a portion of a clockwise listing in S of edges incident to v. Therefore, since $g_1, g_2, \ldots, g_{d_G(v)}$ is a portion of the clockwise listing in S of neighbors incident to v, $g_1, \ldots, g_{d_G(v)}, s_1, \ldots, s_p$ is a clockwise listing in S of neighbors incident to v. This completes the proof of the Lemma 4.3.3.

Lemma 4.3.4 *Let K be a configuration that appears in S, where S is either a free completion of K or a triangulation and let $v \in V(G(K))$ be a cut-vertex of $G = G(K)$. Let $g_1, \ldots, g_{d_G(v)}$ be a clockwise listing in G of edges incident to v such that g_1 and $g_{d_G(v)}$ are both incident to the infinite face of G. There is a clockwise listing in S of edges incident to v of the following form:*

$g_1, \ldots, g_p, s_1, g_{p+1}, \ldots, g_{d_G(v)}, s_2$

where $p < d_G(v)$, $\gamma_K(v) = d_G(v) + 2$, and where the endpoints of s_i are in $(V(S) - V(G)) \bigcup \{v\}$.

Proof: From Lemma 4.3.2 we know that there is an index in $\{1, 2, \ldots, d_G(v) - 1\}$ such that g_p and g_{p+1} are incident to the infinite face of G. Here, we are as usual interpreting g_{i+1} as g_1 when $i = d_G(v)$. Since there are exactly two indices with this property, we know that there exists an integer p such that for $1 \leq l < p$ and $p + 1 \leq l < d_G(v)$ the edges g_l and g_{l+1} are both incident to a common finite face of G. Therefore by Lemma 4.3.1, $g_1, g_2, \ldots, g_p$ and $g_{p+1}, \ldots, g_{d_G(v)}$ are portions of the clockwise listing in S of edges incident to v. By property (i) of configurations, and by the fact that K appears in S, $d_S(v) = \gamma_K(v) = d_G(v) + 2$. Let s_1, s_2 be the edges in $E(S) - E(G)$ that are incident to v. If $v_i \in V(S) - \{v\}$ are the endpoints of s_i for $i = 1, 2$, then the fact that G is an induced subdrawing of S implies that $v_1, v_2 \in V(S) - V(G)$. Thus there are without loss of generality, three possibilities for the clockwise listing in S of edges incident to v:

1) $g_1, \ldots, g_p$,s_1,g_{p+1},. . .,$g_{d_G(v)}$,s_2.

2) $g_1, \ldots, g_p$,s_1, s_2,$g_{p+1}, \ldots, g_{d_G(v)}$.

3) $g_1, \ldots, g_p$,$g_{p+1}, \ldots, g_{d_G(v)}$,$s_1, s_2$.

By Lemma 4.2.1, v is not incident to an infinite face of S. Therefore any two consecutive edges in a clockwise listing in S of edges incident to v are incident to a common finite face of S. Thus, if 2) is a clockwise listing in S of edges incident to v, then $g_{d_G(v)}$ and g_1 are incident to a common finite triangular face r of S. This means that if $h_{d_G(v)}$ and h_1 are the endpoints of $g_{d_G(v)}$ and g_1 in $V(G) - \{v\}$, then there is an edge $e \in E(S)$ with endpoints $h_{d_G(v)}$ and h_1 and incident to the face r. Now e has both endpoints in G and since G is an induced subdrawing of S, e is an edge of G, and thus r is a face of G. Now it must be that r is designated as the infinite face of G, since by hypothesis, $g_{d_G(v)}$, g_1 are both incident to the infinite face of G. Therefore, the infinite face of G is a triangular face which is incident to the edges $g_{d_G(v)}$,g_1, and e. However, the infinite face of G was also assumed to be incident to the edges g_p and g_{p+1} which are both distinct from the edges $g_{d_G(v)}$ and g_1. This is a contradiction. A similar contradiction arises if 3) is a clockwise listing in G. This implies that 1) is a clockwise listing in G of edges incident to v, and this completes the proof of the Lemma.

Let S and S' be drawings and let $f : E(S) \rightarrow E(S')$. Also, let $x \in V(S) \bigcap V(S')$ and suppose that $g_1, \ldots, g_{d_S(x)}$ is a clockwise listing in S of edges incident to x. If $f(g_1), f(g_2), \ldots, f(g_{d(x)})$ is a clockwise listing in S' of edges incident to x then we will say that f *preserves the embedding at* x.

Lemma 4.3.5 *Let K be a configuration with underlying graph $G = G(K)$ and let S be a free completion of K with ring R. Suppose that K appears in S' where S' is a triangulation or a free completion of K. Let E_S (respectively $E_{S'}$) denote the set of edges of S (respectively S') that have at least one endpoint in $V(G)$. There is a function $\phi_n : E_S \rightarrow E_{S'}$ such that*

1) $\phi_n(e) = e$ for every $e \in E(G)$ and $\phi_n(E(S) - E(R) - E(G)) \subset E(S') - E(G)$

2) For every $v \in V(G)$, ϕ_n preserves the embedding at v

3) If e and e' are distinct edges of E_S which have a common endpoint $x \in V(G)$, then $\phi_n(e) \neq \phi_n(e')$

4) The function ϕ_n is one to one and onto.

Proof: Every edge in E_S has at least one endpoint in $V(G)$ so we will define ϕ_n by considering an arbitrary vertex $v \in V(G)$ and defining ϕ_n for every edge of S that is incident to v.

Suppose that $g_1, \ldots, g_{d_G(x)}$ is a clockwise listing in G of edges incident to v.

First consider the case when v is not incident to the infinite face of G. Lemma 4.3.3 insures that a clockwise listing in S is $g_1, \ldots, g_{d_G(v)}$ and that a clockwise listing in S' is $g_1, \ldots, g_{d_G(v)}$. In this case, we simply define $\phi_n(g_i) = g_i$ for $1 \leq i \leq d_G(v)$.

Now consider the case that v is a non cut-vertex of G that is incident to the infinite face of G. By Lemma 4.3.3, a clockwise listing in S of edges incident to v is $g_1, \ldots, g_{d_G(v)}, s_1, \ldots, s_p$, where $s_1, \ldots, s_p \in E(S) - E(G)$. and a clockwise listing in S' of edges incident to v is $g_1, \ldots, g_{d_G(v)}, s'_1, \ldots, s'_p$, where $s'_1, \ldots, s'_p \in E(S') - E(G)$.

In this case, define $\phi_n(g_i) = g_i$ for $1 \le i \le d_G(v)$ and $\phi_n(s_i) = s'_i$ for $1 \le i \le p$.

Finally, if v is a cut-vertex of G, Lemma 4.3.4 again implies that a clockwise listing in S of edges incident to v is $g_1, \ldots, g_p, s_1, g_{p+1}, \ldots, g_{d_G(v)}, s_2$, where $s_1, s_2 \in E(S) - E(G)$, and the clockwise listing in S' of edges incident to v is $g_1, \ldots, g_p, s'_1, g_{p+1}, \ldots,$ $g_{d_G(v)}, s'_2$, where $s'_1, s'_2 \in E(S') - E(G)$. Make the obvious definition $\phi_n(g_i) = g_i$ and $\phi_n(s_i) = s'_i$ for $i = 1, 2$.

First, note that ϕ_n is well defined, despite the fact that each edge $e \in E(G)$ is defined twice, once for each of its endpoints. However, in both cases its image under ϕ_n is e, so there is no ambiguity. This also shows that $\phi_n(E(G)) \subset E(G)$. All other edges in E_S are defined exactly once this way, and have image in $E_{S'} - E(G)$. Also, ϕ_n clearly preserves the embedding at every $x \in V(G)$. Finally, if e and e' have a common endpoint in $V(G)$ then it is clear that $\phi_n(e) \neq \phi_n(e')$. Thus, property 3 holds.

Now we prove that the function ϕ_n is one to one and onto. First, note that $|E_S| = |E(G)| + \sum_{x \in V(G)} \gamma_K(x) - d_G(x) = |E(G)| + \sum_{x \in V(G)} d_{S'}(x) - d_G(x) = |E_{S'}|$. Thus it will suffice to prove that ϕ_n is one to one. Suppose that $\phi_n(e) = \phi_n(f)$. Now the edge $\phi_n(e) = \phi_n(f)$ has a common endpoint $x \in V(G)$. It is clear from the definition of ϕ_n that if $x \in V(G)$ is the endpoint in S of an edge $g \in F_S$, then x is an endpoint in T of the edge $\phi_n(g)$. Thus both e and f have $x \in V(G)$ as one of their endpoints in S. Property 3 then shows $e = f$, and completes the proof that ϕ_n is one to one and onto. This establishes the lemma.

We will refer to the function ϕ_n of Lemma 4.3.5 the *natural edge function.*

Theorem 4.3.1 *Let K be a configuration with free completion S and ring R which appears in S', where S' is either a free completion of K or a triangulation. Then there is a projection ϕ from S to S'.*

Proof: We define $\phi(e) = \phi_n(e)$ for every $e \in E(S)$ which has at least one endpoint in $V(G)$. Thus the values of ϕ are determined for all edges which have at least one endpoint in $V(G)$. It remains then to define ϕ for faces of S, vertices of $V(S)$, and edges of R, and we will do so in that order.

For finite faces $r \in F(G)$, define $\phi(r) = r$. For faces $r \in F(S) - F(G)$, Lemma 4.2.1 shows that r is adjacent to a vertex $x \in V(G)$. Let $e_1, \ldots, e_{\gamma_K(v)}$ be a clockwise listing in S of edges incident to v such that the face r is incident to e_i and e_{i+1} (where if $i = \gamma_K(x)$ then e_{i+1} is interpreted as e_1.) Because K is a configuration, $\gamma_K(v) \geq 5$ and therefore Lemma 4.1.3 implies that r is the unique finite face of $F(S)$ that is incident to e_i and e_{i+1}. Since ϕ preserves the embedding at x, the edge $\phi(e_{i+1})$ follows the edge $\phi(e_i)$ in the clockwise listing in S' of edges incident to x. Therefore, there is at least one finite face r' of $F(S')$ that is incident to both $\phi(e_i)$ and $\phi(e_{i+1})$ in S'. Because $d_{S'}(x) = \gamma_K(x) \geq 5$, Lemma 4.1.3 implies that r' is the unique finite face of $F(S')$ that is incident to $\phi(e_i)$ and $\phi(e_{i+1})$. Since r is the unique finite face of $F(S)$ that is incident to e_i and e_{i+1}, it follows that the assignment $\phi(r) = r'$ is well defined. Since every face of $F(S)$ is incident to at least one vertex of $V(G)$, this completely defines ϕ on the set $F(S)$.

Moreover, for any $e \in E(S) - E(R)$ and for any $f \in F(S)$, if e is incident to f in S, then this construction has the property that $\phi(e)$ is incident to $\phi(f)$ in S'.

We now define ϕ for all vertices in $V(S)$. First, if $v \in V(G)$ define $\phi(v) = v$. We pause at this point to note that ϕ as thus far defined has the property that for any $z \in V(G) \bigcup E(G) \bigcup F(G)$, $\phi(z) = z$. Now let $v \in V(R)$. By Lemma 4.2.1, there is at least one vertex $g \in V(G)$ that is adjacent to v. Let $g_1, g_2, \ldots, g_p$ be the neighbors of v in $V(G)$. By Lemma 4.2.1, a clockwise listing of the edges incident to v is $r_1, e_1, e_2, \ldots, e_p, r_2$, where r_1 and r_2 are edges of the ring R that are both incident to the infinite face of S, and where the endpoint of e_i in $V(S) - \{v\}$ is g_i. For $1 \leq i \leq p$, denote $e_i' = \phi(e_i)$. Now e_i' is an edge of $E(S')$ which has one endpoint g_i and where we denote the other endpoint as x_i'. Now from property 1) of Lemma 4.3.5 and from the fact that G is an induced subdrawing of S', we infer that $x_i' \notin V(G)$ for $1 \leq i \leq p$.

We now claim that $x_i' = x_{i+1}'$ for $1 \leq i \leq p-1$. If $p = 1$ there is nothing to prove so assume $p \geq 2$, $1 \leq i \leq p-1$ and let r be the unique face of $F(S)$ that is incident to both e_i and e_{i+1} in S. This face r is incident to a third edge f in S and this edge must be in $E(G)$ since both its endpoints g_i and g_{i+1} are in $V(G)$ and G is an induced subdrawing of S. Since r is incident to the edges e_i, e_{i+1} and f in S, it follows from the way ϕ was constructed for faces that $\phi(r)$ is incident to $\phi(e_i)$, $\phi(e_{i+1})$ and $\phi(f) = f$ in S'.

We now show that $\phi(e_i)$, $\phi(e_{i+1})$ and $\phi(f)$ are distinct edges in S'. First, $\phi(e_i) \neq \phi(e_{i+1})$ because the edges e_i and e_{i+1} do not share any common vertex in $V(G)$, and the natural edge function maps edges incident to any $u \in V(G)$ to edges again incident to $u \in V(G)$. Moreover, $f = \phi(f) \in E(G)$, along with property 1) of the natural edge function implies that $f \neq \phi(e_i)$ and $f \neq \phi(e_{i+1})$.

Thus far we have shown that the three distinct edges $\phi(e_i)$, $\phi(e_{i+1})$ and $\phi(f) = f$ are all incident to the finite face $\phi(r)$ of S', and this implies that $x'_i = x'_{i+1}$, which completes the proof of the claim since i was arbitrary.

By this claim, we may without ambiguity define $\phi(v) = x'_1$. Note that this implies that $\phi(x) \in V(S') - V(G)$ if $x \in V(R)$. This claim also shows that if $v \in V(R)$ is incident in S to an edge $e \in E(S) - (E(G) \bigcup E(R))$, then $\phi(v)$ is incident to $\phi(e)$ in S'. This in turn can be shown to imply that if $v \in V(R)$ is incident in S to $r \in F(R)$ then $\phi(v)$ is incident with $\phi(r)$ in S'. To see this, use Lemma 4.2.1 to find an edge e in $E(S) - (E(G) \bigcup E(R))$ which is incident in S to both v and r. From what we just showed, $\phi(e)$ is incident to $\phi(v)$ in S', and from the construction of ϕ for faces, $\phi(f)$ is incident in S' to $\phi(e)$. This implies that $\phi(f)$ is incident in S' to $\phi(v)$.

It remains to define ϕ for every edge $e \in E(S)$ which has both endpoints in $V(R)$. Let x_1, x_2 be the endpoints of e in $V(R)$. Using Lemma 4.2.1, let $r \in F(S)$ be the unique finite face of S which is incident to e, let x be the unique vertex of $V(G)$ that is incident to r, and for $i = 1, 2$ let e_i be the edges in $E(S)$ which have endpoints x and x_i and are incident in S to r. Now $\phi(r)$ and $\phi(x) = x$ are incident in S' to $\phi(e_i)$ for $i = 1, 2$ and to each other. Since $\phi(r)$ is a finite triangular face of S', it is incident to a unique edge $e' \neq \phi(e_1), \phi(e_2)$. Define $\phi(e) = e'$. This can be done for every $e \in E(R)$ and so ϕ is now completely defined. The endpoints of e' must be the endpoints of the the edges $\phi(e_1)$ and $\phi(e_2)$ in $V(S) - \{x\}$. Since x_i is the endpoint of e_i in S for $i = 1, 2$, $\phi(x_i)$ must be the endpoint of $\phi(e_i)$ for $i = 1, 2$. Hence e incident to x_i in S implies $\phi(e)$ is incident in S' to $\phi(x_i)$. The construction also implies that ϕ preserves incidence between edges in $E(R)$ and faces in $F(S)$.

Clearly ϕ maps $V(S)$ into $V(T)$, $E(S)$ into $E(T)$ and $F(S)$ into $F(T)$. Let $u, v \in V(G)$. From what was noted before, $\phi(V(R)) \subset V(S') - V(G)$. Also, $\phi(x) = x$ for every $x \in V(G)$ so $\phi(V(G)) = V(G)$. Therefore, if $\phi(u) = \phi(v)$ then either $\{u, v\} \subset V(R)$ or $\{u, v\} \subset V(G)$. If the former holds we are done and if the latter holds then $u = \phi(u) = \phi(v) = v$ so $u = v$. By the construction of ϕ for faces $\phi(r) = r$ for $r \in F(G)$ and $\phi(F(S) - F(G)) \subset F(S') - F(G)$, so $\phi(r) \neq \phi(r')$ for $r \neq r'$. Finally, let $e, f \in E(S)$. If both $e, f \in E(G)$, then $e = \phi(e) = \phi(f) = f$. From property 1) of the natural edge function, we may assume that neither e nor f is and edge of $E(G)$. From the definition of ϕ, if $e' \in E(R)$, $\phi(e')$ has both endpoints in $V(S') - V(G)$, and so it cannot be that one of e, f is in $E(R)$ and the other in $E(S) - E(R)$. Thus we may assume that $e, f \in E(S) - (E(G) \bigcup E(R))$. From the definition of ϕ_n, ϕ maps edges in $E(S)$ with endpoint $x \in V(G)$ to edges in $E(S')$ with endpoint x. Hence, e and f must share a common endpoint x in $V(G)$. However, property 3) of the natural edge function, then implies that $\phi(e) \neq \phi(f)$. All of this establishes property (ii) of projections. Property (iii) of projections has been established throughout the proof and so the proof of Theorem 4.3.1 is complete.

4.4 Existence and Uniqueness of Free Completions

This section is devoted to establishing that every configuration has an essentially unique free completion.

Theorem 4.4.1 *Every configuration K has a free completion S with ring R.*

We omit the proof of this theorem.

Lemma 4.4.1 *If S_1 and S_2 are two free completions of a configuration K with rings R_1, R_2 respectively, then there is an isomorphism between S_1 and S_2. Moreover, if $u, v \in V(S_1)$ are incident to a common face of S_1, then the images of u and v under this isomorphism are incident to a common face of S_2.*

Proof: By Lemma 4.2.1, a configuration appears in any free completion of itself. Therefore, Theorem 4.3.1 shows that there is a function $\phi : V(S_1) \ \rightarrow V(S_2)$ which is the restriction to $V(S_1)$ of a projection of $V(S_1)$ into $V(S_2)$.

First, it will be proved that ϕ is a one to one and onto function. To show that it is onto, let $w \in V(S_2)$. Since $\phi(x) = x$ for every $x \in V(S_1) \bigcup E(S_1) \bigcup F(S_1)$, we may assume that $w \in V(R_2)$. Now w is adjacent in S_2 to some vertex $v \in V(G) \bigcap V(S_2)$ by an edge which we denote e'. By property 3) of the natural edge function, there is an $e \in E(S_1)$ which has v as an endpoint and such that $\phi(e) = e'$. It then follows that there is an endpoint $u \neq x$ of e in S_1 and from the way ϕ is defined, $\phi(u) = w$, so ϕ is an onto function. Now the number of vertices in the ring of a free completion of a configuration equals the ring size of the configuration. Therefore $|V(R_1)| = |V(R_2)|$ and therefore $|V(S_1)| = |V(S_2)|$. Since ϕ is onto, it follows that it is also one-to-one.

Having established that ϕ is a one-to-one and onto function, the fact that ϕ is the restriction to $V(S_1)$ of a projection from S_1 into S_2 proves that ϕ is an isomorphism of the graph S_1 to S_2. The properties of projections also imply that any two vertices $u, v \in V(S_1)$ that are incident to a common face of S_1 have the property that $\phi(u)$ and $\phi(v)$ are incident to a common face of S_2. This completes the proof of Lemma 4.4.1.

One of the fundamental results of this section is that each configuration has essentially one free completion. This will be made more precise in Theorem 4.4.3. More precisely, we will show that if S_1 and S_2 are two free completions of a configuration K, then there is a homeomorphism of Σ which fixes $G(K)$ pointwise and which maps S_1 into S_2. We will need a result of topology to establish this. A reference for this result is [40].

Theorem 4.4.2 *(Jordan-Schönflies Theorem) If f is a homeomorphism of a simple closed curve C in the plane onto another closed curve C' in the plane, then f can be extended to a homeomorphism of the entire plane.*

Let $A \subset \sigma$. Recall that $bd(A)$ is defined to be the set $\bar{A} - A$. Define the *interior of* A, denoted $interior(A)$, to be the set $\bar{A} - bd(A)$. We now state a lemma which is based on a standard result in topology about how to "paste" two distinct continuous functions together to construct a continuous function that extends both of them [41].

Lemma 4.4.2 *Let A_1 and A_2 be closed subsets in a topological space Ω, where $\Omega = A_1 \bigcup A_2$. Suppose that $h_i : A_i \to B_i$ is a homeomorphism such that*

1) $h_1(interior(A_1)) \bigcap h_2(interior(A_2)) = \emptyset$ and

2) $h_1(x) = h_2(x)$ for every $x \in \overline{A_1 \bigcap A_2}$.

There is a homeomorphism from $A_1 \bigcup A_2$ to $B_1 \bigcup B_2$ which agrees with h_i on A_i for $i = 1, 2$.

Theorem 4.4.3 *Let K be a configuration and suppose that S_1 and S_2 are two free completions of K. There exists a homeomorphism of the sphere which fixes $G(K)$ and maps S_1 into S_2.*

Proof: By the proof of Lemma 4.4.1, there is a projection ϕ from S_1 to S_2 whose restriction to $V(S_1)$ is a graph isomorphism of S_1 onto S_2. Moreover, because ϕ is a projection, it has the property that $\{x_1, \ldots, x_k\}$ is a circuit bounding a finite face in S_1 if and only if $\{\phi(x_1), \ldots, \phi(x_k)\}$ is a facial circuit in S_2. We now show that there is a homeomorphism h from the drawing S_1 (considered as a topological space) to the drawing S_2, with the property that for every $x \in \Sigma \bigcap S_1$,

1) if $y \in V(S_1)$ then $x = y$ if and only if $h(x) = \phi(y)$ and

2) if $y \in E(S_1)$, then $x \in y$ if and only if $h(x) \in \phi(y)$.

Defining h for vertices $x \in V(S_1)$ is easy; set $h(x) = \phi(x)$. To define h for edges, we build a homeomorphism mapping $\bar{e}$ to $\overline{\phi(e)}$ for each $e \in E(S_1)$. For each edge $e \in E(S_1)$ there are homeomorphisms $h_e : [0,1] \rightarrow \bar{e}$ and $h_{\phi(e)} : [0,1] \rightarrow \overline{\phi(e)}$. The composition $h_{\phi(e)}(h_e^{-1}x)) : \bar{e} \rightarrow \overline{\phi(e)}$ is a homeomorphism and so we define $h(x) = (h_{\phi(e)}h_e^{-1})(x)$ to be this composition. Since no two edges intersect, $h(x)$ is well defined, and h satisfies the property that $x \in y$ if and only if $h(x) \in \phi(y)$. It remains to show that h is a homeomorphism of S_1 onto S_2.

The topological space S_1 can be thought of as a union of closed sets consisting of the vertices of S_1 and the closure $\bar{e}$ of each edge in S_1. The function h restricted to each of these closed sets is a homeomorphism of that closed set onto its image in $V(S_2)$. Moreover, the intersection of any of these closed sets consists only of vertices, and thus h is well defined on these intersections. Thus by Lemma 4.4.2, h is a homeomorphism of the entire space S_1 onto the space S_2. By the construction, h satisfies properties 1) and 2).

Let r be a face in S and r' its image under h in S'. Because ϕ is a projection, the image of $bd(r)$ under h is $bd(r')$. Thus the restriction of h to $bd(r)$ is a homeomorphism of the simple closed curve $bd(r)$ to the simple closed curve $bd(r')$. By the Jordan-Schönflies Theorem, this homeomorphism can be extended to a homeomorphism h' of Σ onto Σ which agrees with h on $bd(r)$. Moreover, we may arrange things so that the set $h'(r)$ equals the set r'. The restriction then of h' to $\overline{r}$ is also a homeomorphism of $\overline{r}$ to $\overline{r'}$ which we henceforth denote h_r. We know then that $h_r(x) = h(x)$ for $x \in bd(r)$.

If r_1 and r_2 are two faces of S for which $\overline{r_1} \bigcap \overline{r_2} \neq \emptyset$, then $h_{r_1}(x) = h(x) = h_{r_2}(x)$ for any $x \in \overline{r_1} \bigcap \overline{r_2}$. We can thus use the collection of homeomorphisms $\{h_r : r$ is a face in $S_1\}$ along with Lemma 4.4.2 to build a homeomorphism of Σ onto Σ which agrees with h on S. This completes the proof of Theorem 4.4.3.

Chapter 5

Reducibility for the Fiorini-Wilson-Fisk Conjecture

5.1 Introduction

In this section we define what we mean by reducibility for the Fiorini-Wilson-Fisk Conjecture and outline the logic used to establish the reducibility of every configuration in $\mathcal{U}$. Of course, a computer actually verifies the reducibility of each configuration, as it would be too difficult using the present techniques to do so by hand. Essentially, reducibility for the Fiorini-Wilson-Fisk Conjecture is just a strengthening of reducibility for the Four Color Theorem, and in fact many of the configurations that were reducible for the Four Color Theorem are also reducible for the Fiorini-Wilson-Fisk Conjecture.

5.1.1 Tricolorings and Notation

Recall that two functions c and c' with identical domain and range=$\{1, 2, \ldots, k\}$ are *equivalent* if $\{c^{-1}(\{1\}), c^{-1}(\{2\}), \ldots, c^{-1}(\{k\})\} = \{c'^{-1}(\{1\}), c'^{-1}(\{2\}), \ldots, c'^{-1}(\{k\})\}$. We will use this frequently when the functions represent colorings. If $\mathcal{A}$ is a set of functions with domain D and range $R = \{1, \ldots, k\}$, then $\eta(\mathcal{A})$ will denote the set of all functions with domain D and range R that are equivalent to some coloring in $\mathcal{A}$.

Let T be a triangulation or near-triangulation, and let $F(T)$ denote the set of all faces of T that are bounded by exactly 3 edges. A *tricoloring* of T is a function $c : F(T) \to \{-1, 0, 1\}$ such that for every $f \in F(T)$, and for any two distinct edges i and j incident to f, $c(i) \neq c(j)$. The next theorem establishes a connection between tricolorings and vertex colorings of a graph and edge colorings of the dual of the graph.

Theorem 5.1.1 *(Tait) Let G be a triangulation or near-triangulation. The following statements are equivalent:*

(i) The vertices of G can be 4-colored.

(ii) The drawing G has a tricoloring.

(iii) The dual of G can be edge-3-colored.

5.1.1.1 Tricolorings and Contracts

The reducibility part of the recent Robertson et al. proof of the Four Color Theorem essentially proceeds by induction. Without going into details, the contraction of edges is critical in their argument to produce smaller graphs. To avoid notational difficulties, they introduced the idea of a *tricoloring of T modulo X*, where T is a triangulation or near-triangulation and X is a set of edges in T. As the definition will reveal, the set X represents the set of edges to be contracted.

Following the definitions of Robertson et al. [8], a set $X \subset E(T)$ is said to be *sparse* if no two edges of X are incident to a common finite face of T, and if T is a near-triangulation, then no edge of X is incident to the infinite face of T. If X is sparse, then a *tricoloring of* T *modulo* X is a coloring $\kappa\colon E(T) - X \to \{-1, 0, 1\}$ such that for every finite face $r \in F(T)$

1.) If r does not have any edges in common with X then κ assigns distinct colors to the three edges of r.

2.) If r has exactly one edge in common with X, then $\kappa(e) = \kappa(f)$ for the other two edges e and f of r.

Recall that a counterexample is defined to be a planar graph which is not a vertex Fiorini-Wilson-Fisk graph and has at most one vertex-4-coloring. A minimum counterexample is a counterexample with a minimum number of vertices. The following theorem, adapted from [8], captures the idea that if one can contract edges in a minimum counterexample so that no loops are created, then the resulting graph has a tricoloring.

Theorem 5.1.2 *Let* T *be a minimum counterexample, and let* $X \subset E(T)$ *be a nonempty, sparse set such that there is no circuit* C *of* T *for which* $|E(C) - X| = 1$*. Then there is a tricoloring of* T *modulo* X*.*

Proof: Let $T(X)$ be the subgraph of T consisting of the vertices of T and the edges of X. Let V_1,V_2,. . ., V_p be the vertex sets of the components of $T(X)$. Let H be the graph obtained by deleting multiple edges in the graph with vertex set $\{V_1, \ldots, V_p\}$ and with V_i adjacent to V_j if and only if there is an edge in $E(T) - X$ which joins two vertices v_i and v_j with $v_i \in V_i$ and $v_j \in V_j$. Claim: H is loopless. If there was a loop f joining the vertex V_i to itself then there would be an edge $f' \in E(T) - X$ which joins two distinct vertices x, y of T that are both in V_i. Since V_i is a vertex set of a component of the graph $T(X)$, there is a path in T joining x and y and consisting entirely of vertices in T. The circuit $P \bigcup \{f'\}$ violates the condition that $|E(C) - X| \neq 1$ for every circuit C in T. Thus H is loopless. Since X is nonempty, $p < |V(T)|$ and since T is a minimum counterexample and H is loopless, H is either a vertex Fiorini-Wilson-Fisk graph or H has at least two vertex-4-colorings that are not permutations of one another. Either way, H has a vertex-4-coloring c. Use the standard Tait coloring to define a coloring $\kappa : E(T) - X \to \{-1, 0, 1\}$; that is for an edge e of $E(T) - X$ with endpoints $u \in V_i$ and $v \in V_j$, define:

$\kappa(e) = -1$ if $\{c(V_i), c(V_j)\} = \{1, 2\}$ or $\{c(V_i), c(V_j)\} = \{3, 4\}$.

$\kappa(e) = 0$ if $\{c(V_i), c(V_j)\} = \{1, 3\}$ or $\{c(V_i), c(V_j)\} = \{2, 4\}$.

$\kappa(e) = 1$ if $\{c(V_i), c(V_j)\} = \{1, 4\}$ or $\{c(V_i), c(V_j)\} = \{2, 3\}$.

Claim: κ is a tricoloring of T modulo X. To prove this, let r be a triangular face in $F(T)$ incident to the vertices $\{x, y, z\}$ and edges $e = \{x, y\}$,$f = \{x, z\}$ and $g = \{y, z\}$. If $\{e, f, g\} \bigcap X = \emptyset$ then x, y and z are in three distinct vertices of H, say $x \in V_i$, $y \in V_j$ and $z \in V_k$. In the vertex-4-coloring of H which defines κ, V_i, V_j and V_k receive different colors and thus κ can be seen to assign different colors to e,f, and g. If one of the edges incident with r is in X, say $g \in X$, then the vertices y and z are in the same vertex, say V_i of H. If x were in the same component of $T(X)$ as y or z, then since $e, f \notin X$, there would be a circuit C in T such that $|E(C) - X| = 1$. Since X is sparse, x is in a distinct vertex. Hence, $\kappa(e)$ and $\kappa(f)$ are well defined and equal to each other. This completes the proof of the claim that κ is a tricoloring of T modulo X and hence completes the proof of the theorem.

If $X \subset E(T)$ is sparse and $|E(C) - X| \geq 2$ for all circuits C in T then we say that X is *contractible in T*.

5.1.2 Colorings of a Ring

Let S be a free completion of a configuration K with ring R. If c is a tricoloring of S, the restriction of c to the ring defines a coloring of that ring. A basic part of the theory of reducibility is the consideration of these ring colorings. Let the vertices of R be 1,2,...,r and the edges of R be e_1,e_2,...,e_r, where e_i has endpoints i and $i+1$ for $i = 1, \ldots, r-1$ and e_r has endpoints r and 1. A *coloring* of R is a function $\kappa : E(R) \to \{-1, 0, 1\}$. Let $\mathcal{C}^*(R)$ denote the set of colorings of R. We will sometimes abbreviate $\mathcal{C}^*(R)$ by $\mathcal{C}^*$. By a restriction to R of a tricoloring c of S, we mean the function $c_{|R}$ with domain $E(R)$ and range $\{-1, 0, 1\}$ that agrees with c on the edges of R. Also, if $\kappa : E(R) \to \{-1, 0, 1\}$ is a coloring of R, then we define an extension of κ into a tricoloring of S, to be a tricoloring of S which agrees with κ on $E(R)$. We let $\mathcal{C}(S)$ denote the set of restrictions to R of tricolorings of S. A restriction to R of a tricoloring of S has either one or more extensions into tricolorings of S. Let the set of restrictions to R of tricolorings of S which have exactly one extension into a tricoloring of S be denoted by $\mathcal{U}(S)$ or just $\mathcal{U}$ if the free completion S is understood from the context. If T is a triangulation that is uniquely vertex-4-colorable, and if the free completion S appears in T then it follows that the restriction to R of the corresponding tricoloring of T must be an element of $\mathcal{U}$.

The following definitions are taken from Robertson et. al. [8] A *match* is a an unordered pair $\{e, f\}$ of distinct edges of $E(R)$. A *matching* is a nonempty set of matches $\{\{e_1, f_1\}, \{e_2, f_2\}, \ldots, \{e_k, f_k\}\}$ such that for any $i \neq j$, the edges e_j and f_j are in the same component of $R - \{e_i, f_i\}$. Finally, a *signed matching* is a collection of ordered pairs $\{(\{e_1, f_1\}, \mu_1), (\{e_2, f_2\}, \mu_2), \ldots, (\{e_k, f_k\}, \mu_k)\}$, where the collection

$\{\{e_1, f_1\}, \{e_2, f_2\}, \ldots, \{e_k, f_k\}\}$ is a matching, and where $\mu_i \in \{-1, 1\}$ for $1 \leq i \leq k$. The sign of a match is used to differentiate whether both ends of a kempe chain have the same color or distinct colors.

If $\theta \in \{-1, 0, 1\}$ and κ is a coloring of R we say that κ *θ-fits* a signed matching $M = \{(\{e_1, f_1\}, \mu_1), (\{e_2, f_2\}, \mu_2), \ldots, (\{e_k, f_k\}, \mu_k)\}$ if

(i) $E(R) - \bigcup_{1 \leq i \leq k} \{e_i, f_i\} = \{e \in E(R) : \kappa(e) = \theta\}$ and

(ii) For each $(\{e_i, f_i\}, \mu_i) \in M$, $\kappa(e_i) = \kappa(f_i)$ if and only if $\mu_i = 1$.

A set $\mathcal{C}$ of colorings of R is *consistent* if for every $\kappa \in \mathcal{C}$ and every $\theta, \theta' \in \{-1, 0, 1\}$ there is a signed matching M such that

i) κ $\theta-$fits M.

ii) $\mathcal{C}$ contains every coloring of R that $\theta'-$fits M.

Let $\mathcal{A} \subset \mathcal{C}^*$. A set of colorings $\mathcal{C}$ of R is said to be *$\mathcal{A}$-critical* if for every $\kappa \in \mathcal{C}$ and every $\theta, \theta' \in \{-1, 0, 1\}$, there is a signed matching M such that

i) κ $\theta-$fits M,

ii) $\mathcal{C}$ contains every edge that $\theta'-$fits M, and

iii) there are not two colorings α, $\alpha' \in \mathcal{A}$ and integers $\gamma, \gamma' \in \{-1, 0, 1\}$ such that both α $\gamma-$fits M and α' $\gamma'-$fits M and α is not equivalent to α'.

Lemma 5.1.1 *If $|\mathcal{A} \bigcap \mathcal{C}| \leq 1$, then $\mathcal{C}$ is $\mathcal{A}-$critical if and only if $\mathcal{C}$ is consistent.*

Proof: If $\mathcal{C}$ is $\mathcal{A}-$critical, then it is clearly consistent. Conversely, let $\mathcal{C}$ be a consistent set. We must show that under the hypothesis, $\mathcal{C}$ is critical. Let $\kappa \in \mathcal{C}$ and $\theta \in \{-1, 0, 1\}$. Since $\mathcal{C}$ is consistent, there is a signed matching M such that κ $\theta-$fits M and $\mathcal{C}$ contains every coloring that $\theta'-$fits M. Now let $\alpha, \alpha' \in \mathcal{A}$, and let

$\gamma, \gamma' \in \{-1, 0, 1\}$. Since $|\mathcal{A} \bigcap \mathcal{C}| \leq 1$, it follows that one of either α or α' is not in $\mathcal{C}$. Without loss of generality, assume that $\alpha' \notin \mathcal{C}$. Therefore α' does not $\gamma'-$fit κ, because if it did, condition ii) of consistency would imply that $\alpha' \in \mathcal{C}$. This establishes that $\mathcal{C}$ is $\mathcal{A}-$critical and completes the proof of Lemma 5.1.1.

Consistency and criticality are defined in terms of colorings of a circuit, but the near triangulations to which we want to apply the ideas of consistency may have their infinite face bounded by something other than a circuit. This does not turn out to be a serious obstacle to using consistency as we shall now see. Let R be a circuit with vertices $\{1, 2, \ldots, r\}$ and edges e_1, e_2, ..., e_r where edge e_i joins vertex i to vertex $i + 1$ for $1 \leq i < r$, and edge e_r joins vertex r to vertex 1. Let H be a near triangulation with outer-facial walk $W = v_1, f_1, v_2, f_2, v_3, \ldots, v_r, f_r, v_1$, where $v_1, v_2, \ldots, v_r$ are vertices, not necessarily distinct and where $\{f_1, f_2, \ldots, f_r\}$ are edges such that f_i joins v_i and v_{i+1} for $1 \leq i \leq r - 1$ and where f_r joins the vertices v_r and v_1. Let $\phi : E(R) \rightarrow \{f_1, f_2, \ldots, f_r\}$ be defined by $\phi(e_i) = f_i$. Also suppose that κ is a tricoloring of H and define a function λ on the edges of the circuit $E(R)$ by $\lambda(e) = \kappa(\phi(e))$. Following [8], we say that ϕ *wraps* R around H and that the coloring λ of $E(R)$ is a *lift* of κ.

The next theorem is an important result which uses ideas of both Kempe and Birkhoff.

Theorem 5.1.3 *Let H be a near triangulation with outer facial walk W as above, and let ϕ wrap the circuit R around H. The set $\mathcal{C}$ of all lifts of tricolorings of H is consistent.*

Proof: Let $\kappa \in \mathcal{C}$ and let $\theta \in \{-1, 0, 1\}$. We will construct a signed matching

$M = M(\kappa, \theta)$ such that κ θ-fits M. Following Robertson et. al. [8], we define a $\theta-$*rib* to be a sequence g_0,r_1,g_1,r_2,. . .,r_t,g_t such that

(i) $g_0, g_1, \ldots, g_t$ are distinct edges of H.

(ii) r_1,r_2,. . .,r_t are distinct finite faces of H.

(iii) If $t > 0$ then g_0,g_t are both incident with the infinite face of H, and if $t = 0$ then g_0 is incident with no finite face of H.

(iv) For $1 \leq i \leq t$, r_i is incident with g_{i-1} and with g_i.

(v) For $0 \leq i \leq t$, $\kappa(g_i) \neq \theta$.

Any two distinct $\theta-$ribs $\rho = g_0, r_1$,g_1,r_2,g_3,. . .,g_t and $\rho' = g'_0$,r'_1,g'_1,. . .,$r'_{t'}$,$g'_{t'}$ must have $\{g_0, g_1, \ldots, g_t\} \bigcap \{g'_0, g'_1, \ldots, g'_{t'}\} = \emptyset$ and $\{r_0, r_1, \ldots, r_t\} \bigcap \{r'_0, r'_1, \ldots, r'_{t'}\} = \emptyset$. To see this, note two things. First, if ρ and ρ' share a common non-infinite face r, then ρ and ρ' must also share both of the *unique* (because each of the finite faces is a triangle) edges incident to r that are colored with the colors in $\{-1, 0, 1\} - \{\theta\}$, because of (iv). Second, if ρ and ρ' share a common edge g then ρ and ρ' also share both of the faces that are incident to g, because of (iv) and (v). Using these two facts, we can show that if any two $\theta-$ribs share either an edge or a finite face, then the two $\theta-$ribs are identical.

Because of (iii), unless a rib consists of a single edge, it contains at least two edges incident to the infinite face. Because of (ii) and (iv), a rib does not contain more than two edges which are incident to an infinite face. Thus, if a rib is not a single edge, then it has exactly two edges that are incident to the infinite face and colored with colors in $\{-1, 0, 1\} - \{\theta\}$.

Conversely, we claim every edge that is incident to the infinite face and is colored with a color in $\{-1, 0, 1\} - \{\theta\}$ is in some rib. To see this, let e_0 be an edge incident to the infinite face which is colored $\alpha \in \{-1,0,1\} - \{\theta\}$ and let $\gamma \in \{-1,0,1\} - \{\theta, \alpha\}$. If e_0 is not incident to a finite face, then e_0 is itself a rib by (iii). So suppose that e_0 is incident to a unique finite face s_1 that is a triangle. Because the given coloring is a tricoloring, s_1 has exactly one edge $e_1 \neq e_0$, which receives the color γ. This edge is incident to a face $s_2 \neq s_1$. If s_2 is the infinite face, then e_0,s_1,e_1 is a rib and we have proven that e_0 is in a rib. If s_1 is not an infinite face, then because the coloring is a tricoloring and because s_1 is a triangle, there is an edge $e_2 \neq e_1$ which receives the color θ and is incident to a face $s_3 \neq s_2$. In this way we generate an alternating sequence of edges and faces e_0,s_1,e_1,s_2,.... If the sequence ever selects an infinite face s_{k+1} it terminates. We now show that the construction of this sequence guarantees that all of the edges $e_0, e_1, \ldots,$ are distinct. If not, there would be integers $0 \leq i < j$ such that $e_i = e_j$. Of all such pairs (i, j) choose one with the smallest j and subject to that choose among those the one with the largest i. As a first case, assume $i = 0$. Thus e_j is incident to the infinite face. It follows that $s_1 = s_j$ and $e_1 = e_{j-1}$. By the choice of j, it cannot be that $e_{j-1} \in \{e_0, e_1, \ldots, e_{j-2}\}$ and so $j - 2 = 0$. It follows that $s_1 = s_j = s_2$. This however, contradicts the construction. Thus we have shown that $i > 0$, and in addition that if $e_0, \ldots, e_{j-1}$ are distinct and e_j is incident to the infinite face, then $e_j \neq e_0$.

Of the two faces incident to e_i, the face s_{i-1} precedes the face s_{i+1} is the sequence. Similarly, the face s_j precedes the face s_{j+1} and both are incident to $e_i = e_j$. Clearly $\{s_i, s_{i+1}\} = \{s_j, s_{j+1}\}$. The face s_j is incident to an edge $e_{j-1} \neq e_j$ that receives a color in $\{-1,0,1\} - \{\theta\}$. Since $\{s_i, s_{i+1}\} = \{s_j, s_{j+1}\}$, it follows that the edge e_{j-1} is in one of the faces s_i or s_{i+1}. Therefore $e_j \in \{e_{i-1}, e_i, e_{i+1}\}$. Now the choice of j insures that $e_{j-1} \notin \{e_0, e_1, \ldots, e_{j-2}\}$. Thus, it must be that $i + 1 = j - 1$, so

$j = i + 2$ and $s_{i+1} = s_{j-1}$. Thus $e_i = e_j$ is incident to s_i, $s_{i+1} = s_{j-1}$ and $s_{i+2} = s_j$ and since each edge is incident to at most two faces, two of the three faces s_i,s_{i+1} and s_{i+2} must actually be the same face. By the construction, $s_i \neq s_{i+1}$ which forces $s_i = s_{i+2}$. But then $e_{j-1} = e_{i-1}$ which contradicts the choice of j. Thus, every edge of the sequence is distinct. Since the graph is finite, this means that the sequence must terminate on an edge e_k other than e_0 which is incident to the infinite face. The sequence e_0,s_1,e_1,s_2,. . .,s_k,e_k is a rib which contains e_0, as desired.

This shows that each $\theta-$rib ρ defines a pair of edges $\{e_\rho, f_\rho\}$ which are both incident to the infinite face and which both receive colors from $\{-1, 0, 1\} - \{\theta\}$. We can thus use ρ to define a signed match, namely $(\{e_\rho, f_\rho\}, \mu_\rho)$ where $\mu_\rho = -1$ if $\kappa(e_\rho) \neq \kappa(f_\rho)$ and $\mu_\rho = 1$ otherwise. Now we will show the set of ribs $\{\rho_1, \ldots \rho_p\}$ defines a signed matching. First of all, the set $\{\rho_1, \ldots \rho_p\}$ defines a set of signed matches $M = M(\kappa, \theta) = \{(\{e_{\rho_1}, f_{\rho_1}\}, \mu_{\rho_1}), \ldots, (\{e_{\rho_p}, f_{\rho_p}\}, \mu_{\rho_p})\}$ in the manner defined above. Because of the planarity of the graph, and the fact that two ribs are either disjoint or identical, it must be that for every $i \neq j$, $1 \leq i, j \leq p$, 1) $\{e_{\rho_i}, f_{\rho_i}\} \bigcap \{e_{\rho_j}, f_{\rho_j}\} = \emptyset$ and 2) the removal of $\{e_{\rho_i}, f_{\rho_i}\}$ from the ring could not separate e_{ρ_j} from f_{ρ_j}. Thus M is a signed matching. Now we show that κ $\theta-$fits M. First, because every edge incident to the infinite face and receiving a color in $\{-1, 0, 1\} - \{\theta\}$ must be in a rib, it follows that $\{e_{\rho_1}, \ldots, e_{\rho_p}, f_{\rho_1}, \ldots, f_{\rho_p}\}$ equals $\{f \in E(R) : \kappa(f) \in \{-1, 0, 1\} - \{\theta\}\}$. The definition of μ_{ρ_i} also shows that for every integer i $(1 \leq i \leq p)$, $\kappa(e_{\rho_i}) = \kappa(f_{\rho_i})$ if and only if $\mu_{\rho_i} = 1$. This proves that κ $\theta-$fits $M = M(\kappa)$.

We now finish the proof that $\mathcal{C}$ is consistent. First, for every $\kappa \in \mathcal{C}$ and every $\theta \in \{-1, 0, 1\}$, our construction using ribs has produced a signed matching $M = M(\kappa, \theta)$ such that κ $\theta-$fits M. So let $\theta' \in \{-1, 0, 1\}$ and let κ' be another coloring that $\theta'-$fits

$M(\kappa,\theta) = M$. Define the coloring κ'' as follows: $\kappa''(e) = \theta$ if $k'(e) = \theta'$, $\kappa''(e) = \theta'$ if $\kappa'(e) = \theta$ and $\kappa''(e) = \kappa'(e)$ if $\kappa'(e) \in \{-1,0,1\} - \{\theta,\theta'\}$. It follows that κ'' $\theta-$fits M. Let c be the coloring of H whose lift is κ and let $\{\rho_1,\dots\rho_p\}$ be the set of $\theta-$ribs induced by c which define M. If $(\{e_i,f_i\},\mu_i)$ is the signed match associated with the $\theta-$rib ρ_i, then the fact that κ'' $\theta'-$fits implies that

1a) Either $\kappa''(e_i) = \kappa''(f_i) \neq \kappa(e_i)$ or

1b) $\kappa''(e_i) = \kappa''(f_i) = \kappa(e_i)$ or

1c) $\kappa''(e_i) \neq \kappa''(f_i)$, and $\kappa''(e_i) \neq \kappa(e_i)$ or

1d) $\kappa''(e_i) \neq \kappa''(f_i)$, and $\kappa''(e_i) = \kappa(e_i)$.

where either 1a) or 1b) hold if $\mu_i = 1$ and either 1c) or 1d) hold of $\mu_i = -1$.

In the $\theta-$rib $\rho_i = g_0,r_1,g_1,r_2,\dots,r_t,g_t$, we have $\kappa(g_0) = \kappa(g_2) = \kappa(g_4) = \dots = \kappa(g_{2\lfloor\frac{t}{2}\rfloor}) = \alpha$ and $\kappa(g_1) = \kappa(g_3) = \kappa(g_5) = \dots = \kappa(g_{2\lfloor\frac{t-1}{2}\rfloor+1}) = \beta$ where $\{\alpha,\beta\} = \{-1,0,1\} - \{\theta\}$. There is another tricoloring γ of H that can be obtained by exchanging the colors α and β along H, namely $\gamma(e) = \kappa(e)$ if e is not in ρ_i, $\gamma(e_0) = \gamma(e_2) = \gamma(e_4) = \dots = \gamma(e_{2\lfloor\frac{t}{2}\rfloor}) = \beta$ and $\gamma(g_1) = \gamma(g_3) = \gamma(g_5) = \dots = \gamma(g_{2\lfloor\frac{t-1}{2}\rfloor+1}) = \alpha$. Using this idea, we define a new tricoloring c'' of H by exchanging the colors $\alpha,\beta \in \{-1,0,1\} - \{\theta\}$ along each rib ρ_i for which either 1a) or 1c) holds. The lift of c'' will be κ'' and thus $\kappa'' \in \mathcal{C}$. Moreover, by defining a coloring c' of H from the coloring c'' by swapping the colors θ and θ', that is defining $c'(x) = \theta$ if $c''(x) = \theta'$, $c'(x) = \theta'$ if $c''(x) = \theta$ and $c'(x) = c''(x)$ otherwise, we see that c' is also tricoloing of H whose lift equals κ'. Thus $\kappa' \in \mathcal{C}$ as desired.

This shows that $\mathcal{C}$ is consistent and completes the proof of Theorem 5.1.3.

Lemma 5.1.2 *Let R be a ring and let $\mathcal{A} \subset \mathcal{C}^*(R)$. The empty set is an $\mathcal{A}$-critical set. Also, the union of two $\mathcal{A}-$critical sets is an $\mathcal{A}-$critical set and in particular, the* *union of two consistent sets is consistent. Finally, for any subset B of colorings of R, the maximally $\mathcal{A}$-critical subset of B exists, that is, there is a subset of B which is $\mathcal{A}$-critical, and such that every other $\mathcal{A}$-critical subset of B is contained in it.*

Proof: Let $\mathcal{A} \subset \mathcal{C}^*(R)$. The statement that the empty set is $\mathcal{A}-$critical is vacuously true. Let $\mathcal{C}_1$, $\mathcal{C}_2 \subset \mathcal{C}^*(R)$ be two $\mathcal{A}-$critical sets, and assume that for $i = 1, 2$, and any $\theta, \theta', \gamma, \gamma' \in \{-1, 0, 1\}$ and any $\kappa \in \mathcal{C}_i$, there is a signed matching M such that

i) κ $\theta-$fits M and

ii) every κ' that $\theta'-$fits M is in $\mathcal{C}_i$ and

iii) no two non-equivalent colorings $\alpha_i \in \mathcal{A}$ γ_i-fit M for both $i = 1$ and $i = 2$.

Let $\kappa \in \mathcal{C}_1 \bigcup \mathcal{C}_2$ and let $\theta \in \{-1, 0, 1\}$. Without loss of generality, we may assume $\kappa \in \mathcal{C}_1$. Therefore there is a matching M that κ $\theta-$fits and such that every other coloring κ' which $\theta'-$fits M is in $\mathcal{C}_1 \subset \mathcal{C}_1 \bigcup \mathcal{C}_2$.

Now suppose by way of contradiction that there are two non-equivalent colorings α_1, α_2 and two integers $\gamma_1, \gamma_2 \in \{-1, 0, 1\}$ such that α_1 γ_1-fits M and α_2 γ_2-fits M. This however violates the $\mathcal{A}-$criticality of either $\mathcal{C}_1$ or $\mathcal{C}_2$.

This proves that $\mathcal{C}_1 \bigcup \mathcal{C}_2$ is $\mathcal{A}$-critical. If $|\mathcal{A}| \leq 1$ the set $\mathcal{C}$ is $\mathcal{A}-$critical if and only if it is consistent and so by choosing $\mathcal{A} = \emptyset$, we deduce that the union of two consistent sets is consistent.

Now let $B \subset \mathcal{C}^*(R)$. The union of all $\mathcal{A}$-critical subsets of B is $\mathcal{A}$-critical and certainly contains every $\mathcal{A}-$critical subset of B. This completes the proof of Lemma 5.1.2.

Lemma 5.1.3 *If $|\mathcal{A} \bigcap \mathcal{C}| \leq 1$, then the maximal consistent subset of $\mathcal{C}$ equals the maximal $\mathcal{A}-$critical subset of $\mathcal{C}$.*

Proof: Let $MCS_{\mathcal{A}}(\mathcal{C})$ denote the maximal $\mathcal{A}-$critical subset of $\mathcal{C}$ and let $MCS_{\emptyset}(\mathcal{C})$ denote the maximal consistent subset of $\mathcal{C}$. We know that $MCS_{\mathcal{A}}(\mathcal{C})$ is consistent so $MCS_{\mathcal{A}}(\mathcal{C}) \subset MCS_{\emptyset}(\mathcal{C})$. Also, by Lemma 5.1.1 and the fact that $|\mathcal{C}_{\mathcal{A}}| \leq 1$, $MCS_{\emptyset}(\mathcal{C})$ is $\mathcal{A}-$critical. Hence $MCS_{\emptyset}(\mathcal{C}) \subset MCS_{\mathcal{A}}(\mathcal{C})$, and thus the theorem holds.

5.2 Proving Reducibility

5.2.1 Using a Corresponding Projection

Let K be a configuration that appears in a triangulation T and has free completion S and ring R. In general S will not appear in T, but suppose for illustration that it does. The ring R will naturally split T up into two near triangulations, one of them S and the other which we denote by H. However, it may be the case that S does not appear in T. The next lemma is a technical result to show that we may still in a certain sense decompose T into the near triangulation H and the free completion S.

Lemma 5.2.1 *Let K be a configuration which appears in a triangulation T and has free completion S with ring R and let ϕ be the corresponding projection of S into T. Let H be the graph obtained from T be deleting the vertex-set $\phi(V(G(K)))$. Then*

1) H is a near triangulation and ϕ wraps R around H.

2) If $X \subset E(S)$ is sparse in S, then $\phi(X)$ is sparse in T.

Proof: Since ϕ fixes $G(K)$ and $G(K)$ is connected, all of $G(K)$ lies in the same face of the drawing $T - V(G(K))$. This and the fact that T is a triangulation implies that $H = T - \phi(V(G(K))) = T - V(G(K))$ is a near triangulation.

Let $V(R) = \{r_1, r_2, \ldots, r_q\}$ and $E(R) = \{e_1, e_2, \ldots, e_q\}$ where for $i = 1, 2, \ldots, q-1$, e_i has endpoints r_i and r_{i+1} and $e_q = \{r_q, r_1\}$. Suppose that r_1,r_2,...,r_q is the clockwise order of appearance of the vertices of $V(R)$. Consider the alternating sequence W of vertices and edges in T : $\phi(r_1)$, $\phi(e_1)$, $\phi(r_2)$, $\phi(e_2)$, ..., $\phi(r_q)$, $\phi(e_q)$, $\phi(r_1)$. By property (iii) of projections, $\phi(r_i)$ is incident to $\phi(e_i)$ in T for each $i \in 1, \ldots, q-1$ because r_i is incident to e_i in S for each $i \in 1, \ldots, q-1$. For the same reason $\phi(e_q)$ is incident to $\phi(r_1)$. Thus W is a closed walk in H.

We now prove some things that will help in establishing that W is a facial walk. We claim that every finite face $r \in F(S) - F(G)$ has the property that $\phi(r)$ is contained in the infinite face of H. It suffices to show that $\phi(r)$ is incident to some vertex of $V(G)$, because the infinite face of H will contain that vertex in its interior. By property 2) of Lemma 4.2.1, r is incident to a vertex $x \in V(G)$. By property (iii) of projections, $\phi(r)$ is incident to $\phi(x) = x$ in T. This proves that claim.

Let $v \in V(R)$. By property 6) of Lemma 4.2.1, a clockwise listing in S of edges incident to v is $g_0, g_1, \ldots, g_p$ where g_0, $g_p \in E(R)$, $p \geq 2$ and $g_1, \ldots g_{p-1} \in E(G)$. We assume that the endpoints of g_i in $V(S) - v$ are x_i for $0 \leq i \leq p$. Also, for $1 \leq i \leq p$, we label the unique finite face of S that is incident to g_{i-1} and g_i as r_i and the unique edge in $E(S)$ that is incident to r_i as h_i. Thus, for $1 \leq i \leq p-1$, h_{i+1}, g_i, h_i is a portion of a clockwise listing in S of edges incident to x_i. From the fact that ϕ is the extension of the natural edge function, and the fact that the natural edge function preserves the embedding in S at x_i, it follows that $\phi(h_{i+1}), \phi(g_i), \phi(h_i)$ is a portion of the clockwise listing in T of edges incident to $\phi(x_i) = x_i$ for $1 \leq i \leq p-1$. Also, the fact that ϕ is a projection, implies that $\phi(r_i)$ is a face of T that is incident to the edges $\phi(g_{i-1}), \phi(h_i), \phi(g_i)$ for $1 \leq i \leq p$. We claim that for every $1 \leq i \leq p$, $\phi(g_i)$

follows $\phi(g_{i-1})$ in the clockwise listing in T of edges incident to $\phi(v)$. This however follows from three facts:

1) $\phi(h_i)$ follows $\phi(g_i)$ in any clockwise listing in T of edges incident to x_i for $1 \leq i \leq p$.

2) $\phi(g_{i-1})$ follows $\phi(h_i)$ in any clockwise listing in T of edges incident to x_{i-1} for $1 \leq i \leq p$.

3) The edges $\phi(g_i)$, $\phi(h_i)$ and $\phi(g_{i-1})$ are all incident to the finite face r_i in T, for $1 \leq i \leq p$.

All of this implies that $\phi(g_0), \phi(g_1), \ldots, \phi(g_p)$ is a portion of any clockwise listing in T of edges incident to $\phi(v)$ in T. Moreover, from the claim above, $\phi(r_i)$ is a face of T that is contained in the infinite face of H. Hence, the edges $\phi(g_1), \phi(g_2), \ldots, \phi(g_{p-1})$ are all edges of the infinite face of H, and it therefore follows that $\phi(g_p)$ follows $\phi(g_0)$ in any clockwise listing in H of edges incident to $\phi(v)$.

We now show that the sequence $\phi(r_1)$, $\phi(e_1)$, $\phi(r_2)$, $\phi(e_2)$, …, $\phi(r_q)$, $\phi(e_q)$, $\phi(r_1)$ is a facial walk in H that bounds the infinite face. From what we have just shown, $\phi(e_{i-1})$ follows $\phi(e_i)$ in the clockwise listing in T of edges incident to $\phi(r_i)$ for $1 \leq i \leq q$ (and where when $i = 1$, we interpret e_{i-1} as e_r). This completes the proof that W is a facial walk of the infinite face of G and thus shows that ϕ wraps R around H.

Now let $X \subset E(S)$ be a sparse set of edges. Each edge in X must have at least one endpoint in $V(G)$. Therefore, every edge of X is in the domain of the natural edge function of Lemma 4.11. Since ϕ is an extension of the natural edge function, ϕ preserves that embedding at every $x \in V(G)$, which implies that if $e, f \in X$ share a common endpoint $x \in V(G)$, but do not share a common face, then $\phi(e)$ and $\phi(f)$ have common endpoint x in T but are not in a common face of T. This implies that if $\phi(e)$ and $\phi(f)$ are in the same face r' of T for some distinct edges $e, f \in X$, then $\phi(e)$ and $\phi(f)$ do not have a common endpoint in $V(G)$. However, $\phi(e)$ and $\phi(f)$ both have endpoints in $V(G)$ since e and f do. Let x_e denote the endpoint of e in $V(G)$ and x_f the endpoint of f in $V(G)$, and let z be the common endpoint of $\phi(e)$ and $\phi(f)$ in T.

We digress briefly to show that there is an $r \in F(S)$ such that $\phi(r) = r'$. Now r' is adjacent to a vertex in $V(G)$, (in fact two, x_e and x_f). Property 4) of the natural edge function guarantees that ϕ restricted to the edges of S which have at least one endpoint in $V(G)$ is a one to one and onto function into the set of edges T with at least one endpoint in $V(G)$. From the way ϕ was defined for faces, this implies that $\phi(r) = r'$.

Let g be the edge in S which has endpoints x_e and x_f and which is incident to the face r. Because ϕ is a projection, $\phi(g)$ is incident to r'. Also, since G is an induced subdrawing of S, $g \in E(G)$ and so $\phi(g) = g$. Suppose without loss of generality that g follows $\phi(e)$ in every clockwise listing in T of edges incident to x_e, and that $\phi(f)$ follows g in every clockwise listing in T of edges incident to x_f. Now the edge e either precedes or follows the edge g in any clockwise listing in S of edges incident to x_e. If e follows g in every clockwise listing in S of edges incident to x_e then $\phi(e)$ would both precede and follow $\phi(g)$ in every clockwise listing in T of edges incident to x_e. This however is impossible because $d_T(x_e) = \gamma_K(x_e) \geq 5$. Therefore, it must be the case that e precedes g in any clockwise listing in S of edges incident to x_e. For similar reasons, f must follow g in any clockwise listing in S of edges incident to x_f. Thus e and f share a triangular face with each other and with g, which contradicts that X is sparse. This completes the proof of the lemma.

5.2.2 Defining Various Types of Reducibility

We now introduce the notation that will be used for the rest of this chapter. Let K be a configuration with free completion S and ring R. Suppose that K appears in the triangulation T, that ϕ is a corresponding projection of S into T, that H is the near triangulation $T - V(K(G))$, and that ϕ wraps R around the outer facial walk of H. Finally, let X be a sparse subset of $E(S)$. We now define various sets of colorings of R. Let $\mathcal{C}^*$ be the set of all colorings of the ring R, let $\mathcal{C}_S$ be the set of restrictions to R of tricolorings of S, and let $\mathcal{U} \subset \mathcal{C}$ be the set of colorings of R which extend to a unique tricoloring of S. Note that $\mathcal{C}^*(R) - \mathcal{C}_S$ is the set of colorings of R which do not extend into S. The set $\mathcal{C}_S(X)$ will denote the set of restrictions to R of tricoloring of S modulo X. Also, let $\mathcal{C}_H$ denote the set of lifts of tricolorings of H. By Lemma 5.1.2, for any $\mathcal{B} \subset \mathcal{C}$, there is a maximal $\mathcal{U}-$critical subset of $\mathcal{B}$ which we denote by $MCS_{\mathcal{U}}(\mathcal{B})$ or just $MCS(\mathcal{B})$ for short. The notation $MCS_{\emptyset}(\mathcal{B})$ will denote the maximal consistent subset of B. Finally, for $u \in \mathcal{U}$ we denote the set $MCS_{\mathcal{U}}((\mathcal{C}^* - \mathcal{C}_S) \bigcup \{u\})$ by $MCS(u)$. With these definitions in place, we now define various types of reducibility, the first two of which appear in the literature and are sufficient to prove the Four Color Theorem, and the third, fourth and fifth of which are introduced to prove the Fiorini-Wilson-Fisk Conjecture.

1. The configuration K is *D-reducible* if $MCS_{\emptyset}(\mathcal{C}^* - \mathcal{C}_S) = \emptyset$.

2. The configuration K is *C(k)-reducible* if there exists a sparse set set $X \subset E(S)$ such that $|X| = k$, $\phi(X)$ is contractible and no tricoloring of S modulo X is in the set $MCS_{\emptyset}(\mathcal{C}^* - \mathcal{C}_S)$.

3. If $u \in \mathcal{U}$ and $u \notin MCS(u)$ then we say that u is *D-removable*.

4. If $u \in \mathcal{U}$ and there is a sparse set $X \subset E(S)$ such that $\phi(X)$ is contractible and $MCS(u) \bigcap \mathcal{C}_S(X) = \emptyset$, then we say that u is *C-removable.*

5. A configuration K is *U-reducible* if

1) every $u \in \mathcal{U}$, is either D-removable or C-removable.

2) At least one $u \in U$ is C-removable or the configuration K is either D-reducible or C(4)-reducible.

Notice that each of the above types of reducibility depends only on R and S and not on H. This has the very practical application that if $|V(R)|$ is relatively small, say 14 or 15, it is feasible computationally to calculate Maximum Critical Subsets like $MCS(u)$. This coupled with the following observations which relate $\mathcal{C}_H$ to $MCS(u)$ are the key to reducibility because calculations on a small piece of the triangulation T yield information about the rest of the triangulation which could be immense. Let us recall that the operator η was defined at the beginning of the chapter.

1) If it is not the case that $\mathcal{C}_H \bigcap \mathcal{C}_S \not\subseteq \eta(\{u\})$ for some $u \in \mathcal{U}$, then it can be shown that T has at least two vertex-4-colorings.

2) If $\mathcal{C}_H \bigcap \mathcal{C}_S \subset \eta(\{u\})$ for some $u \in \mathcal{U}$, then $\mathcal{C}_H \subset MCS(u)$ by Lemma 5.1.3.

Thus if T is a minimum counterexample, then $\mathcal{C}_H \subset MCS(u)$. This will turn out to be valuable because the induction hypothesis can be used to color H.

Robertson et al. used D-reducibility and C(k)-reducibility for $1 \leq k \leq 4$ to prove the Four Color Theorem [8]. Notice that reducibility for the Four Color Theorem (Types 1. and 2.) is defined for entire configurations while reducibility for the Fiorini-Wilson-Fisk Conjecture must first be defined in terms of individual colors in $\mathcal{U}$ (types 3. and 4.) and only then defined for an entire configuration (type 5.) This means that proving reducibility for the Fiorini-Wilson-Fisk conjecture will tend to be more computationally intensive than proving it for the Four Color Theorem because in principle, each color in $\mathcal{U}$ needs to be considered. In practice, it is often possible to simultaneously get rid of groups of colors in $\mathcal{U}$ as we shall show in Section 6.8.1.

5.2.3 Proving Reducibility

As noted above, S will not, in general, appear in T, but suppose again for illustration that it does. The ring R will appear in T and will naturally split T up into two near triangulations, one of them S and the other which we denote by H. Denoting by $\mathcal{C}_S$ the set of restrictions to R of tricoloring of S and $\mathcal{C}_H$ the set of restrictions to R of tricolorings of H, it is clear that T will have a tricoloring if and only if $\mathcal{C}_S \bigcap \mathcal{C}_H \neq \emptyset$. Many of the results of this section will use this simple principle in one way or another. The next theorem proves that this simple principle can be applied even when only a projection ϕ of S appears in T.

Lemma 5.2.2 *Let d be a coloring of R. Then $d \in \mathcal{C}_S \bigcap \mathcal{C}_H$ if and only if T has a tricoloring whose restriction to $\phi(R)$ is d.*

The proof of this is straightforward and we omit it.

The usefulness of our definitions of reducibility hinge on the following lemma, as was alluded to in Section 4.2.2.

Lemma 5.2.3 *Either T has at least two non-equivalent vertex-4-colorings or there is a $u \in \mathcal{U}$ such that $\mathcal{C}_H \subset MCS(u)$.*

Proof: By Lemma 5.2.2, if $|\mathcal{C}_H \bigcap \mathcal{C}_S| \geq 2$, or if $\mathcal{C}_H \bigcap (\mathcal{C}_S - \mathcal{U}) \neq \emptyset$, then T has at least two distinct vertex-4-colorings. Hence, we may assume there is a $u \in \mathcal{U}$ such that $\mathcal{C}_H \subset (\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\})$. Now $MCS(u) \subset (\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\})$ and also $MCS(u)$ is consistent by Lemma 5.1.3. Thus Theorem 5.1.3 implies that $\mathcal{C}_H \subset MCS(u)$ since $\mathcal{C}_H$ is consistent.

Lemma 5.2.4 *Let ϕ be a corresponding projection of S into T. If c is a tricoloring of T modulo $\phi(X)$, then there are functions c_X and c_H, such that c_X is a tricoloring of S modulo X and c_H is a tricoloring of H. In addition, $c_X(e) = c(\phi(e))$ for every $e \in E(S)$, and $c_H(e) = c(e)$ for all $e \in E(H)$. Finally, the restriction of c to $\phi(E(R))$, the restriction of c_X to $E(R)$ and the lift of c_H by ϕ are all the same ring coloring, and this ring coloring is in $\mathcal{C}_H \bigcap \mathcal{C}_S$.*

Proof: Note that by Lemma 5.2.1, H is a near-triangulation, ϕ wraps R around H and $\phi(X)$ is sparse in T. Since $E(H) \bigcap \phi(X) = \emptyset$, c restricted to H is a tricoloring of H, which we henceforth denote c_H. The tricoloring c also defines a tricoloring c_X of S modulo X as follows: $c_X(e) = c(\phi(e))$ for $e \in E(S)$. By property (iii) of projections, if $r \in F(S)$, and r is incident to the distinct edges $e, f, g \in E(S)$ then $\phi(r)$ is a face in $F(T)$ which is incident to edges the edges $\phi(e)$, $\phi(f)$ and $\phi(g)$. If $X \bigcap \{e, f, g\} = \emptyset$, then $\phi(X) \bigcap \{\phi(e), \phi(f), \phi(g)\} = \emptyset$ so $\{1, 0, 1\} = \{c(\phi(e)), c(\phi(f)), c(\phi(g))\} = \{c_X(e), c_X(f), c_X(g)\}$. If $X \bigcap \{e, f, g\} \neq \emptyset$, say $e \in X$, then $\phi(e) \in \phi(X)$, so $c_X(f) = c(\phi(f)) = c(\phi(g)) = c_X(g)$. Thus c_X is a tricoloring of S modulo X. From the definitions of c_X and c_H, $c_X(e) = c(\phi(e)) = c_H(\phi(e))$ for $e \in E(R)$. Hence the restriction of c_X to R (which equals the restriction of c_H to $\phi(E(R))$ is in the set $\mathcal{C}_S \bigcap \mathcal{C}_H$. This completes the proof of Lemma 5.2.4

Let $\mathcal{A} \subset \mathcal{C}^*$. Generalizing Robertson et al. we say that a set $X \subset E(S) - E(R)$ is an $\mathcal{A}-$*contract* if it is a nonempty, sparse set and if no tricoloring modulo X of

S is in the set $MCS(\ (\mathcal{C}^* - \mathcal{C}) \bigcup \mathcal{A}\ )$. If $\mathcal{A} = \{a\}$ we call an $\mathcal{A}$-contract simply an $a-$*contract*. If $\mathcal{A} = \emptyset$, then we say that X is a *contract*.

The free completion S of a configuration does not necessarily appear in the triangulation T even if the configuration does. However, Theorem 4.3.1 shows that there is a projection of S into T. It is conceivable that a contract X in S might produce loops if the corresponding edges were contracted in T and Theorem 5.1.2 would not be applicable. The following method of Robertson et al. gives an easy to check sufficient condition for a contract $X \in S$ not to produce loops after being projected into T.

An edge e is said to *face* a vertex v if v is not an endpoint of e and both v and e are incident to a common face. A vertex $v \in V(S)$ is a *triad* for X if

(i) $v \in V(G(K))$

(ii) There are at least three vertices of S adjacent to v and incident to a member of X

(iii) If $\gamma_K(v) = 5$, then there is an edge of X that does not face v.

Theorem 5.2.1 *Let K be a configuration with free completion S and ring R and suppose that K appears in an internally* 6 *connected triangulation T. Let ϕ be a corresponding projection of S into T and let $X \subset E(S)$ be a sparse subset with $|X| = 4$ such that there is a vertex of $G(K)$ which is a triad for X. Then for every circuit C in T, $|E(C) - \phi(X)| \geq 2$ or there is a short circuit in T.*

Proof: Let $Y = \phi(X)$. Lemma 5.2.1 guarantees that Y is sparse in T. Let C be a circuit in T. Since T is loopless, $|E(C)| > 1$. If $|E(C)| = 2$ then because all faces are triangles, C cannot bound a face and must therefore be a short circuit. If

$|E(C)| = 3$ then C must bound a face, for otherwise it would be a short circuit. Thus, the sparseness of Y implies that Y has at most one edge in common with $E(C)$ and so the desired inequality holds. If $|E(C)| = 4$ and $C = \{x_1, x_2, x_3, x_4\}$ then either C is a short circuit or some pair of diagonally opposite vertices of C, say x_1 and x_3 are adjacent to each other and $\{x_1, x_2, x_3\}$ and $\{x_3, x_4, x_1\}$ form triangular faces in T. Since Y is sparse, Y has at most one edge in common which each of these two faces and so $|E(C) \bigcap Y| \leq 2$ and the inequality follows. If $|E(C)| \geq 6$, then $|X| \leq 4$ implies $|E(C) - \phi(X)| \geq 2$. So we may assume $|E(C)| = 5$ and thus that $|X| = 4$. Let $C = x_1, x_2, x_3, x_4, x_5$. We may assume $|E(C) - Y| = 1$ and so all 4 edges of Y are in $E(C)$. Let $int(C)$ denote the subdrawing of T induced by the vertices in one of the arc-wise connected components of $\Sigma - C$ and let $ext(C)$ denote the the subdrawing of T induced by the vertices in the other arc-wise connected component of $\Sigma - C$. We may assume that either $|V(int(C))| \leq 1$ or $|V(ext(C))| \leq 1$, or else C is a short circuit. If $|V(int(C))| = 0$ or $|V(ext(C))| = 0$, then there are only edges in one of the two disjoint regions of the sphere defined by C, but this will create triangular faces containing two edges of Y, a violation of the sparseness of Y. Thus we must have $|V(int(C))| = 1$ or $|V(ext(C))| = 1$. By symmetry, we may assume the former and we will let y denote the vertex for which $V(int(C)) = \{y\}$. Note that y has degree 5 and faces all the edges of Y and so cannot be a triad for Y. Since there is a triad v for Y, $v \in V(ext(C))$, v is incident to at least three vertices $x_{i_1}, x_{i_2}, x_{i_3} \subset \{x_1, x_2, x_3, x_4, x_5\}$ which are, in turn endpoints of edges in Y. By relabeling, we may assume $x_{i_1} = x_1$ and $x_{i_2} = x_2$ and that $i_3 \in \{3, 4\}$. If $i_3 = 4$ then $\{v, x_4, y, x_1\}$ form a short circuit. So assume that $i_3 = 3$, and deduce that $\{v, x_3, x_4, x_5, x_1\}$ is either a short circuit, or there is a degree 5 vertex w that is adjacent to $\{v, x_3, x_4, x_5, x_1\}$. We may assume

$\{v, x_3, x_4, x_5, x_1\}$ is not a short circuit in T, so the later holds and v has neighbors $\{x_1, x_2, x_3, w\}$ which form a short circuit. This completes the proof of the theorem.

We will call any $A-$ contract X with $|X| = 4$ and for which X has a triad a *safe contract*.

Theorem 5.2.2 *Every configuration in Appendix A is U-reducible. Moreover, for every $u \in \mathcal{U}$ that is C-removable, there is a safe u-contract X.*

Proof: Let K be a configuration in Appendix A. The computer verifies that for every $u \in \mathcal{U}$, u is either D-removable or C-removable. When the color is C-removable, the computer finds a u-contract X and verifies that X is safe. After showing that every $u \in \mathcal{U}$ is either D-removable or C-removable, the computer verifies that the configuration K is U-reducible. If at least one of the $u \in \mathcal{U}$ was C-removable, then U-reducibility for K is immediately established. Otherwise, the computer verifies that K is either D-reducible or C(4)-reducible. Details about how the computer does this can be found in Chapter 6.

Theorem 5.2.3 *Let T be a minimum counterexample. Then no configuration isomorphic to one in Appendix A appears in T.*

Proof: Let T be a minimum counterexample, and suppose that K is a configuration in Appendix A which appears in T. By Theorem 3.3.1, we know that T is internally $6-$connected.

Let H and S be as at the beginning of Section 5.2.2. We first notice that if $\mathcal{C}_H \bigcap \mathcal{C}_S$ includes two non-equivalent colorings, or if $\kappa \in \mathcal{C}_H \bigcap \mathcal{C}_S$ for some $\kappa \in \mathcal{C}_S - \mathcal{U}$, then Lemma 5.2.2 implies that T would have at least two non-equivalent vertex-4-colorings. Therefore, we may assume that $\mathcal{C}_H \bigcap \mathcal{C}_S \subset (\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\})$, for some $u \in \mathcal{U}$.

Assume first that no $u \in \mathcal{U}$ is C-removable. Therefore, every $u \in \mathcal{U}$ is D-removable, which implies that $u \notin MCS(u)$ for every $u \in \mathcal{U}$. Since K appears in Appendix A, Theorem 5.2.2 implies that K is either D-reducible or C-reducible. We first consider the case that K is D-reducible, and therefore that $MCS(\mathcal{C}^* - \mathcal{C}) = \emptyset$. Since T is a minimum counterexample, H has a vertex-4-coloring and thus $\mathcal{C}_H \neq \emptyset$. If $\mathcal{C}_H \bigcap \mathcal{C}_S = \emptyset$, then $\mathcal{C}_H \subset (\mathcal{C}^* - \mathcal{C}_S)$ which implies that $\mathcal{C}_H \subset MCS(\mathcal{C}^* - \mathcal{C}_S)$ since Lemma 5.1.3 implies the latter is consistent. This however, is a contradiction. Assume then that $\mathcal{C}_H \bigcap \mathcal{C}_S = \eta(\{u\})$ for some $u \in \mathcal{U}$. This also gives rise to a contradiction, because then $\mathcal{C}_H \subset (\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\})$ which implies $\mathcal{C}_H \subset MCS(u)$ since by Lemma 5.1.3, $MCS(u)$ equals the maximal critical subset of $(\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\})$ and $\mathcal{C}_H$ is critical by Theorem 5.1.3. Thus $u \in \mathcal{C}_H \subset MCS(u)$ which contradicts that u is D-removable.

Now consider the case that K is C(4)-reducible, and let X be a sparse subset of S such that $\phi(X)$ is contractible in T and that $\mathcal{C}_S(X) \bigcap MCS(\mathcal{C}^* - \mathcal{C}_S) = \emptyset$. By Theorem 5.1.2, there is a tricoloring of T modulo X which we denote by c. Let c_H and c_X be the colorings that are guaranteed to exist by Lemma 5.2.4, let $c_H(R)$ be the lift of c_H by ϕ and let $c_X(R)$ be the restriction to R of the coloring c_X. Lemma 5.2.4 says that $c_H(R) = c_X(R)$ and that $c_X(R) \in \mathcal{C}_H \bigcap \mathcal{C}_S$. Also, we know that $c_X(R) \in \mathcal{C}_S(X)$. Therefore $c_X(R) \in \mathcal{C}_H \bigcap \mathcal{C}_S \bigcap \mathcal{C}_S(X)$. From this and our assumption that $\mathcal{C}_H \subset (\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\})$ for some $u \in \mathcal{U}$, it follows that $c_X(R) \in \mathcal{U}$ and that $\mathcal{C}_H \subset MCS(c_X(R))$ since by Theorem 5.1.3, $\mathcal{C}_H$ is consistent and by Lemma 5.1.3, $MCS(c_X(R))$ equals the maximal consistent subset of $(\mathcal{C}^* - \mathcal{C}_S) \bigcup \{c_X(R)\}$. Since we are assuming that every $u \in \mathcal{U}$ is D-removable, it follows that $c_X(R)$ is D-removable, and hence that $c_X(R) \notin MCS(c_X(R))$. This however is a contradiction because we know that $c_X(R) \in \mathcal{C}_H \subset MCS(c_X(R))$. This completes the proof of Theorem 5.2.3 in the case when no $u \in \mathcal{U}$ is C-removable.

We may assume then that there is a $u \in \mathcal{U}$ which is not D-removable and hence is C-removable. By Theorem 5.2.2, we know that there is a safe $u-$contract X. We now show that we may assume $\mathcal{C}_H \subset (\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\})$. If not, then from our previous assumptions we know that there is a $u' \in \mathcal{U}$ with $u \neq u'$ such that $\mathcal{C}_H \subset (\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u'\})$. Now $\mathcal{C}_H \not\subset \mathcal{C}^* - \mathcal{C}_S$; otherwise $\mathcal{C}_H \subset \mathcal{C}^* - \mathcal{C}_S \subset (\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\})$. It follows then that $u' \in \mathcal{C}_H \subset MCS(u')$, so u' is not D-removable. Therefore u' is C-removable, by Theorem 5.2.2. Thus, we could let u' play the role of u. This proves that that we may assume $u \in \mathcal{C}_H \subset MCS(u) \subset (\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\})$.

Since X is a safe contract, Theorem 5.1.2 guarantees that T has a tricoloring modulo X which we denote by c. Using Lemma 5.2.4 and its notation, we write $c_H(R)$ for the lift of c_H by ϕ, and $c_X(R)$ for the restriction to R of the coloring c Lemma 5.2.4 guarantees that $c_X(R) = c_H(R)$ and that $c_X(R) \in \mathcal{C}_H \bigcap \mathcal{C}_S(X)$. Since $\mathcal{C}_H \subset MCS(u)$, it follows that $c_X(R) \in MCS(u) \bigcap \mathcal{C}_S(X)$. This is a contradiction however, because X is a u-contract implies that $\mathcal{C}_S(X) \bigcap MCS(u) = \emptyset$.

This completes the proof of Theorem 5.2.3.

Chapter 6

The Reducibility Program for the Fiorini-Wilson-Fisk Conjecture

6.1 Introduction

In this section we give a description of the computer program used to prove Theorem 5.2.2, that is, to show that every configuration in the unavoidable set is U-reducible. The code for this program is written in `C` and is available upon request.

6.1.1 Notation

The following notation will be used throughout this chapter. Let K be a configuration, which appears in a triangulation T, let G be the underlying graph of K and let S be the free completion of K with a ring R having r edges. Let ϕ denote a corresponding projection of S onto T that fixes G and let X be a sparse subset of $E(S)$. Let $\mathcal{C}^*$ denote the set of all ring colorings, let $\mathcal{C}_S$ denote the set of ring colorings of S which extend to a tricoloring of all of S and let $\mathcal{C}_S(X)$ denote the set of ring colorings of R that extend to a tricoloring of S modulo X. Let $\mathcal{U}$ be those elements of $\mathcal{C}_S$ which extend to exactly one tricoloring of S and for $\mathcal{A} \subset \mathcal{U}$, let $MCS(\mathcal{A})$ denote the maximal critical subset of $(\mathcal{C}^* - \mathcal{S}) \bigcup \mathcal{A}$ with respect to $\mathcal{U}$. When $\mathcal{A} = \{u\}$ we write

$MCS(u)$ instead of $MCS(\{u\})$. We will assume that $V(S) = \{1, 2 \ldots, |V(S)|\}$ and that $V(R) = \{1, \ldots, |V(R)|\}$.

6.1.2 High Level Description

Here is a high level description of what the program does:

Having verified that $K_1, \ldots, K_k \in \mathcal{K}$ are U-reducible, the program:

1. Reads the next block of data from the file and checks that it is a configuration. This block of data will represent the free completion of the configuration K_{k+1}. If it is not a valid configuration, the program stops and warns the user that an error has occurred.

2. Let the current configuration be K and its free completion S. Assume that the ring of S is R. The program calculates all tricolorings of S, and for each tricoloring c of S, records its restriction to R by means of a unique code. The program notes which tricolorings of R have no extensions into S, exactly one extension into S, (we denote the set of these by U), and which have two or more extensions into S. For notation, we let $\mathcal{C}^*(R) = \mathcal{C}^*$ denote the set of colorings of R, and $\mathcal{C}(S)$ denote the set of restrictions to R of colorings of S.

3. For each $u \in \mathcal{U}$, the program tries to establish that every coloring $u \in \mathcal{U}$ is either D-removable or C-removable. It does this by calculating $MCS(A)$ for various subsets $A \subset \mathcal{U}$, trying to show that for each $u \in A \subset \mathcal{U}$ either

   (a) $u \notin MCS(A)$ (Which we will show implies u is D-removable) or

(b) There is a contract $X = X_A$ such that no tricoloring of S modulo X is in $MCS(A)$ (Which we will show implies u is C-removable)

If every color $u \in \mathcal{U}$ satisfies either condition 3a or condition 3b, then the program proceeds to step 4. If some coloring $u \in \mathcal{U}$ does not satisfy either condition 3a or condition 3b when $A = \{u\}$, then the program terminates unsuccessfully, concluding that the current configuration K is not U-reducible.

4. If during the course of step 3 the program discovers that some coloring in $\mathcal{U}$ is C-removable, then the program concludes that the configuration is U-reducible. If not, then the program tries to verify that that the configuration is either D-reducible or C(4)-reducible. If this verification is successful, the program concludes that the current configuration is U-reducible and proceeds back to step 1 for the next configuration. If the verification is not successful, the program halts, concluding that the current configuration is not U-reducible.

6.1.3 Balanced Colorings

A coloring $\kappa\colon E(R) \to \{-1, 0, 1\}$ is *balanced*, if $|\kappa^{-1}(-1)|$, $|\kappa^{-1}(0)|$, $|\kappa^{-1}(1)|$ and $|R|$ all have the same parity. The next result is well-known, and we omit the proof.

Lemma 6.1.1 *Let H be a near triangulation with outer facial walk W, and let ϕ wrap the circuit R around H. The set $\mathcal{C}$ of all lifts of tricolorings of H by ϕ contains only balanced colorings.*

The main value of $MCS(u)$ is that it is a superset of $\mathcal{C}_H$ provided that $|\mathcal{C}_H \bigcap \mathcal{C}_S| \leq 1$. This follows because $|\mathcal{C}_H \bigcap \mathcal{C}_S| \leq 1$ implies $\mathcal{C}_H \subset (\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\})$ which in turn implies that $\mathcal{C}_H \subset MCS(u)$ since $MCS(u)$ is the maximal consistent subset of $(\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\})$. By Lemma 6.1.1, $\mathcal{C}_H$ contains only balanced colorings, and so when considering the set $\mathcal{C}^*$ in the calculation of $MCS(u) = MCS((\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\})$, it suffices for computational purposes to only include balanced colorings. Then when only balanced colorings are included in $\mathcal{C}^*$, the set $MCS(u) = MCS((\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\}))$ remains a superset of $\mathcal{C}_H$.

6.1.4 Subroutines

There will be 6 subroutines helpful to this end goal.

1. Data Reading Subroutine

   Input: A file consisting of block of specified format, a pointer to some place in this file, a matrix A.

   Output: The free completion of a configuration K written into matrix A in a manner specified later, or a warning that the data did not represent a free completion.

2. $TriColor(S, \mathcal{C})$

   Input: A free completion S with ring R of a configuration K and a vector which has $|\mathcal{C}|$ elements.

   Output: A recording for each ring coloring $c \in \mathcal{C}^*(R)$ of whether c extends to 0, 1 or ≥ 2 tricolorings of S.

3. $TriModCon(S, \mathcal{A}, X)$

Input: A free completion S with ring R of some configuration K, a set $A \subset \mathcal{U}$ and a proposed $A-$contract X.

Output: Either a warning that the proposed A-contract is invalid , or if the contract is valid then a set containing every tricoloring of S modulo X.

4. <u>$Critical(S, \mathcal{A}, \mathcal{D}, \mathbf{v})$</u>

   Input: Two subsets $\mathcal{D} \subset \mathcal{C}^*(R) - \mathcal{C}_S$, and $\mathcal{A} \subset \mathcal{U}$ and a vector $\mathbf{v}$ consisting of $|\mathcal{A}|$ ones.

   Output: The maximal critical subset $MCS_{\mathcal{U}}(\mathcal{A}, \mathcal{D})$ of the set $\mathcal{A} \bigcup \mathcal{D}$, and the vector $\mathbf{v}$ such that the i^{th} element of $\mathbf{v}$ is 0 if the i^{th} element of $\mathcal{A}$ is not in $MCS_{\mathcal{U}}(\mathcal{A}, \mathcal{D})$ and 1 otherwise.

5. <u>Contracts Subroutine</u>

   Input: A free completion S.

   Output: A sparse set and safe set $X \subset E(S)$ satisfying $|X| = 4$.

6. <u>$UniqueReduce(S, \mathcal{A}, \mathcal{C}))$</u>

   Input: A free completion S with ring R of some configuration K, and a subset $\mathcal{A}$ of the set $\mathcal{U}$ of ring colorings that extend to exactly one tricoloring of S.

   Output: A set $\mathcal{A}'$ with the following properties:

1. $\mathcal{A}' \subset \mathcal{A}$ is empty if every coloring $\gamma \in \mathcal{A}$ is either $D-$removable or $C-$removable for K and otherwise $\mathcal{A}' = \{\gamma\}$ where γ is a ring coloring of R such that γ is neither D-removable nor C-removable.

6.1.5 Running The Program

It is suggested that the code be compiled with some sort of optimization option. Once it is compiled, a guide to usage can be printed by typing the name of the compiled program without any arguments. There is one mandatory argument, the name of the file containing the set $\mathcal{K}$, and this must be the first argument. There are three optional arguments, the first specifying which configuration in the file the program should start with, and the next two giving the ranges of those codes of colorings in $\mathcal{U}$ which the user wants to eliminate. Of course, a given configuration will be U-reducible only if all of the colorings in $\mathcal{U}$ can be eliminated with either D-removability or C-removability. These last two arguments then are used if the user wants to investigate the D-removability or C-removability of a limited range of colors.

The output can be controlled by either defining or not defining the constants `NORMALPRINT`, `PRINTHOPE`, `PRINTSMART`. To keep output to a minimum, leave all of these undefined. To keep track of how the program is progressing towards its goal of proving that the current configuration is U-reducible, `NORMALPRINT` should be defined by including the pre-processing statement `#define NORMALPRINT` in the code before the function main(). If `PRINTHOPE` is defined, then some information about the matrix `hope` will be printed, (see 6.7 and 6.8.3.1 for details about `hope`) and if `PRINTSMART` is defined, then information about the contracts which `smart_contract` generates (again see 6.7 and 6.8.3.1 for details) is displayed. Likewise, leaving `PRINTGOODCONTRACT` defined enables printing $\mathcal{A}-$contracts as they are discovered (see 6.7). Various other options about how the program is run can be controlled by either defining or leaving undefined the variables `HOPE` and `SMARTCONTRACT`. We refer the reader to Section 6.7 for more details.

6.2 Global Variables and Data Structures

6.2.1 Important Constants

First, the program defines a number of constants `EDGES`, `VERTS`, `MAXRING`, and `MAXSET`. They represent respectively the maximum numbers of edges, the maximum number of vertices, the maximum ringsize of any of the free completions in the unavoidable set, and the maximum size of any subset $\mathcal{A} \subset \mathcal{U}$, which implies that $|\mathcal{U}|$ is assumed to be at most `MAXSET`. The constant `DEG` is a generous upper bound on the maximum degree of a free completion, and will be used to allocate the number of columns in the matrix which will store S.

These constants are later used to allocate matrices and vectors for other data structures.

6.2.2 Storing the Free Completion

The global variable `graph` will be the data structure used to store information about a given free completion. The entries `graph[0][0]`, and `graph[0][1]`, `graph[0][2]`, `graph[0][3]` and `graph[0][4]` will store respectively, $|V(S)|$, $|V(R)|$, $|\mathcal{C}_S|$, $|\mathcal{C}^*|$ and $|X|$ respectively. The information in $|\mathcal{C}_S|$, $|\mathcal{C}^*|$ and $|X|$ is not used during the course of calculations nor is what is originally read from the input file into the entries `graph[0][3+2i]` and `graph[0][4+2i]` for $1 \leq i \leq |X|$. However, these entries will be used later during run time to store potential contracts. For each $i \in V(S)$, `graph[i][0]` will denote the degree of i in S, and `graph[i][1]` ... `graph[i][graph[i][0]]` will be a clockwise listing in S of neighbors of i. Further properties of `graph` can be found by consulting Section 6.3.1.

6.2.3 Storing Colorings of R

The global variables used to keep track of colorings of R are the scalars `ring`, `nlive`, `ncodes`, and the vectors used are `live`, `fixedlive`, and `power`. The description of what follows uses some ideas from [42]. Suppose that $E(R) = \{e_1, \ldots, e_r\}$ and e_i has ends i and $i-1$ if $2 \leq i \leq r$ and e_1 has ends r and 1. Two ring colorings κ, γ are said to be *equivalent* if $\{\kappa^{-1}(-1), \kappa^{-1}(0), \kappa^{-1}(1)\} = \{\lambda^{-1}(-1), \lambda^{-1}(0), \lambda^{-1}(1)\}$. To find a representative colorings amongst the set of similar colorings we call a coloring κ *canonical* if $\kappa(e_i) = 0$ for $1 \leq i \leq r$ or there is an integer k with $1 \leq k < r$ satisfying $\kappa(e_r) = \kappa(e_{r-1}) = \ldots = \kappa(e_{k+1}) = 0$ and $\kappa(e_k) = 1$. By a suitable permutation of colors it can be shown that every ring coloring is equivalent to a unique canonical coloring. Given a coloring κ, its *code* is defined as $\sum_{i=1}^{r} \kappa'(e_i)3^{i-1}$ where κ' is the canonical coloring that is equivalent to κ. If κ and γ are two distinct canonical colorings, and if the largest index at which they differ is $1 \leq p < r$, where $\kappa(e_p) > \gamma(e_p)$, then $code(\kappa) - code(\gamma) = \sum_{i=1}^{k} (\kappa(e_i) - \gamma(e_i))3^{i-1} \geq 3^{k-1} - \sum_{i=1}^{k-1} 2\ 3^{i-1} \geq 1$ so no two distinct colorings have the same code. Notice that every code is non-negative and has value at most $\frac{3^{r-1}-1}{2}$.

The global variable used to keep track of colorings of R are the scalars `ring`, `nlive`, `ncodes`, and the vectors `live`, `fixedlive`, and `power`.

The global variables `ring`, and `ncodes` store $|V(R)|$, and $|\mathcal{C}^*|$ respectively.

Let i be the code of a ring coloring κ of R. The entry `live[i]` will contain information about κ, but as to just what this information is will depend on what stage the algorithm is in. It is safe to assume that if $\kappa \in \mathcal{C}_S - \mathcal{U}$, or if $\kappa \in \mathcal{U}$ has been eliminated via D-removability or C-removability, then `live[i]`$= 0$.

The global variable `nlive` is used to store the number of non-zero elements in the vector `live`. Sections 6.4.4 and 6.8.2 may be consulted for further information about how to interpret the entry `live[i]`.

The global variable `fixedlive` will be used to store $\mathcal{U}$ and to keep track of which elements of $\mathcal{U}$ have been eliminated by either establishing $D-$removability or $C-$removability. The meaning of the entry `fixedlive[i]` will also vary at the different stages of the algorithm, but with these general guidelines:

$0 \leq$`fixedlive[i]`≤ 18 means the coloring with code i is not in $\mathcal{C}_S$

`fixedlive[i]`$= 19$ means the coloring with code i is in $\mathcal{C}_S - \mathcal{U}$.

`fixedlive[i]`$= 20$ means the coloring with code i is in $\mathcal{U}$ and has been has been eliminated by either D- or C-removability.

$21 \leq$`fixedlive[i]` means the coloring with code i is in $\mathcal{U}$ and has not yet been eliminated through D-removability or C-removability.

See Sections 6.4.4, and 6.9.4 for more information on `fixedlive`. Finally, `power[i]` simply stores the value 3^{i-1} for $i \geq 1$.

6.2.4 Storing Signed Matchings

Suppose that $\mathcal{M}$ is a set of signed matchings. The algorithm will keep track of such a set $\mathcal{M}$ with the the array `real`, which consists of `nchar` elements of C data type `char` and with the variables `nrealterm` and `bit` and their respective pointers `pnreal` and `pbit`. To conserve memory, the algorithm views each such element of type `char` as a binary string of a length equal to the number of bits that that particular computer uses to represent the C data type `char`. Membership in $\mathcal{M}$ for a given signed matching M that is represented by the i^{th} bit of some element of `real` is tracked by the rule that $M \in \mathcal{M}$ if and only if the i^{th} bit equals one. The static array `simatchnumber` gives the number of signed matchings of rings having between 0 and 16 edges, which explains the allocation made in main:

```
nchar = simatchnumber[MAXRING] / 8 + 1;
real = (char *) malloc((long unsigned) nchar * sizeof(char));
```

Suppose that a particular machine represents the data type `char` with n bits. The variable `bit` is of type `char` and will cycle through the values 2^0, 2^1, 2^2, ..., 2^{n-1}, being incremented by the command `*pbit<<1` (found only once in the entire algorithm in the function `checkreality`) which multiplies bit by 2 and does arithmetic modulo $2^n - 1$. If `bit`$= 2^i$, then lines like

```
if (!(*pbit & real[*prealterm]))
```

found in `checkreality` test whether the signed matching represented by the i^{th} bit of the `realterm`th element of `real` are in $\mathcal{M}$.

6.2.5 Storing Information Related to Edges and Contracts

The function `strip` orders the edges of $E(S)$ so that the ring edges are the first r edges in the order they appear around the ring. The variable `edgeends` satisfies the rule that `edgeends[i][1]` and `edgeends[i][2]` are the endpoints of the i^{th} edge e_i in the ordering. Based on this labeling, the global variable `edgeid[i][j]` will be 1 if there is an edge in S with endpoints i and j and 0 otherwise. The global variables `angle`, `sameangle` and `diffangle` are all matrices whose i^{th} row contains information pertaining to the edge e_i. The global variable `contract` is a $0-1$ vector such that $\texttt{contract}[\mathrm{i}] = 1$ if and only if the edge $e_i \in X$. The entries `angle[i][1-4]` are reserved for those indices j such that $j > i$ and the edge e_j is in a common facial triangle of the configuration with the edge e_i. The integer `angle[i][0]` gives the number of such indices j. Suppose now that e_i is not in X. The entries `diffangle[i][1-4]` are the indices $j > i$ such that e_j is in a common facial triangle with e_i and some third edge e_k, and $e_j, e_k \notin X$. The entries of `sameangle[i][1-4]` are the indices $j > i$ such that e_j is in a common facial triangle with e_i and some third edge e_k, and such that $e_j \notin X$ but $e_k \in X$. Both `diffangle[i][0]` and `sameangle[i][0]` store quantities analogous to `angle[i][0]`. In addition, the entries `angle[0][p]`, `diffangle[0][p]` are equal and are used to store the quantities $|V(S)|$, $|V(R)|$ and $|E(S)|$ respectively for $p = 0, 1, 2$.

6.3 Reading Configurations

6.3.1 Problem Statement, Notation and Data Structures

In this section we discuss Subroutine 1 of Section 6.1.4. An obvious preliminary step of our task is to read in the data of the next potential configuration that is to be proved reducible from a file, prove that it is a configuration, and store it in an appropriate data structure.

Let K be the next configuration to be read and let its free completion be S with ring R. Actually the data stored will represent S in the following form:

name

$n\ r\ n_1\ n_2$

$x\ y_1\ y_2\ y_3\ \cdots y_{2x}$

$1\ d_1\ y_{1,1}\ y_{1,2}\ \cdots\ y_{1,d_1}$

$2\ d_2\ y_{2,1}\ y_{2,2}\ \cdots\ y_{2,d_2}$

$3\ d_3\ y_{3,1}\ y_{3,2}\ \cdots\ y_{3,d_3}$

.

.

.

$n\ d_n\ y_{1,1}\ y_{1,2}\ \cdots\ y_{1,d_1}$

$c_1\ c_2, \ldots, c_8$

$c_9\ c_{10}, \ldots, c_{16}$

$c_{17}\ \ldots\ c_n$.

Here $n = |V(S)|$, r is the ringsize of S, n_1 is the number of restrictions to R of tricolorings of S, n_2 is an undefined number, $X = \{\{y_1, y_2\}, \{y_3, y_4\}, \{y_5, y_6\}\ \{y_7, y_8\}\}$

is a contract containing 4 edges. As mentioned above, the initial information in $\texttt{graph}[0][4-12]$ that is read from the file will not be used, but later on the program will store potential contracts in these 9 entries. Also, d_i is the degree of vertex i, and $y_{i,1}\ y_{i,2}\ \ldots\ y_{i,d_i}$ is the clockwise listing in S of neighbors of vertex i. If the data represents a configuration with ring size r, then the data will be written into the global variable `graph` by the function `ReadConf` in the following manner

$\texttt{graph}[0][0] = n$, $\texttt{graph}[0][1] = r$, $\texttt{graph}[0][2] = \kappa$, $\texttt{graph}[0][3] = n_1$,

$\texttt{graph}[0][4] = x$, $\texttt{graph}[0][4+2i-1] = y_{2i-1}$, $\texttt{graph}[0][4+2i] = y_{2i}$ for $1 \leq i \leq x$

$\texttt{graph}[i][0] = d_i$ for $1 \leq i \leq n$, $\texttt{graph}[i][j] = y_{i,j}$ for $1 \leq j \leq d_i$

For $1 \leq i \leq n$, the number c_i is the coordinate of vertex i in a drawing of the free completion. Here, $c_i = 1024x + y$, where x is the $x-$coordinate of vertex i and y is the $y-$coordinate of vertex i in the drawing.

6.3.2 Algorithm

We now list 7 tests taken from Robertson et. al. that will be applied to the the input as it is read in.

1: $2 \leq r < n$

2: $3 \leq d_i \leq n$ for $1 \leq i \leq r$ and $5 \leq d_i \leq n-1$ for $r+1 \leq i \leq n$.

3: $1 \leq y_{i,j} \leq n$ for $1 \leq i \leq n$ and $1 \leq j \leq d_i$.

4: If $1 \leq i \leq r$ then $y_{i,1} = i+1$ (except $y_{r,1} = 1$), $y_{i,d_i} = i-1$, (except $y_{1,d_1} = r$) and $r+1 \leq y_{i,j} \leq n$ for $2 \leq j \leq d_i - 1$.

5:$d_1 + d_2 + \ldots d_n = 6n - 6 - 2r$

6:For each integer i satisfying $r+1 \leq i \leq n$, there are at most two integers j such that $y_{i,j} > r$ and $y_{i,j+1} \leq r$, and if there are exactly two, then $y_{i,j+2} > r$ for both such integers. Here $y_{i,d_i+k} = y_{i,k}$ for $k = 1, 2$.

7: Let $k = y_{i,j}$ where either $1 \le i \le r$ and $1 \le j \le d_i - 1$ or $r + 1 \le i \le n$ and $1 \le j \le d_i$. There exists an integer p such that $y_{i,j+1} = y_{k,p}$ (or $y_{i,1} = y_{k,p}$ if $j = d_i$) and $i = y_{k,p+1}$ (or $i = y_{k,1}$ if $p = d_k$, in which case $k > r$).

Here then are the basic steps performed to read the next free completion's data.

Step 1. Read the data of the file into the matrix A so that $A[0][0]$ through $A[0][4+2x]$ are defined as above. Check test 1. and if it fails, output warning message and exit the program.

Step 2. For $1 \le i \le n$ read the information for vertex i so that the $A[i][0]$ through $A[i][d_i]$ are defined as above. While this is being done check that the data satisfy tests 2-7. If any test is violated, output a warning message and exit the program immediately.

6.3.3 Correctness

The correctness of the algorithm is proved in the following lemma. taken from Robertson et. al.[42]

Lemma 6.3.1 *Suppose that the data of the matrix A satisfy tests 1 through test 7. Then there is a configuration K with free completion S and ring R such that:*

(i) $V(S) = \{1, 2, \ldots, n\}$, $V(R) = \{1, \ldots, r\}$, the degree of vertex i in S is d_i, and a clockwise listing in S of neighbors of i is $A[i][1]$,..., $A[i][d_i]$. Also, if $1 \le i \le r$, then $A_{i,1}$ and $A_{i,A[i][0]}$ are in $V(R)$.

(ii) $A[0][0] = n$ and $A[0][1] = r$.

6.4 Tricoloring Subroutine

6.4.1 Problem Statement, Notation and Data Structures

In this section we describe an algorithm for:

Input: A free completion S with ring R of a configuration K and a vector which has $|\mathcal{C}|$ elements.

Output: A recording in the vector for each ring coloring $c \in \mathcal{C}^*(R)$ of whether c extends to 0, 1 or ≥ 2 tricolorings of S.

6.4.2 Algorithm

We use the notation introduced in Section 6.2.3. The strategy will be to compute all tricolorings of S and record their restriction to R with the appropriate code. Number the edges of $E(S)$ $e_1, e_2, \ldots, e_m$ so that e_{m-1} and e_m are in the same triangle and so that $e_1, \ldots, e_r$ are the edges of $E(R)$ as they were defined above. The algorithm will compute *virtual tricolorings of* S, that is, all mappings of $E(S) - \{e_1, \ldots, e_r\}$ to $\{1, 2, 4\}$ such that no triangle of S has two edges which are colored the same color. Each such virtual tricoloring defines a unique tricoloring (up to permutation of colors) because R being chord-less means that each edge of R is in a unique triangle of $F(S)$, both of whose other edges are in $E(S) - E(R)$ and are colored with two distinct colors in $\{1, 2, 4\}$. We have used to $\{1, 2, 4\}$ instead of $\{-1, 0, 1\}$ to stay as close as possible to the code. Suppose that c is a coloring of some subset of the edges $E(S) - E(R)$ that satisfies the condition. For each edge e_i with $i < m - 1$, the set of *forbidden* *colors for* e_i, denoted F_i, is defined to be the set of all colors $c(e_j) \in \{1,2,4\}$ such that $j > i$ and e_i is in a common triangle. This set will change throughout the algorithm.

The strategy we use is to compute all virtual tricolorings of S, and record what their restriction to R is. We impose a lexicographic ordering on the set $\mathcal{T}$ of $m - r$ element vectors $\mathbf{c}$ whose elements are members of $\{1,2,4\}$, whose i^{th} element is indexed $i+r$ and whose last two elements c_{m-1} and c_m are 2 and 1 respectively as follows: Declare $\mathbf{c} > \mathbf{c}'$ if at the largest index $m-2 > i \geq r+1$ at which c_i and c'_{i-1} differ, we have $c'_i < c_i$. The algorithm will consider all sequences in the set $\mathcal{T}$ in lexicographic order, from least to greatest, and record the restriction to R of those which correspond to virtual tricolorings. The algorithm follows:

Initialize $c(e_m) = 1$, $c(e_{m-1}) = 2$, $F_{m-1} = \{1,4\}$, $F_j = \emptyset$ and $c(e_j) = 8$ for $r+1 \leq j \leq m-2$ and $j = m-2$. Keep repeating steps 1, 2, 3 below.

Step 1. While $c(e_j) \in F_j$, keep repeating steps (i) and (ii) below.

(i) Double $c(e_j)$, and

(ii) while $c(e_j) = 8$ repeat the following steps:

(a) if $j \geq m-1$ terminate computation,

(b) increase j by one and double $c(e_j)$.

If $j = r+1$ do step 2, otherwise do step 3.

Step 2. Do the following:

(i) A tricoloring is uniquely defined by this virtual tricoloring. Record the code of its restriction to $E(R)$, taking into account multiplicity by increasing

`fixedlive`[code]++ by one.

(ii) Double $c(e_j)$

(iii) while $c(e_j) = 8$ repeat steps (a) and (b) below:

(a) if $j \geq m - 1$ terminate computation,

(b) increase j by one and double $c(e_j)$.

Step 3. If $j > r + 1$ decrease j by one, set $c(e_j) = 1$ and compute F_j.

6.4.3 Correctness

To prove correctness, we use the fact that e_{m-1} and e_m are in the same triangle. Essentially, the algorithm builds a function $c : \{e_{r+1}, e_{r+2}, \ldots e_m\} \to \{1, 2, 4\}$ by progressing lexicographically through partial sequences of the form $c_j, c_{j+1}, \ldots, c_m$ where $j \geq m + 1$ and $c_i \in \{1, 2, 4\}$ represents the color $c(e_i)$ that the algorithm has currently assigned to the edge e_i for $j \leq i \leq m$. To prove correctness we must prove that 1) Everything that the algorithm writes to output in step 2 is a tricoloring of the free completion S, and 2) for every tricoloring c of S, the algorithm writes the restriction of c to R to output in step 2.

We first prove 1). Let $c_{r+1}, c_{r+2}, \ldots, c_m$ be a sequence that is written to output at step 2.1 and suppose that it does not induce a tricoloring of S. There must be two indices i, j with $r + 1 \leq j < i \leq m - 2$ such that $c_i = c_j$ and e_i and e_j are in the same triangular face of S. Consider the last time the algorithm assigns the edge e_j the color c_j before recording the tricoloring induced by $c_{r+1}, \ldots, c_m$. For ease of reference, we label this time t and we also label the first time that the algorithm records the restriction to R of the tricoloring induced by $c_{m+1}, c_{m+2}, \ldots, c_r$ by u. We may assume that t occurs before u. By definition of t, the value of c_j will not be altered by the algorithm between the time t and the time u.

We now prove that the edge e_i must be assigned the value c_i at any time between t and u. Assume then that e_i is assigned value other than c_i at some time between time t and time u. Therefore the algorithm must at some point assign c_i to e_i before recording the restriction of the induced tricoloring. For this to happen, step $1(ii)(b)$ or $2(iii)(b)$ must be executed at some point between time t and time u or else the fact that $i > j$ would make it impossible to change the value assigned to e_i. The first time step $1(ii)(b)$ or $2(iii)(b)$ is executed after time t the condition tested in step $1(ii)$ or $2(iii)$ respectively will be whether the current value assigned to e_j is 8. Moreover, this condition must be true since step $1(ii)(b)$ or $2(iii)(b)$ is executed. Since $c_j \in \{1, 2, 4\}$, this contradicts the choice of t. Thus, the value of e_i between time t and time u must be c_i.

It can be shown that there will be a time strictly between time t and time u that the algorithm tests the condition in step 1. The first time it does so, it will be testing if $c_j \in F_j$, and since the value assigned to e_i is c_i, this test will be true. Thus step $1.i$ will be executed and will change the value assigned to e_j, in contradiction to the choice of t. This completes the proof of 1).

To prove 2), we make some helpful observations. For $j > r + 1$ we will say that the algorithm *assumes the assignment* $c'_j, c'_{j+1}, \ldots, c'_m$ if at some point the algorithm has made the assignment $c(e_i) = c'_i$, for every $j \leq i \leq m$. We also say that that the algorithm *attains the assignment* $c'_j, c'_{j+1}, \ldots, c'_m$ if it assumes the assignment $c'_{j-1}, c'_j, c'_{j+1}, \ldots, c'_m$ where $c'_{j-1} = 1$. We first observe that if the algorithm attains an assignment $c'_j, c'_{j+1}, \ldots, c'_m$, then at some point the algorithm assumes the assignment $2c'_j$, c'_{j+1}, …, c'_m. Using this, we then notice that if the algorithm attains an assignment $c'_j, c'_{j+1}, \ldots, c'_m$ then for every $x \in \{1, 2, 4\}$ the algorithm assumes the assignment $c'_{j-1} = x$, c'_j, …, c'_m. Using these two observations, we notice that if

I) the algorithm attains an assignment $c'_j, c'_{j+1}, \ldots, c'_m$, and

II) if for $x \in \{1,2,4\}$ the assignment of x to e_{j-1} and c'_i to e_i for every $j \leq i \leq i$ has the property that there is no facial triangle in S with two edges in $\{e_{j-1}, e_j \ldots, e_m\}$ that have the same value under this assignment then the algorithm attains the assignment $c'_{j-1} = x, \ldots, c'_m$.

We now prove 2). Let $c_{r+1}, c_{r+2}, \ldots, c_m$ represent an arbitrary sequence which induces a tricoloring of S. We may assume by a suitable permutation of colors that $c_m = 1$ and $c_{m-1} = 2$. From the facts proved above, it can be shown by downward induction on j that for $m+1 < j \leq m-2$, the algorithm attains the assignment $c_j, c_{j+1}, \ldots, c_m$. From this and the definition of F_k it follows that $c_{r+1}, c_{r+2}, \ldots, c_m$ induces a tricoloring of S which is recorded in step *ii*. This proves 2) and establishes the correctness of Tricolor.

6.4.4 Implementation

The implementation is primarily carried out by the function `findlive`. The local variables `j,i,u,*am` are all auxiliary to the main task. The local variable `edges` equals m, the number of edges in the configuration, and the local variable `ring` gives the ring size of S, Both of the local variables `extent` and `bigno` are passed to the function `record`; `extent` keeps track of the number of colorings of the ring that extend into S and `bigno` equals the total number of ring colorings. The variable `bigno` is pre-calculated to save time in the function `record`. The local variable `c[j]` represents the algorithm's current color assignment to the edge e_j and the local variable `forbidden[j]` represents the content of the set F_j. If $d_2 d_1 d_0$ are the rightmost three digits in the base 2 representation of `forbidden[j]` then for $0 \leq i \leq 2$, $2^i \in F_j$ if and only if $d_i = 1$. This representation allows the set membership test in step 1 of the algorithm to be carried out by evaluating `c[j] & forbidden[j]`. The portion of code

```
c[j]=1;
for(u=0,i=1;i<=am[0];i++)
u |= c[am[i]];
```

is responsible for the recalculation of F_j in step 3 of the algorithm.

The function `record` is called every time a virtual tricoloring of S is found. Using the information that is passed to it, `record` converts the virtual tricoloring given by `c` to an actual tricoloring of S and computes the code of the restriction to R of this tricoloring. If the code of this ring coloring is denoted by i, then `record` sets `live[i]` to 0 and sets `fixedlive[i]` appropriately. Thus, just after `findlive` finishes its execution `live[i]` and `fixedlive[i]` will have the following properties:

`live[i]`$=0$ if the coloring with code i is in $\mathcal{C}_S$

`live[i]`$=1$ if the coloring with code i is in $\mathcal{C}^* - \mathcal{C}_S$

`fixedlive[i]`$=18$ if the coloring with code i is in $\mathcal{C}^* - \mathcal{C}_S$.

`fixedlive[i]`$=19$ if the coloring with code i is in $\mathcal{C}_S - \mathcal{U}$.

`fixedlive[i]`$=21$ if the coloring with code i is in $\mathcal{U}$.

The main purpose of the function `printstatus` is to print $|\mathcal{C}_S|$ to output. The function `findlive` then terminates by returning the quantity $|\mathcal{C}_S|$. The function `strip` uses a heuristic to choose the ordering of the edges (see Section 6.2.5) so as to reduce the computation time of both `findlive` and `checkcontract`.

6.5 Tricoloring Modulo a Contract

6.5.1 Problem Statement, Notation, and Data Structures

In this section we describe the algorithm:

Tricoloring Modulo A Contract Subroutine ($TriModCon(S, \mathcal{A}, X)$)

Input: A free completion S with ring R of some configuration K, a set $\mathcal{A} \subset \mathcal{U}$ and a proposed safe $\mathcal{A}-$contract X.

Output: Notification of whether the proposed A-contract is valid or not.

There are three parts to the algorithm. The first will check that $\phi(X)$ is contractible for T and this done by using Theorem 5.2.1 in the natural way to show that X is safe. The second is to calculate all tricolorings of S modulo X and the last is to check that none of these tricolorings modulo X are in the set $MCS((\mathcal{C}^* - \mathcal{C}_S) \bigcup \{\mathcal{A}\})$. Since the algorithm in the previous section essentially found all tricolorings of S modulo X when $X = \emptyset$, the first part of the task will be accomplished using an algorithm similar to the one in that section.

6.5.2 Algorithm

Step 1. While $c(e_j) \in F_j$, keep repeating steps (i) and (ii) below.

(i) Double $c(e_j)$, and

(ii) while $c(e_j) = 8$ repeat the following steps:

(a) if $j \geq m - 1$ terminate computation,

(b) increase j by one and set $c(e_j) = min\{2c(e_j), 8\}$.

If $j = r + 1$ do step 2, otherwise do step 3.

Step 2. Do the following:

(i) A tricoloring is uniquely defined by this virtual tricoloring. Record the code of its restriction to $E(R)$.

(ii) Double $c(e_j)$

(iii) while $c(e_j) = 8$ repeat step (a) and (b) below:

(a) if $j \geq m - 1$ terminate computation,

(b) increase j by one and set $c(e_j) = min\{2c(e_j), 8\}$.

Step 3. If $j > r + 1$ decrease j by one, and if $e_j \notin X$ set $c(e_j) = 1$ and compute F_j.

6.5.3 Correctness and Implementation

The correctness of this algorithm follows in a similar manner as the proof of correctness of the algorithm $TriColor$ that is found in Section 6.4.3.

The primary function for this algorithm is `checkcontract`.

The function itself as well as its local variables are very similar to the function `findlive` discussed in Section 6.4.3. The function `inlive` that is called by `checkcontract` is analogous to the function `record` that is called by `findlive`. At the beginning of its execution, the pointer `pneedscontract` points to a variable `needscontract` which is 1 as long as there is no known $\mathcal{A}-$contract. The function `inlive` is called each time a tricoloring modulo X is discovered. If this tricoloring modulo X has a restriction to R that is also in $MCS((\mathcal{C}^* - \mathcal{C}_S)\bigcup \mathcal{A})$ then `inlive` returns a 1 and `checkcontract` thus immediately terminates with the variable `needscontract` set to 1. This corresponds to the case that the proposed contract X was not an $\mathcal{A}$-contract. If on the other hand the variable j is eventually equal to or exceeds `start`, then this means that every tricoloring of S modulo X has been found, and none of these tricolorings is in $MCS((\mathcal{C}^* - \mathcal{C}_S)\bigcup \mathcal{A})$ because otherwise `checkcontract` would have terminated. This means that X is actually an $\mathcal{A}-$contract. In this case, `checkcontract` sets the variable `needscontract` to 0, thus indicating that an $\mathcal{A}-$contract has been found.

6.6 Finding Critical Sets

6.6.1 Problem Statement, Notation, and Data Structures

In this section, we describe the algorithm $Critical(S, \mathcal{A}, \mathcal{D}, \mathbf{v})$:

Input: Two subsets $\mathcal{D} \subset \mathcal{C}^*(R) - \mathcal{C}_S$, and $\mathcal{A} \subset \mathcal{U}$ and a vector $\mathbf{v}$ consisting of $|\mathcal{A}|$ ones.

Output: The maximal critical subset $MCS_{\mathcal{U}}(\mathcal{A}, \mathcal{D})$ of the set $\mathcal{D} \bigcup \mathcal{A}$, and the vector $\mathbf{v}$ such that the i^{th} element of $\mathbf{v}$ is 0 if the i^{th} element of $\mathcal{A}$ is **not in** $MCS_{\mathcal{U}}(\mathcal{A}, \mathcal{D})$ and 1 otherwise.

The techniques used are those of Robertson et al. in [42]. We call a signed matching $P = \{(\{e_1, f_i\}, \mu_1), \ldots, \{e_k, f_k\}, \mu_k)\}$ *balanced* if $r - \sum_{i=1}^{k} \frac{(\mu_i - 1)}{2}$ is even. We define some sets in the same way that Robertson et. al. do with the exception of $\mathcal{M}_0$ which is defined differently to reflect that we are calculating Critical Sets rather than Consistent sets.

Let $\mathcal{M}_0$ be the set of all balanced signed matchings having the property that for every $M \in \mathcal{M}_0$, there are not two distinct colorings u_1,$u_2 \in \mathcal{U}$ and two integers γ_1, $\gamma_2 \in \{-1, 0, 1\}$ such that u_1 θ_1-fits M and u_2 θ_2-fits M.

Let $\mathcal{C}_0 = \mathcal{D} \bigcup \mathcal{A}$. Given $\mathcal{C}_i$ and $\mathcal{M}_i$, define $\mathcal{M}_{i+1}$ to be the of all signed matchings M in $\mathcal{M}_i$ having the property that for every $\theta \in \{-1, 0, 1\}$, $\mathcal{C}_i$ contains every coloring that $\theta-$fits M. Also, given $\mathcal{M}_i$ and $\mathcal{C}_{i-1}$, define $\mathcal{C}_i$ to be the set of all colorings in $\mathcal{C}_{i-1}$ such that for every $\theta \in \{-1, 0, 1\}$, there is a signed matching $M \in \mathcal{M}_i$ such that κ $\theta-$fits M. The next theorem will be the foundation for proving the correctness of $Critical$.

Lemma 6.6.1 *If $\mathcal{M}_0$, $\mathcal{M}_1$, ..., and $\mathcal{C}_0$, $\mathcal{C}_1$, ... are defined as above, and if there is a positive integer n such that $\mathcal{C}_{n-1} = \mathcal{C}_n$, then the maximally critical subset of $\mathcal{D} \bigcup \mathcal{A}$ with respect to $\mathcal{U}$ equals $\mathcal{C}_n$.*

Proof: Let $\kappa \in \mathcal{C} \in \mathcal{C}_n$, $\theta, \theta', \eta, \eta' \in \{-1, 0, 1\}$ and let $\alpha, \alpha' \in \mathcal{U}$ be distinct. We show that there is an $M \in \mathcal{M}_n$ such that

i) κ $\theta-$fits M.

ii) Every coloring κ' that θ' fits M is in $\mathcal{C}$.

iii) It is not the case that α $\eta-$fits M and α' $\eta'-$fits M.

Since $\kappa \in \mathcal{C}_n = \mathcal{C}_{n-1}$, the definition of $\mathcal{C}_n$ shows that there is a signed matching $M \in \mathcal{M}_n$ which κ $\theta-$fits and so property i) holds. Now suppose that some κ' $\theta'-$fits M. Since $\mathcal{M}_n = \mathcal{M}_{n+1}$, $M \in \mathcal{M}_{n+1}$ and so it follows that $\kappa' \in \mathcal{C}_n$, which proves property (ii) of critical. Finally, since $\mathcal{M}_n \subset \mathcal{M}_0$, it cannot be the case that α $\eta-$fits M and α' $\eta'-$fits M simultaneously. Thus $\mathcal{C}_n$ is critical with respect to $\mathcal{U}$.

To show that $MCS_{\mathcal{U}}(\mathcal{A}, \mathcal{D}) = \mathcal{C}_n$ it will suffice to show $MCS_{\mathcal{U}} \subset \mathcal{C}_i$ for $i = 0, 1, \ldots$, since the maximality of $MCS_{\mathcal{U}}(\mathcal{A}, \mathcal{D})$ implies that $\mathcal{C}_n \subset MCS_{\mathcal{U}}(\mathcal{A}, \mathcal{D})$.

Assume then that this is not the case and let j be the smallest integer such that $MCS_{\mathcal{U}}(\mathcal{A}, \mathcal{D}) \nsubseteq \mathcal{C}_j$. Clearly $j > 0$. Let $\kappa \in (\mathcal{C}_{j-1} \bigcap MCS_{\mathcal{U}}(\mathcal{A}, \mathcal{D})) - \mathcal{C}_j$. It follows that there is a $\theta \in \{-1, 0, 1\}$ and an $M \in \mathcal{M}_{j-1} - \mathcal{M}_j$ such that κ $\theta-$fits M. Since $\kappa \in MCS_{\mathcal{U}}(\mathcal{A}, \mathcal{D})$ and $\mathcal{M} \subset \mathcal{M}_{j-1}$, we may choose M so that $M \in \mathcal{M}$. Since

$M \notin \mathcal{M}_j$, there must be a $\theta' \in \{-1, 0, 1\}$ and a $\kappa' \notin \mathcal{C}_{j-1}$ such that κ' θ'−fits M. Now $M \in \mathcal{M}_{j-1}$ implies that $\kappa' \in \mathcal{C}_{j-2}$. There is a coloring κ'' that is equivalent to κ' such that κ'' θ−fits M. By the definition of equivalent colorings, it follows that $\kappa'' \in \mathcal{C}_{j-2} - \mathcal{C}_{j-1}$. However, $\kappa'' \in MCS_{\mathcal{U}}(\mathcal{A}, \mathcal{D})$ because $M \in \mathcal{M}$ and κ'' θ−fits M. Therefore, by the choice of j, $\kappa'' \in \mathcal{C}_{j-1}$, which contradicts that $\kappa'' \in \mathcal{C}_{j-2} - \mathcal{C}_{j-1}$. This completes the proof of the lemma.

6.6.2 Algorithm

6.6.2.1 High Level Description

Here is an overview of the Subroutine $Critical(\mathcal{M}, \mathcal{D}, \mathcal{A}, \mathbf{v})$:

$Critical(\mathcal{M}, \mathcal{D}, \mathcal{A}, \mathbf{v})$**:**

Step 1. Compute $\mathcal{M}_0$, the set of all balanced signed matchings having the property that for every $M \in \mathcal{M}_0$, there is no $\theta \in \{-1, 0, 1\}$ such that there are two distinct colorings of $u_1,\ u_2 \in \mathcal{U}$ which θ−fit M. Set $\mathcal{C}_0 = \mathcal{D} \bigcup \mathcal{A}$.

Step 2. Repeat the following: having calculated $\mathcal{M}_i$ and $\mathcal{C}_i$, do:

2a) Compute $\mathcal{M}_{i+1}$.

2b) Compute $\mathcal{C}_{i+1}$

2c) If $\mathcal{C}_i = \mathcal{C}_{i-1}$, halt.

2d) For every $\kappa \in \mathcal{C}_{i-1} - \mathcal{C}_i$, if $\kappa \in \mathcal{A}$, set the appropriate entry of $\mathbf{v}$ to 0.

This can be accomplished by the following Do While loop:

Do(ALLMATCHINGS($\mathcal{M}_i, \mathcal{C}_i, \mathcal{A}$))

While(ADJUST($\mathcal{M}_i, \mathcal{C}_i, \mathcal{A}$) = 1)

$\{i = i + 1\}$

Here ALLMATCHINGS generates all signed matchings, and discards those $\mathcal{M}_i$ that are incompatible with the current state of $\mathcal{C}_i$ while ADJUST updates $\mathcal{C}_i$ to $\mathcal{C}_{i+1}$, returning a 1 if $\mathcal{C}_{i+1} \neq \mathcal{C}_i$ and a 0 if $\mathcal{C}_i = \mathcal{C}_{i+1}$ or if it is discovered that every color in $\mathcal{A}$ is either D-removable or C-removable. More details about this follow in the next section.

The correctness of the algorithm follows from Lemma 6.6.1 above.

6.6.2.2 Calculating $\mathcal{M}_{i+1}$ from $\mathcal{M}_i$

We now describe in more detail how $\mathcal{M}_{i+1}$ is calculated from $\mathcal{M}_i$ and $\mathcal{C}_i$. This will be accomplished by recursively calculating every signed matching in the same order, including those we know to not be $\mathcal{M}_i$. Let R be a ring and suppose that $M = \{\{e_1, f_1\}, \{e_2, f_2\}, \{e_3, f_3\}, \ldots, \{e_k, f_k\}\}$ is a fixed unsigned matching, and that $\mathcal{I} = \{[s_1, t_1],\ [s_2, t_2],\ \ldots,\ [s_l, t_l]\}$ are disjoint intervals satisfying $s_1 \leq t_1 < s_2 \leq t_2 < s_3 \leq t_3 < \ldots s_l \leq t_l$ and $\{s_1, \ldots, s_l, t_1, \ldots t_l\} \subset \{1, 2, \ldots, r\} - \{e_1, \ldots, e_l, f_1, \ldots, b_l\}$. For convenience, if $0 \leq h \leq l$, let $\mathcal{I}_h$ denote the collection $\{[s_1, t_1], \ldots, [s_h, t_h]\}$. Let $\mathcal{M}$ be a set of signed matchings, and $\mathcal{C}$ a set of ring colorings. If P is a signed matching, then let $M(P)$ denote the set of individual matches that make up P.

We will call functions that the subroutines of Section 6.1.4 use procedures, although it would have been appropriate to refer to them as sub-subroutines.

We now describe a recursive procedure MATCHING$(M, \mathcal{I}, \mathcal{M}, \mathcal{C})$ which updates the set $\mathcal{M}$ to a set $\mathcal{M}'$ such that

1) $\mathcal{M}$ contains all and only those signed matchings P which

1a) Satisfy $M \subset M(P)$ and if $\{a, b\} \in M(P)$ then there is a p with $1 \leq p \leq l$ such that $s_p \leq a < b \leq t_p$.

1b) There is a $\theta \in \{-1, 0, 1\}$ such that $\mathcal{C}$ contains every coloring that $\theta-$fits P.

There are two main tasks which in the implementation are woven together. The first is to generate all unsigned matchings which satisfy 1a). This is accomplished by finding a new match in one of the intervals in $\mathcal{I}$, adding this match to M to get a new unsigned matching M', and recursively calling MATCHING on M' and an appropriate adjustment of $\mathcal{I}$. This is done in step 2.ii) below.

As a new unsigned matching M' is generated, the call of SIGN(M') in step 1.ii) will make sure that property 1b) will hold for $\mathcal{C}'$.

MATCHING(M,$\mathcal{I}$, $\mathcal{M}$, $\mathcal{C}$){

1) SIGN(M, $\mathcal{M}$, $\mathcal{C}$);

2) For every $h \in \{1, \ldots, l\}$ and for every pair of integers i, j satisfying $s_l \leq j < i \leq t_l$ do

2.i) Add $M \bigcup \{j, i\}$ to $\mathcal{M}$.

2.ii) If $j \leq s_h + 1$ and $i \leq j + 2$) do

MATCHING($M \bigcup \{\{j, i\}\}, \mathcal{I}_h, \mathcal{M}$);

Else If $j > s_h + 1$ and $i \leq j + 2$ do

MATCHING($M \bigcup \{\{j, i\}, \}, \{\{j, i\}\}, \mathcal{I}_{h-1} \bigcup [s_h, j-1], \mathcal{M}$);

Else If $j \leq s_h + 1$ and $i > j + 2$ do

MATCHING($M \bigcup \{\{j, i\}, \}, \{\{j, i\}\}, \mathcal{I}_{h-1} \bigcup [j+1, i-1], \mathcal{M}$) ;

Else If $j > s_h + 1$ and $i > j + 2$ do

MATCHING($M \bigcup \{\{j, i\}, \}, \{\{j, i\}\}, \mathcal{I}_{h-1} \bigcup [s_h, j-1] \bigcup [j+1, i-1], \mathcal{M}$);

}

The procedure SIGN when called on the unsigned matching M will first generate all possible assignments of signs to the matchings M. If P denotes a signed matching generated from this procedure, so that $M(P) = M$, SIGN will then determine whether or not P has the property that there is some $\theta \in \{-1, 0, 1\}$ such that $\mathcal{C}$ contains every coloring which $\theta-$fits M and will include or remove P from $\mathcal{M}$ accordingly. Finally, if SIGN decides to keep P in $\mathcal{M}$, then there will be some preprocessing done on the set $\mathcal{C}$ to prepare this set for the next call to the procedure ADJUST($\mathcal{C}, \mathcal{M}$)

If $\mathcal{M} = \emptyset$ initially, then when MATCHING($M, \mathcal{I}, \mathcal{M}, \mathcal{C}$) finishes it execution, properties 1a) and 1b) will hold for $\mathcal{M}$. This is not enough however, because condition 1a) demands that a particular matching be included and we want all signed matchings. To generate all signed matchings in the same order every time, we use the procedure ALLMATCHINGS.

ALLMATCHINGS($\mathcal{M}, \mathcal{C}$) {

For every $i \in \{2, \dots, r\}$ and every $j \in \{1, \dots, i\}$ Do 1) and 2):

1) If $j \leq 2$ and $i \leq j + 2$, Let $\mathcal{I} = \emptyset$

Else If $j \geq 3$ and $i \leq j + 2$, Let $\mathcal{I} = \{[1, j - 1]\}$;

Else If $j \leq 2$ and $i \geq j + 3$, Let $\mathcal{I} = \{[j + 1, i - 1]\}$

Else If $j \geq 3$ and $i \geq j + 3$, Let $\mathcal{I} = \{[1, j - 1], [j + 1, i - 1]\}$

2) MATCHING($\{j, i\}, \mathcal{I}, \mathcal{M}, \mathcal{C}$);

}

6.6.2.3 Calculating $\mathcal{C}_{i+1}$ from $\mathcal{C}_i$

After ALLMATCHINGS terminates execution, $\mathcal{M}$ will contain every signed matching that satisfies 1b). At this point the algorithm will call a subroutine ADJUST($\mathcal{C}, \mathcal{M}, \mathcal{A}$) which will remove colors of $\mathcal{C}$ so that when it is finished executing the only colorings which will remain in $\mathcal{C}$ are those colorings κ satisfying:

2) for every $\theta \in \{-1, 0, 1\}$, there is an $P \in \mathcal{M}$ such that κ $\theta-$fits P.

As it runs, ADJUST($\mathcal{C}, \mathcal{M}, \mathcal{A}$) will also remove any colors from $\mathcal{A}$ that it also removed from $\mathcal{C}$ (which Section 6.9.1 shows implies D-removability for such removed colorings), and will occasionally call a procedure SHORTCUT, explained in Section 6.9.3 that attempts to find an $\mathcal{A}-$contract.

ADJUST returns a zero if (i) it removes every color in $\mathcal{A}$ or (ii) if it discovers an $\mathcal{A}-$contract or (iii) if in a particular iteration it does not remove any color from $\mathcal{C}$. Possibility (i) implies every color in $\mathcal{A}$ being D-removable, possibility (ii) implies each color in $\mathcal{A}$ being C-removable, and possibility (iii) corresponds to the condition $\mathcal{C}_n = \mathcal{C}_{n-1}$ and so implies that $\mathcal{C}_n$ is now a maximal critical subset. Otherwise ADJUST returns a one and the algorithm calls ALLMATCHINGS again to start the process of again generating all signed matchings.

6.6.3 Implementation

We now discuss the process in terms of the code actually used. The Do- While loop at the end of Section 6.6.2 corresponds to the following Do-While loop found in the function `control`:

```
do
      testmatch(ring, real, power, live,fixedlive, nchar);
while(updatelive(live, fixedlive,ncodes, pnlive,set,
             &needscontract,&progress));
```

We have omitted the body of this loop and will discuss its role in Section 6.9.4.

The task of computing $\mathcal{M}_{i+1}$ from $\mathcal{M}_i$ by generating all signed matchings and discarding appropriately is carried out by the functions `testmatch`, `augment`,

`checkreality` and `stillreal` as follows: `testmatch` is responsible for some necessary initialization and corresponds to the procedure ALLMATCHINGS; `augment` is responsible for recursively generating unsigned matchings and corresponds to the procedure MATCHING; `checkreality` generates all possible signings of a given unsigned matching and corresponds to the first part of the procedure SIGN; `stillreal` generates all ring colorings which fit a given signed matching and corresponds to the second part of the procedure SIGN as well as doing preprocessing for the function `updatelive`. This function `updatelive` corresponds to the procedure ADJUST and has the task of updating $\mathcal{C}_i$ to $\mathcal{C}_{i+1}$.

The variable `real` tracks the sets $\mathcal{M}_j$ by the rule that the bit in `real` corresponding to some signed matching P is 1 if and only and if $P \in \mathcal{M}_j$. The variable `live` is usually governed by the rule that at the end of the j^{th} iteration of the Do-While Loop, `live`$[i] = 1$ if and only if there is a ring coloring κ with code i and $\kappa \in \mathcal{C}_j$; otherwise `live`$[i] = 0$. More information about `real` and `live` can be found in Sections 6.2.3 and 6.2.4.

The function `testmatch` is straightforward, corresponding closely to ALLMATCHINGS. In the function `augment`, the array `interval` corresponds to the collection $\mathcal{I}$ of intervals in the procedure MATCHING. The array `weight` of `augment` essentially encodes the fixed unsigned matching M of the procedure MATCHING. To understand what exactly the encoded information of `weight` is and to better understand `checkreality` and `stillreal` we make some definitions.

Let $P = \{(\{e_1, f_1\}, \mu_1), (\{e_2, f_2\}, \mu_2, \ldots, (\{e_k, f_k\}, \mu_k)\}$ be a signed matching. The *code* of P is is given by $\sum_{i=1}^{k}(3^{e_i-1}+\mu_i 3^{f_i-1})$ if $e_1 < r$, and $\frac{(3^i-1)}{2} - \sum_{i=1}^{k}(3^{e_i-1}+\frac{(3-\mu_i)3^{f_i-1}}{2})$ if $e_1 = r$. Here we are assuming $e_i > f_i$ for $1 \leq i \leq k$ and that $e_1 = max\{e_1, e_2, \ldots, e_k\}$. For $i = 2, \ldots, k$, define $h_i = 2(3^{e_i-1}+\mu_i 3^{f_i-1})$ if $e_1 < r$ and $h_i = 3^{e_i-1}+\mu_i 3^{f_i-1}$ if $e_1 = r$ and define the vector $(h_2, h_3, \ldots, h_k)$ to be the *choice sequence of* M.

The information of `weight` that was originally assigned to the variable `mw` in the function `testmatch` is all of the various combinations of powers of 3 that are needed to calculate the code and the choice function of a signed matching. The variable `choice` in the function `checkreality` is just the choice function for a particular signing of the unsigned matching M that was passed to `checkreality`. The variables `pbit` and `prealterm` are discussed in Section 6.2.4 and are used to access and update `real`. Once `checkreality` fixes a particular signing of M, call it P, `stillreal` is called to see if all of the colorings which fit P are in `live` and if any of them are not, `stillreal` returns a zero and so the line

```
real[*prealterm] ^= *pbit;
```

of `checkreality` is executed, thus setting the bit corresponding to P to zero, and thereby denying P the right to remain in $\mathcal{M}$. The following lemma found in Robertson et. al. [42] is used by `stillreal` to accomplish the task of generating all ring colorings which fit a given signed matching. The statement here is a little different then the statement in some versions of [42] which seemed to have some typographical errors.

Lemma 6.6.2 *Let P be a signed matching with code c and choice sequence $(h_2, \ldots, h_k)$. Let $\mathcal{C}_M$ be the set of codes of canonical ring colorings κ for which there is a $\theta \in \{-1, 0, 1\}$ such that κ $\theta-$fits M. Then $\mathcal{C}_M = \{|c - \sum_{i=2}^{k}\epsilon_i h_i| : \epsilon_i \in \{0,1\}\}$. Moreover, if κ is a canonical ring coloring which $\theta-$fits M for some $\theta \in \{-1, 0, 1\}$ and which has code $|d|$ where $d = c - \sum_{i=2}^{k}\epsilon_i h_i$ where $\epsilon_i \in \{0,1\}$ for $i = 2, \ldots, k$, then the* *identity of θ is determined as follows: if $e_i < r$ then $\theta = 0$ and if $e_i = r$ then $\theta = 1$ if $d > 0$ and $\theta = -1$ if $d < 0$.*

Proof: We give an idea of how the proof proceeds for the case when $e_1 < r$. Fix an assignment of $(\epsilon_2, \ldots, e_k)$. Define $\epsilon_1 = 1$ and for $\alpha \in \{-1, 1\}$ the sets

$E_\alpha := \{e_j : \epsilon_j = 1 + \alpha\}$, and

$F_\alpha := \{f_j : (\epsilon_j = 0 \ \& \ \mu_j = -\alpha) \ \text{ or } (\epsilon_j = 1 \ \& \ \mu_j = \alpha)\}$.

Define a coloring $\kappa : E(R) \ \rightarrow \ \{-1, 0, 1\}$ as follows:

$\kappa(i) = \alpha$ if $i \in E_\alpha \bigcup F_\alpha$ for $\alpha \in \{-1, 1\}$ and $\kappa(i) = 0$ otherwise. It can be shown that c $0-$fits P.

Conversely, it follows from the fact that there are $k - 1$ pairs of matched edges in P (excluding the one with the largest index) which allow 2 choices of coloring that there are 2^{k-1} different canonical ring colorings which fit P. Thus, every ring coloring which fits P is included. This completes the proof of the lemma.

Practically, the for loop in `stillreal` which is initialized by

```
for (i = 2, twopower = 1, mark = 1; i <= depth; i++, twopower <<= 1)
```

and `for (j = 0; j < twopower; j++, mark++)`

simply calculates $c - \sum_{i=2}^{k} \epsilon_i h_i$ for every $(\epsilon_2, \epsilon_3, \ldots, \epsilon_k) \in \{0, 1\}^{k-1}$.

Knowing the code i of every ring colorings κ which $\theta-$fit a signed matching P that has been passed to `stillreal`, lines like

```
if (!live[col])
    return ((long) 0);
```

check if $\kappa \in \mathcal{C}_j$ and if not, `stillreal` returns 0. If every ring coloring that $\theta-$ fits P for some $\theta \in \{-1, 0, 1\}$, then end of `stillreal` performs preprocessing for `updatelive` by treating the elements of live as four digit binary strings and marking the 2, $3rd$ or $4th$ digit of live$[i]$ with a 1 depending on whether κ either $-1-$fits, or $0-$fits or 1-fits P.

When testmatch and all the functions it calls finish their recursion, execution in the Do-While loop passes to updatelive. In updatelive, the line

```
if(live[i] != 15)
```

uses the preprocessing done by stillreal as a basis of whether to keep or remove the coloring with code i from the set $\mathcal{C}_j$ because live[i]$\neq$ 15 if and only if at least one of the second, third or fourth bits of live[i] was not set to one by stillreal which happens if and only if there is a $\theta \in \{-1, 0, 1\}$, such that no signed matching of $\mathcal{M}_{j+1}$ is $\theta-$fit by κ. When updatelive removes a ring coloring with code i from $\mathcal{C}_j$, it checks to see whether κ was in $\mathcal{A}$ with the lines

```
if(in_set(i,testset,1,testset[0]) ){
    fixedlive[i]=20;
    leftinset--;
    (*pprogress)++;
},
```

setting fixedlive[i] to 20 because κ is D-removable.

If all of the colorings of $\mathcal{A}$ are eliminated this way, or if in a particular iteration, no colors are removed from $\mathcal{C}_j$, then updatelive returns a zero and so terminates the Do-While loop. In the case when all colors of $\mathcal{A}$ are removed, updatelive sets *pneedscontract to zero, a fact of some relevance to Section 6.9.4.

One other comment about some of the code in stillreal. Lines like

```
if( ( 20<=fixedlive[-b] ) && unique){
    tally++;
```

```
    if(tally>=2) return( (long) 0);

}/* end if !fixedlive && unique */
```

are designed to eliminate matchings M for which there are two distinct colorings $\alpha, \alpha' \in \mathcal{U}$ and integers $\eta, \eta' \in \{-1, 0, 1\}$, such that α $\eta-$fits M, and α' $\eta'-$fits M simultaneously. This is necessary to comply with the definition of $\mathcal{M}_0$.

6.7 Finding Contracts

In this section, we discuss the Subroutine 5 of Section 6.1.4. To find a safe contract, we simply run through all possible sparse sets $|X|$ on four edges, and for those which are safe, we calculate $\mathcal{C}_S(X)$ using the subroutine $TriConMod$ and then see whether or not $\mathcal{C}_S(X) \bigcap MCS(\mathcal{A}) = \emptyset$. The functions used for this are `trycontract`, `updateangles`, and `checkcontract`. The function `trycontract` generates all possible 4 element subsets of edges, and uses the information in the array `angle` (see Section 6.2.5), to exclude non-sparse sets. At this point, the function `updateangles` is called to change the information in the arrays `diffangle`, `sameangle` and `contract`, and to see whether or not the contract is safe by trying to find a triad. If the proposed set X of 4 edges is not safe then `updateangles` returns a zero, and it returns a one otherwise. At this point, `checkcontract` is called to see whether or not $\mathcal{C}_S(X) \bigcap MCS(\mathcal{A}) = \emptyset$, and thus whether or not X is a safe $\mathcal{A}-$contract. The variable `needscontract` and its associated pointer `pneedscontract` will be set to one until a safe contract is found, at which time it is set to zero. The vertices which are the endpoints of the i^{th} edge in a potential contract are stored in `graph[0][4+2i-1]`, and `graph[0][4+2i]`. In addition, all of the edges are ordered, and the i^{th} element of the array `contract` is set to 1 if the i^{th} edge is in the potential contract. The array `contractindices` records the indices under this order of the edges of the contract.

Certain contracts are much better than others. When a contract X is discovered that has promise of being favorable, the array `hope` stores certain X by a call to the program `prepare_matrix`. The function `check_hope` can then be called to quickly see if there is some contract stored in `hope` that might be a valid contract. The function `record_contract` which appears in the function `main` is used by `prepare_matrix` to record the set $\mathcal{C}_S(X)$ into the matrix `hope`. The function `smart_contract` is designed to find contracts which would be expected to be good, using a heuristic. Roughly, this heuristic attempts to find those safe contracts which have the effect of identifying distinct ring vertices. When `smart_contract` finds such a contract X, it stores X and $\mathcal{C}_S(X)$ in `hope`. Leaving `HOPE` undefined disables any attempt to store colorings in `HOPE` or make calls to `check_hope`, and leaving `SMARTCONTRACT` undefined insures that the program does not call the function `smart_contract`.

6.8 The Controlling Algorithm

6.8.1 Problem Statement, Notation

We will introduce some generalizations of D-removability and C-removability that are used to speed up the process of showing that a configuration K is U-reducible. Let $\mathcal{C}$ be any subset of $\mathcal{C}^* - \mathcal{C}_S$. We will denote by $MCS(\mathcal{A}, \mathcal{C})$ the maximal critical subset of $\mathcal{C} \bigcup \eta(\mathcal{A})$ with respect to $\mathcal{U}$. When $\mathcal{C} = \mathcal{C}^* - \mathcal{C}_S$ we simply denote this set $MCS(\mathcal{A})$ as we have been doing before. We say that a coloring $\kappa \in \mathcal{U}$ is *D-removable with* *respect to* $\mathcal{C}$ if $\kappa \notin MCS(\kappa, \mathcal{C})$ and we say that $\mathcal{A}$ is $D-$*removable with respect to* $\mathcal{C}$ if $\mathcal{A} \bigcap MCS(\mathcal{A}, \mathcal{C}) = \emptyset$. Similarly, if there is a safe contract $X \subset E(S)$ for which $\mathcal{C}_S(X) \bigcap MCS(\mathcal{A}, \mathcal{C}) = \emptyset$, then we say that $\mathcal{A}$ is $C-$*removable with respect to* $\mathcal{C}$ and make obvious allowances in notation if $|\mathcal{A}| = 1$. When $\mathcal{C} = \mathcal{C}^* - \mathcal{C}_S$, these definitions reduce to the original definitions for D-removability and C-removability.

Lemma 6.8.1 *If* $u \in \mathcal{A} \subset \mathcal{U}$*, then* u *is D-removable with respect to* $\mathcal{C}$*, if* $\mathcal{A}$ *is* $D-$*removable. Similarly,* u *is* $C-$*removable if* $\mathcal{A}$ *is C-removable. Also, let* $\mathcal{A}' = \mathcal{A} \bigcap MCS(\mathcal{A}, \mathcal{C})$ *and let* $\mathcal{C}' = \mathcal{C} \bigcap MCS(\mathcal{A}, \mathcal{C})$*. Then* $MCS(\mathcal{A}', \mathcal{C}') = MCS(\mathcal{A}, \mathcal{C})$ *and so* $\mathcal{A}$ *is D-removable (or C-removable respectively) with respect to* $\mathcal{C}$ *if and only if* $\mathcal{A}$ *is D-removable (or C-removable respectively) with respect to* $\mathcal{C}'$*.*

Proof: From Lemma 5.1.2 and the fact that $\eta(\{u\}) \bigcup (\mathcal{C}^* - \mathcal{C}_S) \subset \eta(\mathcal{A}) \bigcup (\mathcal{C}^* - \mathcal{C}_S)$, it follows that $MCS(u) \subset MCS(\mathcal{A})$. Therefore, $\mathcal{A} \bigcap MCS(\mathcal{A}) = \emptyset$ implies $u \notin MCS(u)$. Also, since $\mathcal{C} \subset \mathcal{C}^* - \mathcal{C}_S$, it follows that $\mathcal{C} \bigcup \eta(\{u\}) \subset (\mathcal{C}^* - \mathcal{C}_S) \bigcup \eta(\{u\})$, and so by Lemma 5.1.2, $MCS(u, \mathcal{C}) \subset MCS(u)$. Hence, $u \notin MCS(u, \mathcal{C})$, which proves the first statement about $D-$removability. The first statement about C-removability follows in a similar vein because for some sparse X, $\mathcal{C}_S(X) \bigcap MCS(\mathcal{A}) = \emptyset$.

Now we establish that $MCS(\mathcal{A}', \mathcal{C}') = MCS(\mathcal{A}, \mathcal{C})$. Clearly $MCS(\mathcal{A}', \mathcal{C}') \subset MCS(\mathcal{A}, \mathcal{C})$. If $x \in MCS(\mathcal{A}, \mathcal{C})$ then $x \in \mathcal{A}$ or $x \in \mathcal{C}$ which implies $x \in \mathcal{A}' \bigcup \mathcal{C}'$, and so $MCS(\mathcal{A}, \mathcal{C}) \subset \mathcal{A} \bigcup \mathcal{C}$. Thus $MCS(\mathcal{A}, \mathcal{C}) \subset MCS(\mathcal{A}', \mathcal{C}')$. From this it follows that $\mathcal{A}$ is D-removable (respectively, C-removable) if and only if $\mathcal{A}$ is D-removable (respectively, C-removable) with respect to $\mathcal{C}^{-}\mathcal{D}$ where $\mathcal{D} = (\mathcal{A} \bigcup \mathcal{C}) - (\mathcal{A}' \bigcup \mathcal{C}')$. This completes the proof of the lemma.

The above ideas will be exploited computationally by attempting to simultaneously eliminate (i.e. show to be either D or C removable) large subsets $\mathcal{A} \subset \mathcal{U}$ rather than simply individual colors $u \in \mathcal{U}$. They also give a natural way to recurse.

We now describe the algorithm used to prove that every color $u \in \mathcal{U}$ is either D-removable or C-removable.

$\underline{\mathit{UniqueReduce}(S, \mathcal{A}, \mathcal{C}, \mathbf{v})}$

Input A free completion S with ring R of some configuration K, and a subset $\mathcal{A}$ of the set $\mathcal{U}$ of ring colorings that extend to exactly one tricoloring of S, a subset $\mathcal{C} \subset \mathcal{C}^* - \mathcal{C}_S$, and a vector $\mathbf{v}$ consisting of $|\mathcal{A}|$ ones.

Output A set $A \subset \mathcal{A}$ with the following property:

A is empty if every coloring $\gamma \in \mathcal{A}$ is either D-removable or C-removable with respect to $\mathcal{C}$ and otherwise $A = \{\gamma\}$ where γ is a ring coloring of R such that γ is neither D-removable or C-removable with respect to $\mathcal{C}$. Also, each entry in $\mathbf{v}$ is one if and only if the corresponding element of $\mathcal{A}$ is in $MCS(\mathcal{A}, \mathcal{C})$ and zero otherwise.

By calling $\mathit{UniqueReduce}(S, \mathcal{U}, \mathcal{C}, \mathbf{v})$ and checking the results, we can determine either that every color in $\mathcal{U}$ is either C-removable or D-removable or that there is some coloring in $\mathcal{U}$ which is neither $D-$ nor $C-$removable. If every color in $\mathcal{U}$ is C-removable or D-removable, then the only thing needed to do to establish that K is U-reducible is to check that either some coloring $u \in \mathcal{U}$ is C-removable or that K is either D-reducible or C(4)-reducible.

6.8.2 Algorithm

$\underline{\mathit{UniqueReduce}(S, \mathcal{A}, \mathcal{C}, \mathbf{v})}$

Step 1. Call $\mathit{Critical}(\mathcal{A}, \mathcal{C}, \mathbf{v})$ to compute $MCS(\mathcal{A}, \mathcal{C})$, let $\mathcal{B} = MCS(\mathcal{A}, \mathcal{C}) \bigcap \mathcal{A}$, and let $\mathcal{A}' = \mathcal{A} - \mathcal{B}$.

Step 2. If $\mathcal{A}' = \emptyset$ return $\emptyset$. Otherwise, for each sparse set $X \subset E(S)$ with $|X| = 4$ Do

2.i $TriModCon(S, X, \mathcal{A}')$.

2.ii If X is a $\mathcal{A}'$ contract, then $\mathcal{A}'$ is C-removable with respect to $\mathcal{C}$. Return $\emptyset$.

Step 3. If $|\mathcal{A}'| = 1$ let κ be the unique element of $\mathcal{A}'$ and return $\{\kappa\}$. Otherwise partition $\mathcal{A}'$ into two roughly equal size subsets $\mathcal{A}_1$ and $\mathcal{A}_2$. Let $\mathcal{C}'$ be the maximal subset of $\mathcal{C}$ having the property that $\mathcal{C}' \subset MCS(\mathcal{A}', \mathcal{C})$.

Step 4. Do

4.i Call $UniqueReduce(S, \mathcal{A}_1, \mathcal{C}', \mathbf{v}_1)$ and suppose it returns A_1.

4.ii If $A_1 \neq \emptyset$, return A_1. Otherwise, call $UniqueReduce(S, \mathcal{A}_2, \mathcal{C}', \mathbf{v}_2)$ and suppose it returns A_2. Return (A_2).

6.8.3 Implementation

The function which corresponds to $UniqueReduce(S, \mathcal{A}, \mathcal{C})$ is `control`. When run on the arguments `startcolor` and `stopcolor`, the function `control` returns a one if every ring coloring $u \in \mathcal{U}$ satisfying `startcolor` $\leq code(u) \leq$ `stopcolor` is either D-removable or C-removable. The set $\mathcal{C}$ will be encoded in `live` by the rule that $\kappa \in \mathcal{C}$ if and only if `live[`$code(\kappa)$`]`$= 1$ and `live[`$code(\kappa)$`]`$= 0$ otherwise. The set $\mathcal{A}$ is encoded in `fixedlive` and the vector **v** in `set`. The arguments of `control`, like `maxiter`, `hopethresh`, `hopefrec`, `tryconthresh`, `tryconfrec`, `progressmin`, `minprogressiter` and `hope` are simply related to the shortcuts that `control` will attempt to dispose of its task more quickly but do not contribute to the essence of `control`. The other arguments either contain important information about S or are needed to pass to other functions that `control` calls.

When `control` begins, it must do some initialization to appropriately adjust `set` and `live` to make sure that every element of $\mathcal{A}\bigcup\mathcal{C}$ is represented. Then it carries out Step 1 of Section 6.8.2 in the Do-While loop which is discussed in Section 6.6.3. If in the midst of executing the Do-While loop, either an $\mathcal{A}-$contract during a shortcut, or if every color in $\mathcal{A}$ is shown to be D-removable, then the variable `needscontract` is set zero, and `control` returns a one, having successfully completed its task. If not, then by the end of the Do-While loop, the set $MCS(\mathcal{A},\mathcal{C})$ has been computed and the task of Step 2 of Section 6.8.2 commences. If the call `trycontract` produces an $\mathcal{A}-$contract, then `trycontract` sets `needscontract` to zero, and is `control` returns a one. If not, then Step 3 of Section 6.8.2 is carried out, and in particular, if $|\mathcal{A}| = 1$ `control` returns a zero, indicating that the current configuration is not U-reducible. Assuming, $|\mathcal{A}| \neq 1$, `control` proceeds to split up the colors in $\mathcal{A}$ that have yet been proved to be D- or C-removable into two roughly equal size set $\mathcal{A}_1$ and $\mathcal{A}_2$ for a recursive call.

6.8.3.1 Shortcuts in `control`

We now explain the shortcuts that are built into the body of the Do-While loop in `control`. In terms of the notation of Section 6.6.3, a particular cycle of the Do-While loop primarily corresponds to a calculation of $\mathcal{M}_i$, and $\mathcal{C}_i$ from the knowledge of $\mathcal{M}_{i-1}$ and $\mathcal{C}_{i-1}$. Depending on the values of the parameters `maxiter`,`hopethresh`, etc., an attempt is made to find an $\mathcal{A}-$contract, even though $MCS(\mathcal{A},\mathcal{C})$ is not known yet. This is possible because at every iteration, $\mathcal{C}_i \subset MCS(\mathcal{A},\mathcal{C})$, and so if some safe X can be found such that $\mathcal{C}_i \bigcap \mathcal{C}_S(X) = \emptyset$, then $MCS(\mathcal{A},\mathcal{C}) \bigcap \mathcal{C}_S(X) = \emptyset$ as well. Some examples of these attempts occur in the lines

```
if( (iteration > = hopethresh) ) && ...)  {
    check_hope(hope,live, &needscontract,...);
}
```

and

```
if( (iteration>=trycontthresh)&& ...){
    try_contract(0,4,graph,ring,...);
} /* if iteration >= trycontthresh etc */
```

As discussed in Section 6.7, the array `hope` is an array which stores previously calculated contracts that are likely to be successful $\mathcal{A}-$contracts. The function `check_hope` simply checks these favorable contracts against $\mathcal{C}_i$ to see if any of them might be an $\mathcal{A}-$contract, and if it is, `check_hope` set `*pneedscontract` to zero, thus breaking the Do-While loop.

The call to `trycontract` just generates all possible safe sets on 4 edges. There is a tradeoff with this call to `trycontract`; if it is successful, then no more iterations of the Do-While loop need to be performed, whereas if it is unsuccessful, then time is wasted generating all possible safe sets on 4 edges. The arguments like `maxiter`, `hopethresh`, etc are used to tune this tradeoff. For instance, `trycontthresh` represents the minimum number of iterations that the Do-While loop is to perform before making a call to `trycontthresh`. Such a parameter is useful because there is no reason to waste time trying to find $\mathcal{A}-$contracts when $|\mathcal{C}_i|$ is still large.

Another shortcut outside the body of the Do-While loop occurs after $MCS(\mathcal{A}, \mathcal{C})$ has been found. Let $\mathcal{C}' = MCS(\mathcal{A}, \mathcal{C}) - \mathcal{A}$. For the purposes of calculating $MCS(\mathcal{A}, \mathcal{C})$, we may assume $\mathcal{A} \bigcap \mathcal{C} = \emptyset$ and from this it follows that $\mathcal{C}' \subset \mathcal{C}$. By Lemma 6.8.1,

$MCS(\mathcal{A}, \mathcal{C}') = MCS(\mathcal{A}, \mathcal{C})$, and so $u \in \mathcal{A}$ is D-removable (respectively, C-removable) with respect to $\mathcal{C}$ if and only if u is D-removable, (respectively, C-removable) with respect to $\mathcal{C}'$. Thus, the colors of $\mathcal{C} - \mathcal{C}'$ can be eliminated. This is implemented by the following code in `control`:

```
if( (!live[i]) && (depth<= fixedlive[i]) && (fixedlive[i]<= 18) )
    fixedlive[i]=depth;
```

as well as

```
if(fixedlive[i]<depth ) live[i]=0;
```

which appears earlier in `control`. These two lines work together to insure that if some color in $\mathcal{C} - \mathcal{A}$ with code i, has been eliminated during a certain recursive call of `control`, then color i is also considered as eliminated in all recursive calls which are proper descendants of this call in the tree describing the recursion.

Finally, the function `smart_contract`, described in Section 6.7, attempts to find good contracts before any attempt to eliminate colorings from $\mathcal{U}$ and store them in `hope`. The goal of this is to eliminate searching through all possible contracts, many of which are not likely to be effective.

6.9 Checking D-reducibility or C(4)-reducibility of Configurations

We now discuss the implementation of Step 4 in Section 6.1.2. The main function responsible for this implementation is `four_color_reducibility` and its task is to determine that the current configuration is either D-reducible or C-reducible. By the time that the program comes to the following lines in `main`,

```
if (strictlyD && goodflag) {
      printf("WARNING: Although Every Coloring in Script U
   was D-Removable,\n");
      printf("No C-removable Colorings were Found.  \n");
      printf("Now Attempting to Prove Configuration is");
      printf("D-reducible or C(4)-reducible\n");
      four_color_reducibility(live, fixedlive, real, &strictlyD,
          &needscontract, graph, ring, nlive, ncodes, nchar,
          power, angle, diffangle, sameangle, contract,
          edgeid, edgeends, hope);
      if (strictlyD) {
        printf("WARNING: The Configuration is NOT U-Reducible \n");
     }
}
```

the variable `goodflag` will be set to 1 if and only if the program has already established that every color in $\mathcal{U}$ is either C-removableor D-removable, and the variable `strictlyD` will be set to 1 if and only if the program has failed to establish that some color in $\mathcal{U}$ is C-removable. Thus, the function `four_color_reducibility` is called only if every color in $\mathcal{U}$ is known to be D-removable and no color in $\mathcal{U}$ is known to be C-removable. When it is called it sets the variable `strictlyD` to 0 if it determines the configuration to be D-reducible or C(4)-reducible and otherwise it keeps `strictlyD` set to 1.

To determine whether the configuration is D-reducible, `four_color_reducibility` calculates $MCS(\mathcal{C}^* - \mathcal{C}_S)$, and checks if this set is empty. This calculation can be done with the subroutine *Critical* (see 6.6 for a description). If $MCS(\mathcal{C}^* - \mathcal{C}_S) \neq \emptyset$, then all sparse sets X on four edges are generated and the subroutine *TriConMod* (see 6.7) is called to see if any of these will be a contract. If one is found, `four_color_reducibility` concludes that the current configuration is U-reducible and sets `strictlyD` to 0; otherwise `four_color_reducibility` concludes that the configuration is not U-reducible. This completes the description of the function `four_color_reducibility`.

Chapter 7

Discharging

7.1 Introduction to Discharging

In this section we prove that for every internally 6−connected triangulation T there is a configuration K that is isomorphic to a member of $\mathcal{K}$ such that K appears in T. We follow the techniques of [8] in the proof.

A *cartwheel* is a configuration W such that there is a vertex w called a *hub* and two circuits N_1 and N_2 of $G(W)$ with the following properties:

(i) $\{w\}$, $V(N_1)$ and $V(N_2)$ are pairwise disjoint and have union $V(G(W))$,

(ii) N_1 and N_2 are both induced subgraphs of $G(W)$, and $U(N_2)$ bounds the infinite face of $G(W)$, and

(iii) w is adjacent to all vertices of N_1 and no vertices of N_2.

The following is due to Birkhoff [9].

Lemma 7.1.1 *Let T be an internally 6−connected triangulation and let $v \in V(T)$. There is a unique cartwheel appearing in T with hub v.*

If W is a cartwheel, then we say a configuration K *appears* in W if $G(K)$ is an induced subdrawing of $G(W)$, every finite face of K is a finite face of $G(W)$, and $\gamma_K(v) = \gamma_W(v)$ for every $v \in V(G(K))$. A *pass* P is a quadruple (K, r, s, t) where

(i) K is a configuration,

(ii) r is a positive integer,

(iii) s and t are distinct adjacent vertices of $G(K)$, and

(iv) for each $v \in G(K)$ there is an $s, v-$path and a $t, v-$path in $G(K)$ both having length at most 2.

We will occasionally write $r(P) = r$, $s(P) = s$, $t(P) = t$ and $K(P) = K$. The quantity r will be called the *value* of the pass P, s the *source* of P and t the *sink* of P.

A pass P *appears* in a triangulation T if the configuration $K(P)$ appears in T and P *appears* in a cartwheel W if $K(P)$ appears in W. Two passes P_1 and P_2 are isomorphic if the two configurations $K(P_1)$ and $K(P_2)$ are isomorphic, $r(P_1) = r(P_2)$ and if ϕ is a graph isomorphism between $G_1 = G(K(P_1))$ and $G_2 = G(K(P_2))$, that is guaranteed by $K(P_1)$ and $K(P_2)$ being isomorphic, then $\phi(s(P_1)) = s(P_2)$ and $\phi(t(P_1)) = t(P_2)$.

Lemma 7.1.2 *Let T be a triangulation, and let W_s be the cartwheel appearing in T having hub s. If P be a pass appearing in T with source or sink s, then P appears in W_s. Conversely, if a pass P with source or sink s appears in some cartwheel of T having hub s then P appears in T.*

Proof: Denote $K(P), r(P), s(P), t(P)$ and $G(K(P))$ by K, r, s, t and G respectively. Assume that P is a pass appearing in T with source or sink S. We first prove that K appears in W_s. We must show that G is an induced subdrawing of $G(W_s)$, every finite face of G is a finite face of $G(W_s)$ and that $\gamma_K(v) = \gamma_{W_s}(v)$ for every $v \in V(G(K))$.

We first show that G is an induced subdrawing of T. Clearly $s \in V(G) \bigcap V(W_s)$.

For every $v \in V(G)$ there is an $s-v$ path in G of length at most 2. Thus since s is the hub of the cartwheel W_s, $v \in V(G)$ implies $v \in V(W_s)$ and so $V(G) \subset V(W_s)$. Now let $e \in E(G)$ be an edge in G with endpoints u and v. We know by the hypothesis that $e \in E(T)$ and we just showed that $u, v \in V(W_s)$. Therefore, since W_s is an induced subdrawing of T, $e \in E(W_s)$ and so $E(G) \subset E(W_s)$. Now suppose that $u, v \in V(G)$ and that $\{u, v\} \in E(W_s)$. Because W_s is a subdrawing of T, $\{u, v\} \in E(T)$, and since G is an induced subdrawing of T, $\{u, v\} \in E(G)$ which completes the proof that G is an induced subdrawing of W_s.

We now show that every finite face of G is a finite face of W_s. If $G(W_s)$ equals the triangulation T then we are done since every finite face of G is a finite face of T. So assume that there is some face of T that is not a finite face of W_s. Let f be a finite face of G incident to the vertices $\{u, v, w\}$ and the edges $\{u, v\}$, $\{v, w\}$, $\{w, u\}$. From the work above, $\{u, v, w\} \subset V(H)$ and $\{\{u, v\}, \{v, w\}, \{w, u\}\} \subset E(H)$. Because G appears in T, f is a face of T and so $f \bigcap (V(T) \bigcup E(T)) = \emptyset$ which implies that f is a subset of some face of W_s. Since every finite face of W_s is a finite face of T, it follows that either f is a finite face of W_s or that $f \subset f_\infty$ where f_∞ denotes the infinite face of W_s. If f is a finite face of W_s we are done so assume $f \subset f_\infty$.

First consider the case that $f = f_\infty$. Since $T \neq W_s$, the infinite face of G which we denote f_G meets the f_∞ in every face of T that is not a finite face of W_s or G. This however is a contradiction, because it implies $f \bigcap f_\infty \neq \emptyset$.

So assume $f \neq f_\infty$ and let $y \in f$ and let x be in the infinite face $f_\infty - f$. Since $f \subset f_\infty$, there is a (topological) path P in $\Sigma^0 - (V(W_s) \bigcup E(W_s))$ with endpoints x and y. By the Jordan curve theorem, if P does not intersect the circuit defined by $\{u, v, w\}$ then $x \in F$. Therefore we may assume that P intersects the circuit

$\{u, v, w\}$. However $\{u, v, w\} \subset V(H)$ and $\{\{\{u, v\}, \{v, w\}, \{w, u\}\} \subset E(H)$ which contradicts that $P \subset \Sigma^0 - (V(W_s) \bigcup E(W_s))$. All of this establishes that every finite face of G is a finite face of W_s.

Finally, let $v \in V(G)$. Since $v \in V(W_s)$ and since W_s appears in T, $\gamma_K(v) = d_T(v) = \gamma_{W_s}(v)$ and thus the proof that P appears in W_s is complete.

The converse statement follows more easily. This completes the proof of Lemma 7.1.2.

A *rule* is a 6−tuple $(G, \beta, \delta, r, s, t)$ where

(i) G is a near-trinagulation,

(ii) β is a map from $V(G)$ to $\mathbf{Z}$; δ is a map from $V(G)$ to $\mathbf{Z} \bigcup \{\infty\}$ satisfying $\beta(v) \leq \delta(v)$ for every vertex v,

(iii) $r > 0$ is an integer, and

(iv) s and t are distinct, adjacent vertices of G, and for every $v \in V(G)$, there is a v, s−path and a v, t−path, each of length at most two, such that $\delta(w) \leq 8$ for the internal vertex w of the path, if there is one.

A pass P *obeys* a rule $(G, \beta, \delta, r, s, t)$ if P is isomorphic to some (K, r, s, t), where $G(K) = G$ and $\beta(v) \leq \gamma_K(v) \leq \delta(v)$ for every $v \in V(G)$.

A pictorial representation of the rules is given in Figure 7.1. As in the case of the Four Color Theorem, there are only three possibilities for the ordered pair $(\beta(v), \delta(v))$ for each v, listed below.

1) $5 \leq \beta(v) = \delta(v) \leq 8$ or

2) $\beta(v) = 5$ and $6 \leq \delta(v) \leq 8$ or

3) $5 \leq \beta(v) \leq 8$ and $\delta(v) = \infty$.

Case 1) can be described using the convention for vertices (Figure 4.1). Case 2) can be described by placing a plus sign close to the vertex v and case 3 can be described by placing a minus sign close v. The identification of s and t is accomplished by placing placing arrows on the edge joining s to t in such a way that it is directed from the source s to the sink t. The number of arrows placed on this edge represents the value of r, although only in the first rule, where $r = 2$, is more than one arrow required.

These rules are almost identical to the rules for the Four Color Theorem in Robertson et.al., the differences being slight adjustments in the rules $2 - 7$. This change reflects the fact that the fourth configuration (see Figure 7.2) in the unavoidable set used in [8] to prove the Four Color Theorem is not reducible for the Fiorini-Wilson-Fisk conjecture using U-reducibility. The fourth configuration is reducible for the Fiorini-Wilson conjecture using “Block-Count reducibility” [43], but we decided to avoid this technique in order to be able to devise a polynomial time algorithm.

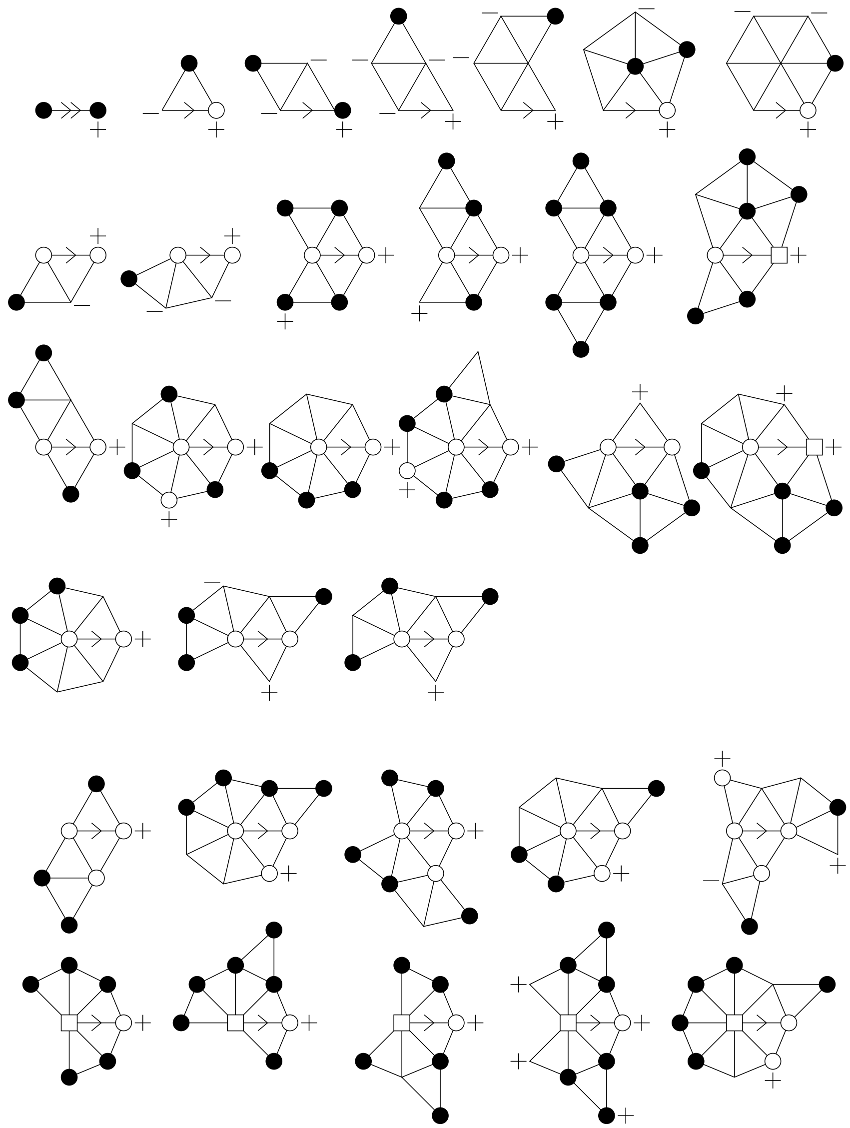

Figure 7.1: Rules For Distributing Charge

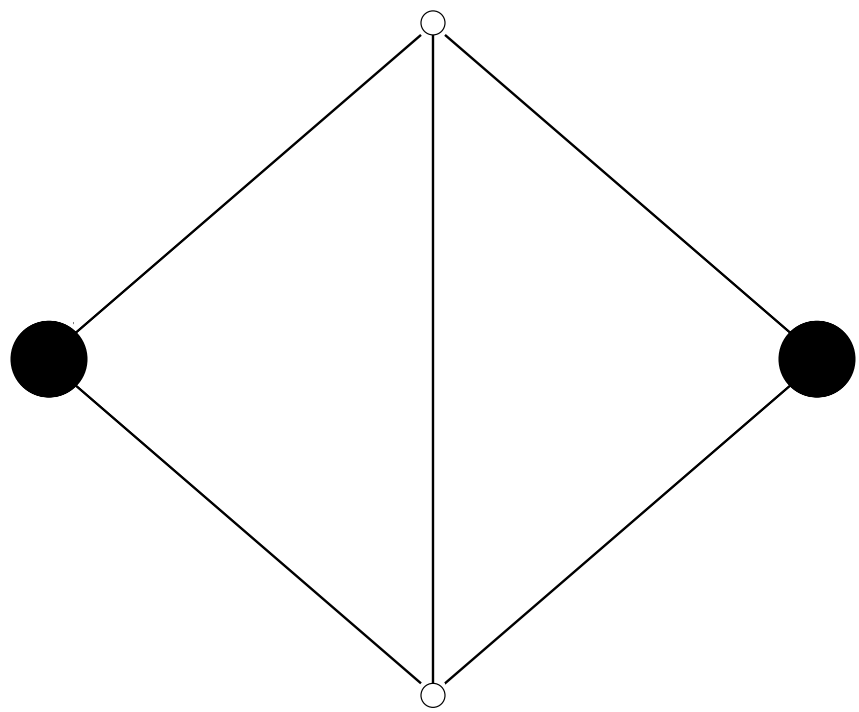

Figure 7.2: A Non U-reducible Configuration

Given a set of passes $\mathcal{P}$ we write $P \sim \mathcal{P}$ if P is isomorphic to some pass in $\mathcal{P}$. If W is a cartwheel with hub w we denote by $N_{\mathcal{P}}(W)$ the quantity $10(6 - \gamma_W(w)) + \sum_{P \sim \mathcal{P}_t(W)} r(P) - \sum_{P \sim \mathcal{P}_s(W)} r(P)$. Here, $P \sim \mathcal{P}_t(W)$ (respectively, $P \sim \mathcal{P}_s(W)$) denotes that we are summing over all passes P that are isomorphic to some pass of $\mathcal{P}$ that appears in W and has $t(P) = w$ (respectively, $s(P) = w$).

This quantity can be interpreted as follows. Every vertex v in a triangulation T is endowed with a charge equal to $10(6 - deg_T(v))$. A discharging procedure is then applied to the whole graph in such a way that all the charges are redistributed amongst the vertices in such a way that no charge is lost. Passes in the set $\mathcal{P}$ correspond to directions about how to redistribute these charges. If a vertex is a source for some pass $P \in \mathcal{P}$ then a charge equal to $r(P)$ flows from that vertex along the edge joined to the sink vertex $t(P)$. The next lemma uses Euler's formula and captures the idea that charge is conserved by the discharging procedure.

Lemma 7.1.3 *Let T be an internally $6-$connected planar triangulation and let $\mathcal{P}$ be any set of passes. Then $\sum N_{\mathcal{P}}(W) = 120$, where the summation is over all cartwheel W that appear in T. Moreover, there is a cartwheel W appearing in T such that $N_{\mathcal{P}}(W) > 0$.*

Proof: For convenience, we drop the subscript $\mathcal{P}$ from $N_{\mathcal{P}}$ throughout the proof. By Lemma 7.1.2, if P has source v and appears in T then P appears in W_v and conversely. Therefore $\sum (r(P) : P$ appears in T and $s(P) = v) = \sum_{v \in V(T)} \sum_{P \sim \mathcal{P}_s(W_v)} r(P)$ and for the same reason $\sum (r(P) : P$ appears in T and $s(P) = v) = \sum_{v \in V(T)} \sum_{P \sim \mathcal{P}_t(W_v)} r(P)$. Therefore, $\sum_{v \in V(T)} \mathcal{N}_P(W_v) = \sum_{v \in V(T)} 10(6 - \gamma_W(v)) = 10(\sum_{v \in V(T)} 6 - d_T(v))$, since W_v appears in T and v is not on the infinite face of W_v, implies that $\gamma_W(v) = d_T(v)$. By Euler's formula, we thus conclude that $\sum_{v \in V(T)} \mathcal{N}_P(W_v) = 120$ and therefore there is some cartwheel W appearing in T with $\mathcal{N}_P(W) > 0$. This completes the proof of Lemma 7.1.3.

The main theorem therefore of this section is

Theorem 7.1.1 *There is a set of passes $\mathcal{P}$ such that if W is a cartwheel appearing in a triangulation T such that $N_{\mathcal{P}}(W) > 0$, then some configuration isomorphic to one in Appendix A appears in W and thus in T.*

This theorem, together with Lemma 7.1.3 and Lemma 7.1.1 imply Theorem 1.2.7 of Chapter 1 and constitue the unavoidability part of the proof.

7.2 Unavoidability when the Hub Degree is Small

We will dispose of the cases when the hub has degree 5, 6 or ≥ 12 in the next few lemmas and theorems which have the same statements as some of the lemmas and theorems in [8] and require slight modifications in the proofs because of the change in rules.

Lemma 7.2.1 *Let W be a cartwheel, with hub w having degree 5 or 6. For $k = 1, \ldots, 32$ let p_k (q_k) be the sum $r(\mathcal{P}_k)$ where $\mathcal{P}_k$ is the set of all passes obeying rule k and appearing in W, with sink (respectively source) w. If no configuration isomorphic to one in Appendix A appears in W then*

(i) $p_1 = q_2 + q_3$

(ii) $p_3 = q_4$

(iii) $p_4 = q_5 + q_6$

(iv) $p_5 = q_7$

Proof: Throughout the proof we abbreviate γ_W by γ. Also, given a near-triangulation G without parallel edges in which each edge is in exactly two faces, and given a triangular face of G that is incident to the distinct vertices x, y, z, the expression $t_G(\{x, y\}, z)$ will denote the unique vertex $u \notin \{x, y, z\}$ that is in the other face of G incident to the edge $\{x, y\}$. If the edge $\{x, y\}$ is in exactly one triangle of G, then $t_G(\{x, y\})$ will denote the unique vertex not equal to x or y that is incident to the face of G containing the edge $\{x, y\}$.

We start by proving (i). Let X be the set off all triples (x, y, z) such that w is adjacent to x,y, and z, y is adjacent to both x and z and $\gamma(x) = 5$. Now X can be written as a disjoint union $X = \{(x, y, z) \in X : \gamma(y) \geq 7\} \bigcup \{(x, y, z) \in X : \gamma(y) \leq 6$

and $\gamma(z) \geq 5\}$. Claim: $|X| = p_1$, $q_2 = |\{(x', y', z') \in X : \gamma(y') \geq 7\}|$ and $q_3 = |\{(x', y', z') \in X : \gamma(y') \leq 6$ and $\gamma(z') \geq 5\}|$. Proof of claim: Let x be a neighbor of w with $\gamma(x) = 5$ and suppose that a portion of the the clockwise listing in $G(W)$ of neighbors of w is x_{-2},x_{-1}, x, x_1, x_2. Then $p_1 = 2|\{x' : x'$ is adjacent to w and $\gamma(x') = 5\}| = |X|$ because x gives rise to exactly one element in $\{x' : x'$ is adjacent to w and $\gamma(x') = 5\}$ (itself) and precisely two triples (x, x_{-1}, x_{-2}) and (x, x_1, x_2) in X. Now suppose a pass P obeys rule 2 where rule 2 is the $6-$tuple $(G_2, \beta_2, \delta_2, 1, s, t)$ where G_2 is a graph consisting of a triangle with vertices s, t and $t_{G_2}(\{s,t\})$ where $\gamma_{G_2}(s) = 6$, $\gamma_{G_2}(t) = 7$ and $\gamma_{G_2}(t_{G_2}(\{s,t\})) = 5$. Now for rule 2, $\beta_2(s) = 5 = \delta_2(s)$, $\beta(t) = 7, \delta(t) = \infty$ and $\beta(t_{G_2}(\{s,t\})) = 5 = \delta(y)$. Such a P can be mapped to the triple (x, y, z), where w will correspond to s, y to t, x to $t_{G_2}(\{s,t\})$ and where z is defined to equal $t_W(\{w,y\}, x)$. This mapping is well defined, because W being a cartwheel with hub w means that the edge $\{w, y\}$ is in exactly two faces. Also, because the pass obeys rule 2, $\gamma(x) = 5$, and $\gamma(y) \geq 7$ so $(x, y, z) \in \{(x', y', z') \in X : \gamma(y') \geq 7\}$. Conversely, if $(x, y, z) \in \{(x', y', z') \in X : \gamma(y') \geq 7\}$ then the triangle formed by the vertices $\{x, y, w\}$ gives rise to a unique pass P that obeys rule 2. Thus $q_2 = |\{(x', y', z') \in X : \gamma(y') \geq 7\}|$.

Similarly, if a pass P obeys rule 3, where G_3 is the graph underlying the picture of the configuration corresponding to rule 3, where s is the source, t is the sink, then we can make a correspondance between P and triple (x, y, z) as follows: w will correspond to s, z to t, y to $t_{G_3}(\{x, y\})$, and x to $t_{G_3}(\{y, w\}, z)$. It follows that $(x, y, z) \in \{(x', y', z') \in X : \gamma(y') \leq 6$ and $\gamma(z') \geq 5\}$. Conversely, if $(x, y, z) \in \{(x', y', z') \in X : \gamma(y') \leq 6$ and $\gamma(z') \geq 5\}$, then the subdrawing of $G(W)$ induced by x,y,z and w gives rise to a unique pass P obeying rule 3, and so $q_3 = |\{(x', y', z') \in$

$X : \gamma(y') \leq 6$ and $\gamma(z') \geq 5\}|$. This completes the proof of the claim. From here, it is easy to see (i) because $X = \{(x', y', z') \in X : \gamma(y') \geq 7\} \bigcup \{(x', y', z') \in X : \gamma(y') \leq 6$ and $\gamma(z') \geq 5\}$.

We now prove (ii) for the case when $\gamma(w) = 5$. Let X now be the set of triples (x, y, z) having the properties that w is adjacent to x, y, z, y is adjacent to x and z, $\gamma(x) \leq 6$, $\gamma(y) \leq 6$, the three vertices x, y, and z are distinct, and the vertex $u = t_W(\{x, y\}, w)$ satisfies $\gamma(u) = 5$.

We will describe a one to one and onto mapping between the set of passes obeying rule 3, appearing in W and having sink w to the set X of ordered triples which will establish that $p_3 = |X|$. . Let P be a pass satisfying rule 3 which appears in W and has sink w. Because P obeys rule 3, $G(K(P))$ is isomorphic to the near triangulation G_3. Define $y' = t$, $x' = t_{G_3}(\{s, t\})$ and $z' = t_W(\{y', w\}, x')$. Strictly speaking, it would have been more proper to define x' by $t_{G(K(P))}(\{s, t\})$, since x' is meant to be a vertex in W whereas $t_{G_3}(\{s, t\})$ is a vertex in the near triangulation G_3. However, because G_3 is isomorphic to $G(K(P)$, and because we want to keep in mind the connection to G_3, we risk this abuse. The definition of G_3 insures that x' is well defined and the fact that W is a cartwheel implies that z' is well defined. We now claim that $(x', y', z') \in X$. First, note that $\gamma_(x') \leq 6$ and $\gamma_(y') \leq 6$ as a because P obey's rule 3. Moreover, P's obedience to rule 3 also implies that the vertex $u = t_W(\{x', y'\}, w)$ satisfies $\gamma(u) = 5$ because, in fact, $u = t_{G_3}(\{x', y'\}, t)$. This implies that $(x', y', z') \in X$ and so establishes a well defined mapping from the set of passes which appear in W, have sink w and obey rule 3.

We now claim that this mapping is a one-to-one correspondence. First, note that this mapping is onto because if (x', y', z') is any triple in X, then a pass P can be constructed which maps to (x', y', z') and which appears in W, has sink w and obeys rule 3. Also, it is one to one because if two distinct passes P, and P' appear in W, have sink w and obey rule 3, they either have different sources, which give rise to different values of y' in the mapping, or if they have the same source, give rise to a different value of z' in the mapping. This proves $|X| = p_3$.

We now prove that $q_4 = p_3$ and thus establish (ii) for the case when $\gamma(z) = 5$. We do so by proving $q_4 = |X|$, again by constructing a one to one and onto mapping between the set of all passes which appear in W, have source w and obey rule 4 and the set X. Let G_4 be the near triangulation associated with rule 4 and let P be a pass which appears in W, has source w, and which obeys rule 4. Define $z' = t$, $y' = t_{G(K(P))}(\{s, t\})$, and $x' = t_{G(K(P))}(\{s, y'\}, t)$, keeping in mind that $G(K(P))$ is isomorphic to G_4. By this definition, and from the fact that $w = s$, z',y' and x' are mutually adjacent to w, and y' is adjacent to x' and z'. Let u be the vertex $t_{G(K(P))}(\{x', y'\}, w)$. Since P obeys rule 4, $\gamma(z') \geq 6$, $\gamma(y') \leq 6$, $\gamma(x') \leq 6$ and $\gamma(u) = 5$. It thus follows that $(x', y', z') \in X$ and this construction gives a well defined mapping from the set of passes appearing in W with source w and obeying rule 4 to the set X. This mapping also is one to one. However, it is not clear yet that it is onto, because a triple $(x', y', z') \in X$ might have $\gamma(z') = 5$ and would have no pass mapped to it. We now show that this is impossible by showing that when $\gamma(w) = 5$, then $\{(x, y, z) \in$

in much the same fashion that we proved p_3

Two distinct passes P and P' which both obey rule

so with a From the associated drawing G_3 of rule 3, identify are assuming that P is a pass which appears in W with

If $\gamma(w) = 5$, then $p_4 = p_5 = p_6 = p_7 = q_5 = q_6 = q_7 = 0$. Therefore (iii) and (iv) hold, so we may assume $\gamma(w) = 6$ and prove (ii), (iii) and (iv).

We first prove (ii). Let X be the set of triples (x, y, z) having the properties that w is adjacent to x, y, z, y is adjacent to x and z, $\gamma(x) \leq 6$, $\gamma(y) \leq 6$, the three vertices x, y, and z are distinct, and the vertex $u = t_W(\{x, y\}, w)$ satisfies $\gamma(u) = 5$. As in the case when $\gamma(w) = 5$, we first show that $|X| = p_3$ and then show that $|X| = q_4$.

Let P be a pass which appears in W, has sink w and which obeys rule 3, and as usual, let G_3 be the near triangulation associated with rule 3 which is isomorphic to $G(K(P))$. Define $y = s$, $x = t_{G(K(P))}(\{s, t\})$ and $z = t_W(\{y, w\}, x)$. We now show $\gamma(z) \geq 6$. Suppose instead that $\gamma(z) = 5$. Remembering that $\gamma(w) = 6$, if $\gamma(x) = 6$ and $\gamma(y) = 6$, then conf(1,1,7) would appear in W, if $\gamma(x) = 6, \gamma(y) = 5$ then conf(1,1,4) would appear in W, if $\gamma(x) = 5, \gamma(y) = 6$, then conf(1,1,5) would appear in W, and if $\gamma(x) = 5 = \gamma(y)$, then conf(1,1,2) would appear in W. Therefore, we may assume $\gamma(z) \geq 6$ and so $X = \{(x, y, z) \in X : \gamma(z) \geq 5\} = \{(x, y, z) \in X : \gamma(z) \geq 6\}$. We will now show that $p_3 = |X|$ and that $q_4 = |\{(x, y, z) \in X : \gamma(z) \geq 6\}$ which will establish (ii).

We first describe a mapping from the set of passes obeying rule 3 and appearing in W to the set X of ordered triples. Let P be a pass satisfying rule 3 which has sink w. From the associated drawing G_3 of rule 3, identify x with s, w with t, y with $t_{G_3}(\{x, w\})$, u with $t_{G_3}(\{x, y\}, w)$ and define z to equal $t_W(\{x, w\}, y)$. Since the pass obeys rule 3, $\gamma(u) = 5$, $\gamma(x), \gamma(y) \leq 6$ and thus $(x, y, z) \in X$. Conversely, a triple $(x, y, z) \in X$ gives rise to a unique pass P which obeys rule 3 and has sink w. Thus, $p_3 = |X|$.

Now we show that there is a one to one correspondence from the set of passes obeying rule 4, having source w and appearing in W to the set $\{(x', y', z') \in X : \gamma(z) \geq 6\}$. Let P be such a pass, and let G_4 be the near triangulation of rule 4 which is isomorphic to $G(K(P))$. We now construct a unique triple $(x, y, z) \in \{(x', y', z') \in X : \gamma(z) \geq 6\}$ from the pass P. Define $z = t$, $y = t_{G_4}(\{w, z\})$, $x = t_{G_4}(\{w, y\}, z)$ and u with $t_{G_4}(\{x, y\}, w)$. In this way, we establish that $q_4 = |\{(x', y', z') \in X : \gamma(z) \geq 6\}$. This completes the proof of (ii) for $\gamma(w) = 6$.

Now we establish (iii). This time let X be the set of triples (x, y, z) such that the three vertices x, y and z are distinct and each adjacent to w, y is adjacent to both x and z, $u = t_W(\{x, y\}, w)$, $v = t_W(\{u, y\}, x)$, $\gamma(x), \gamma(y), \gamma(u) \leq 6$, $\gamma(v) = 5$.

We first show that if $(x, y, z) \in X$ and $\gamma(y) = 5$ then $\gamma(z) \geq 7$. Suppose to the contrary that $\gamma(y) = 5$ and $\gamma(z) \leq 6$, first considering the case that $\gamma(z) = 5$. Since $\gamma(y) = 5$, the vertex z is adjacent to the vertex v in W. If $\gamma(u) = 5$ then conf(1,1,1) appears on the vertices u, v, y, z so we assume $\gamma(u) = 6$. If $\gamma(x) = 6$ then conf(1,1,8) appears on the vertices u, v, w, x, y, z, so we may assume $\gamma(x) = 5$. Then conf(1,1,2) appears on the vertices w, x, y, z. Thus we may may assume $\gamma(z) = 6$. If $\gamma(x) = 5$ then either conf(1,1,1) or conf(1,1,2) appear in W, so we may assume $\gamma(x) = 6$. If $\gamma(u) = 5$ then conf(1,1,8) appears in W on u, v, w, x, y, z and if $\gamma(u) = 6$ then conf(1,3,1) appears in W on u, v, w, x, y, z. Thus we must have $\gamma(z) \geq 7$ which is what we desired to show.

We now show that if $(x, y, z) \in X$ and $\gamma(y) = 6$ then $\gamma(z) \geq 6$. Suppose to the contrary that $\gamma(z) = 5$. If $\gamma(x) = \gamma(u) = 5$ then conf(1,1,2) appears on $\{u, v, x, y\}$ and if $\gamma(x) = 5, \gamma(u) = 6$ then conf(1,2,2) appears on u, v, w, x, y, z. If $\gamma(x) = 6, \gamma(u) = 5$ then conf(1,1,7) appears on u, w, x, y, z and of $\gamma(x) = \gamma(u) = 6$ then conf(1,3,3) appears on u, v, w, x, y, z. This proves that $\gamma(z) \geq 6$.

We may write the set X as a disjoint union $X = \{(x', y', z') \in X \ \gamma(y) = 5\} \bigcup \{(x', y', z') \in X : \gamma(y) = 6\}$. From the results just proved, $\{(x', y', z') \in X : \gamma(y) = 5\} = \{(x', y', z') \in X : \gamma(y) = 5, \gamma(z) \geq 7\}$ and $\{(x', y', z') \in X : \gamma(y) = 6\} = \{(x', y', z') \in X : \gamma(y) = 6, \gamma(z) \geq 6\}$. The assertion (iii) now follows by verifying that $p_4 = |X|$, $q_5 = |\{(x', y', z') \in X : \gamma(y) = 6, \gamma(z) \geq 6\}|$ and $q_6 = |\{(x', y', z') \in X : \gamma(y) = 5, \gamma(z) \geq 7\}|$. The proofs of these facts are similar to the above; for instance when trying to establish $p_4 = |X|$ we will identify w with the sink t, x with the source s, y with $t_{G_4}(\{s, t\})$ etc. and when trying to establish the assertions about q_5 and q_6 we will identify w with the source, z with the sink, with etc. This completes the proof of (iii) for $\gamma(w) = 6$.

We now prove (iv). Let X be the set of triples (x, y, z) such that the three vertices x, y and z are distinct and each adjacent to w, there is a vertex $u \neq w$ adjacent to x and y, there is a vertex $v \neq x$ adjacent to u and y, and finally there is a vertex t adjacent to v, y and z, $\gamma(x) = \gamma(y) = 6$, $\gamma(u), \gamma(v) \leq 6$, $\gamma(t) = 5$.

We first establish that if $(x, y, z) \in X$ then $\gamma(z) \geq 7$. First assume that $\gamma(z) = 5$. Then $\gamma(v) = 6$ or else conf(1,1,2) appears on t, v, z, y. If $\gamma(u) = 6$ then conf(1,4,5) appears on t, u, v, w, x, y, z and so we assume $\gamma(u) = 5$. Therefore conf(1,1,7) appears on u, w, x, y, z which is a contradiction. Therefore we may assume $\gamma(z) = 6$. Now if $\gamma(u) = 5, \gamma(v) = 5$ then conf(1,1,2) appears on t, u, v, y and if $\gamma(u) = 5, \gamma(v) = 6$, then conf(1,3,3) appears on t, u, w, x, y, z. If $\gamma(u) = 6, \gamma(v) = 5$ then conf(1,4,5) appears on t, u, v, w, x, y, z and if $\gamma(u) = \gamma(v) = 6$ then conf(1,5,5) appears on t, u, v, w, x, y, z. Thus we must have $\gamma(z) \geq 7$ if $(x, y, z) \in X$. By this result $X = \{(x', y', z') \in X : \gamma(z) \geq 7\}$. Now it can be shown that $p_5 = |X|$ by identifying w with the sink t,

x with the source s and y with $t_{G_5}(\{s,t\})$ etc. Similarly, by identifying w with the source s, z with the sink t, y with $t_{G_7}(\{s,t\})$ etc. and x with $t_{G_7}(\{w,y\},z)$, it follows that $q_7 = |\{(x',y',z') \in X : \gamma(z) \geq 7\}|$. This completes the proof of (iv) for $\gamma(w) = 6$ and completes the proof of Lemma 7.2.1.

Theorem 7.2.1 *Let W be a cartwheel with hub w which appears in T and which satisfies $\mathcal{N}_{\mathcal{P}} > 0$. If $d_T(w) \leq 6$, then some configuration isomorphic to one in Appendix A appears in W.*

Proof: Suppose to the contrary that no configuration isomorphic to one in Appendix A appears in W. First consider the case that $\gamma(w) = d_T(w) = 5$. By consulting the rules it can be seen that $\mathcal{N}_{\mathcal{P}} = 10 + p_1 + p_3 - q_1 - q_2 - q_3 - q_4$ where we have adopted the notation found in the statement of Lemma 7.2.1. Now the hypothesis of this lemma applies since we are assuming no configuration isomorphic to one in Appendix A appears in W. Thus $p_1 = q_2 + q_3$ and $p_3 = q_4$. Also, $q_1 = 10$. Hence $\mathcal{N}_{\mathcal{P}} = 0$ which contradicts the assumption that $\mathcal{N}_{\mathcal{P}} > 0$. Therefore we may assume $\gamma(w) = 6$.

It can now be seen that $\mathcal{N}_{\mathcal{P}} = p_1 + p_3 + p_4 + p_5 - q_2 - q_3 - q_4 - q_5 - q_6 - q_7$. From (i), (ii), (iii) and (iv) of Lemma 7.2.1, we thus see $\mathcal{N}_{\mathcal{P}} = 0$, which again contradicts that $\mathcal{N}_{\mathcal{P}} = 0$. This completes the proof of Theorem 7.2.1.

7.3 Unavoidability when the Hub Degree is Large

The next lemma and its proof are also taken from [8], with some small changes.

We will sometimes say that a subset $A \subset V(W)$ of vertices *obeys a rule with source s and sink t*, or just that *A obeys a rule* if the source and sink are understood, to mean that the subdrawing induced by them together with the function γ restricted to them forms a pass P with source s and sink t that obeys that rule.

Lemma 7.3.1 *Let W be a cartwheel with hub w in which no configuration isomorphic to one in Appendix A appears and let v be adjacent to w. The sum of $r(P)$ over all passes P appearing W that have source v and sink w is at most* 5.

Proof: Let R_k ($k = 1, 2, \ldots, 32$) denote the sum of $r(P)$ for all passes P which appear in W, have source and sink v and w respectively, and which obey rule k. The proof that $R = R_1 + R_2 + \ldots + R_{32} \leq 5$ will be split up into cases according to whether $\gamma(v) = 5$, 6, 7, 8 or ≥ 9. Before analyzing these cases, notice that $R_i \leq 2$ for $1 \leq i \leq 32$.

If $\gamma(v) = 5$ then $R = R_1 + R_2 + R_3 + R_4$ because the only rules in which the source has degree 5 are rules $1, 2, 3, 4$. Let the clockwise listing in W of neighbors of v be $w = v_1$,v_2, v_3, v_4, v_5, let $s \neq v$ be a vertex in the triangular face containing the edge v_2 and v_3, and let $t \neq v$ be the vertex in a triangular face which includes the edge v_5 and v_4. If $\gamma(v_5) \geq 7$ then $R_2, R_3, R_4 \leq 1$, because there would be no way that $\{w, v_5, v\}$ could obey rule 2, or $\{w, v, v_4, v_5\}$ could obey rule 3 or that $\{w, v, v_4, t\}$ could obey rule 4. Hence we may assume $\gamma(v_5) \leq 6$ and by symmetry that $\gamma(v_2) \leq 6$. If $R_2 = 0$ then $\gamma(v_5) = \gamma(v_2) = 6$ and we may assume $R_3 = R_4 = 2$ and so $\gamma(v_4) = \gamma(v_3) = \gamma(s) = \gamma(t) = 5$. This implies that conf(1,1,2) appears on v, v_5, v_4, t. Therefore we may assume $R_2 \geq 1$.

If $R_2 = 1$ then we may assume by symmetry that $\gamma(v_2) = 5$ and $\gamma(v_5) = 6$ and that one of R_3 or R_4 equals two, and the other is at least one. If $R_3 = 2$ then $\gamma(v_3) = \gamma(v_4) = 5$ and so conf(1,1,1) appears on v_2, v, v_3, v_4. So assume $R_3 = 1$ and

$R_4 = 2$ which means that $\gamma(s) = \gamma(t) = 5$. If $\gamma(v_3) \in \{5, 6\}$ then either conf(1,1,1) or conf(1,1,1) appears on v, v_2, s, v_3. Therefore we may assume that $\gamma(v_3) = 7$ but this implies that $R_4 \leq 1$ which is a contradiction.

So assume that $R_2 = 2$ and therefore that $\gamma(v_2) = \gamma(v_5) = 5$. We may assume that $R_3 + R_4 \geq 2$ or else $R \leq 5$. Therefore, at least two of the values in $\{\gamma(s), \gamma(t), \gamma(v_3), \gamma(v_4)\}$ must equal 5. First consider the case that $\gamma(t) = 5$. If $\gamma(v_4) = 5$ or $\gamma(v_4) = 6$, then conf(1,1,1) or conf(1,1,2) appears on $\{v, v_4, v_5, t\}$. If $\gamma(v_4) \geq 7$ then $R_3 + R_4 \geq 2$ force $\gamma(s) = \gamma(v_3) = 5$ which implies conf(1,1,1) appears on $\{v, v_2, v_3, s\}$. Therefore, we may assume $\gamma(t) \geq 6$ and by symmetry that $\gamma(s) \geq 6$. If $\gamma(t) \geq 7$, then necessarily $\gamma(v_3) \leq 6$ and it can then be seen that either conf(1,1,1) or conf(1,1,2) will appear in W. Hence, we may assume $\gamma(t) = 6$ and by symmetry we may assume $\gamma(s) = 6$. Since $R_3 + R_4 \geq 2$, we must have $\gamma(v_3) = \gamma(v_4) = 5$ and thus conf(1,1,1) appears on $\{v_5, v, v_3, v_4\}$, a contradiction. This completes the case when $R = 2$, and finishes the $\gamma(v) = 5$ analysis.

Now assume that $\gamma(v) = 6$. Then $R = R_2 + R_3 + R_4 + R_5 + R_6 + R_7$ as every other rule has source whose degree does not equal 6. Let $w = v_1, v_2, v_3, \ldots, v_6$ be the counterclockwise listing in W of the neighbors of v. Let $w = v_{2,1}, v = v_{2,2}, v_3 = v_{2,3}, \ldots, v_{2,\gamma(v_2)}$ be the clockwise listing in W of neighbors of v_2, and let $w = v_{6,1}, v = v_{6,2}, v_5 = v_{6,3}, \ldots, v_{6,\gamma(v_6)}$ be the counterclockwise listing in W of the neighbors of v_6.

First, it will be established that we may assume $\gamma(v_2) \leq 6$. If not, then $R_2, R_3, R_4, R_5, R_6, R_7 \leq 1$. Now if $\gamma(v_6) = 5$ then $R_5 = 0$, if $\gamma(v_6) = 6$ then $R_6 = 0$, and if $\gamma(v_6) \geq 7$ then $R = 0$. Either way, $R \leq 5$, and so we may assume that $\gamma(v_2) \leq 6$. This implies that we may also assume $\gamma(v_6) \leq 6$ by a symmetric argument.

By symmetry, it suffices to split the rest of the $\gamma(v) = 6$ proof into three cases, 1)

$\gamma(v_2) = \gamma(v_6) = 5$, 2) $\gamma(v_2) = 5, \gamma(v_6) = 6$, and 3) $\gamma(v_2) = \gamma(v_6) = 6$.

Proof of case 1) ($\gamma(v_2) = \gamma(v_6) = 5$). Under these assumptions, $R_2 = 2$, $R_5 = R_7 = 0$ so we must assume at least one of R_3, R_4 and R_6 equals 2. First assume $R_3 = 2$, so $\gamma(v_5) = \gamma(v_3) = 5$. Now if $\gamma(v_{2,4}) \geq 7$ then $R_4, R_6 \leq 1$ and since equality must hold in both cases, $\gamma(v_{6,4}) = \gamma(v_{6,5}) = 5$ which means that conf(1,1,1) appears on $v_6, v_5, v_{6,4}, v_{6,5}$ which is a contradiction. Thus $\gamma(v_{2,4}) \leq 6$ and by symmetry, $\gamma(v_{6,4}) \leq 6$. This, in turn, implies that $R_6 \leq 0$, for otherwise one of $\gamma(v_{2,5})$, or $\gamma(v_{6,5})$ equals 5 which would force either conf(1,1,1) or conf(1,1,2) to appear on $v_6, v_5, v_{6,4}, v_{6,5}$ or on $v_2, v_3, v_{2,4}, v_{2,5}$. Thus we may assume $R_3 = R_4 = 2$, so $\gamma(v_{2,4}) = \gamma(v_{6,4}) = 5$. Then conf(1,1,3) appears on $v_2, v_3, v_{2,4}, v, v_5, v_6, v_{6,4}$ which is a contradiction. This completes the case when $R_3 = 2$.

It remains to consider the case that $R_3 = 1$ because if $R_3 = 0$ we are done. We must have either $R_4 = 2$ or $R_6 = 2$. If $R_4 = 2$ then $\gamma(v_{2,4}) = \gamma(v_{6,4}) = 5$. Since $\gamma(v_3), \gamma(v_5) \leq 6$, it follows that either conf(1,1,3), conf(1,1,6), or conf(1,2,4) appears on $v_2, v_3, v_{2,4}, v, v_5, v_6, v_{6,4}$. Therefore we may assume $R_6 = 2$, and $\gamma(v_{2,5}) = 5 = \gamma(v_{6,5})$. Then conf(1,1,1) appears on either $v_{2,5}, v_{2,4}, v_3, v_2$ or $v_{6,5}, v_{6,4}, v_5, v_6$, which is a contradiction. This completes the proof of case 1).

Proof of case 2) ($\gamma(v_2) = 5, \gamma(v_6) = 6$) The assumptions insure that $R_2 = 1$, R_5, R_6, $R_7 \leq 1$. Also, it must be that $\gamma(v_{6,4}) \leq 6$; otherwise $R_5 = R_7 = 0$, and $R_4 \leq 1$ which implies $R \leq 5$. It will be shown that we may assume either a) $R_4 \leq 1$ or b) $R_4 = 2$ and $R_3 = 0$. Assume neither a) or b) holds so $R_4 = 2$ and $R_3 \geq 1$. Therefore $\gamma(v_{2,4}) = \gamma(v_{6,4}) = 5$ and one of $\gamma(v_3), \gamma(v_5)$ equals 5. Now $R_4 = 2$ implies that $\gamma(v_3) \leq 6$ and $\gamma(v_5) \leq 6$. We may assume that $\gamma(v_3) = \gamma(v_{2,4}) = \gamma(v_{6,4}) = 5$ and $\gamma(v_5) = 6$, or else conf(1,1,6) or conf(1,2,5) appears on $v, v_2, v_3, v_{2,4}, v_5, v_6, v_{6,4}$ which is contrary to the hypothesis that no configuration isomorphic to one in Appendix A appears in W. Thus $R_3 = 1$. If $\gamma(v_{2,5}) = 5$, then conf(1,1,1) appears on v_2, v_3 $v_{2,4}$, $v_{2,5}$, so it must be that $\gamma(v_{2,5}) \geq 6$ and $R_6 = 0$. Since $R_3 + R_4 = 3$ and $R_6 = 0$, we must conclude that $R_5 = R_7 = 1$ which forces conf(1,1,2) to appear on v_6, $v_{6,4}$, $v_{6,5}$, $v_{6,6}$. This proves that either a) or b) is true.

If b) is true, then we may assume $R_5 = R_6 = R_7 = 1$ and therefore that $\gamma(v_{6,6}) = \gamma(v_{6,5}) = \gamma(v_{6,4}) = 5$, so conf(1,1,2) appears on $v_6, v_{6,4}, v_{6,5}, v_{6,6}$ which is impossible.

Therefore assume a) is true. Now $R_4 = 1$ because if $R_4 = 0$ it must be that $R_3 = 2$ and $R_5 = R_6 = R_7 = 1$, and then conf(1,1,1) would appear in W. Also we may assume $R_3 = 2$ and $\gamma(v_3) = \gamma(v_5) = 5$, otherwise $R \leq 5$ or $R_3 = R_4 = R_5 = R_6 = 1$, the latter implying that either conf(1,1,1), or conf(1,1,2) or conf(1,1,5) would appear in W. Now $R_6 = 0$, or else conf(1,1,1) or conf(1,1,2) appear on $\{v_2, v_3, v_{2,4}, v_{2,5}\}$. Thus, we must have $R_5 = R_7 = 1$ which with the condition that $\gamma(v_{6,4}) \leq 6$ implies that conf(1,1,5) or conf(1,1,2) appears in W.

Proof of case 3) $(\gamma(v_2) = 6, \gamma(v_6) = 6)$ The assumptions imply that $R_2 = R_6 = 0$ so $R = R_3 + R_4 + R_5 + R_7$. First, $R_3 + R_4 \leq 3$ because $R_3 = R_4 = 2$ implies that conf (1,2,4) appears on $v, v_2, v_3, v_5, v_6, v_{2,4}, v_{6,4}$. If $\gamma(v_{2,4}) \geq 7$, then $R_4, R_5, R_7 \leq 1$ and so $R \leq 5$. Therefore we may assume $\gamma(v_{2,4}) \leq 6$ and by symmetry, $\gamma(v_{6,4}) \leq 6$. Also, if $\gamma(v_{2,5}) \geq 7$, then $R_5, R_7 \leq 1$ which implies $R \leq 5$. Therefore, $\gamma(v_{2,5}) \leq 6$ and by symmetry, $\gamma(v_{6,5}) \leq 6$. We may assume $R_5 + R_7 \geq 3$ because $R_3 + R_4 \leq 3$. Therefore, by symmetry, we may assume $\gamma(v_{6,5}) = \gamma(v_{6,6}) = 5$. If either $\gamma(v_{6,4})$ or $\gamma(v_5)$ equals 5, then either conf(1,1,5) appears on $v_6, v_5, v_{6,4}, v_{6,5}, v_{6,6}$ or conf(1,1,2) appears in W. Thus $\gamma(v_5) = \gamma(v_{6,4}) = 6$ and this implies that $R_3 + R_4 \leq 2$. Hence, we must assume that $R_5 + R_7 = 4$, and therefore that $\gamma(v_{6,6}) = \gamma(v_{6,5}) = 5$. By the reasoning just used, this forces $\gamma(v_3)$ and $\gamma(v_{2,4})$ to equal 6 and $R_3+R_4=0$ so $R=4$. This completes the proof of case 3 and the case that $\gamma(v)=6$.

Now we assume that $\gamma(v) = 7$ Let $w = v_1, v_2, v_3, v_4, v_5, v_6, v_7$ be the counter-clockwise listing in W of the neighbors of v. Let $s \neq v$ be the unique vertex in the triangular face containing the edge $\{v_2, v_3\}$ and let t be the unique vertex in the triangular face containing the edge $\{v_6, v_7\}$. We break the proof down into cases with various values for $\gamma(v_2)$ and $\gamma(v_7)$.

If $\gamma(v_2) = \gamma(v_7) = 5$, then the only rules that a pass P with source v, sink w that appears in W could obey are 8, 9, 10, 11, 12, and 13, so $R = R_8+R_9+R_{10}+R_{11}+R_{12}+R_{13}$. Notice that $R_{12} \leq 1$ and if $R_{10} \geq 1$, then $R_9 < 2$ or else conf(1,7,2) appears on v, v_4, v_5, v_6, v_7 or on v, v_2, v_3, v_4, v_5. Assume first that $R_{10} = 2$, so $\gamma(v_3) = \gamma(v_6) = 5$. It follows that $R_{11} = R_{13} = 0$ and $R_9 \leq 1$. If $R_9 = 0$ then $R \leq 5$ so assume $R_9 = 1$. By symmetry, we may assume $\gamma(v_4) = 5$. If $R_{12} = 0$, then $R = R_8 + R_9 + R_{10} \leq 5$ so assume that $R_{12} = 1$. Then conf(1,7,3) appears on $v, v_2, v_3, v_4, v_5, v_6, t$, which is a contradiction. Assume then that $R_{10} = 1$. Again, $R_9 \leq 1, R_{11} = 0$. and since $R_{12} + R_{13} \leq 1$, $R \leq 5$. Finally, assume $R_{10} = 0$. It follows that $R_{12} = R_{13} = 0$. Therefore, we may assume $R_8 = R_9 = R_{11} = 2$. But this is impossible since $R_8 = 2$ implies $\gamma(v_3) = \gamma(v_6) = 5$ which forces $R_{11} = 0$. This completes the proof when $\gamma(u_1) = \gamma(u_2) = 5$.

Now assume $\{\gamma(v_2), \gamma(v_7)\} = \{5, 6\}$. By symmetry we may take $\gamma(v_2) = 6$ and $\gamma(v_7) = 5$. Therefore $R = R_8 + R_9 + R_{14} + R_{15} + R_{16} + R_{17} + R_{18} + R_{19}$, and $R_{14}, R_{15}, R_{16}, R_{17}, R_{18}, R_{19} \leq 1$. If $R_9 = 2$, then $\gamma(v_4) = \gamma(v_5) = 5$ so either conf(1,7,2) or conf(1,7,4) appears on v, v_4, v_5, v_6, v_7 which is impossible, so $R_9 \leq 1$. Also $R_{16} + R_{18} \leq 1$, because $\gamma(v_6)$ must be 5 if the pass obeys rule 16 and 6 if the pass obeys rule 18. Similarly, $R_{15} + R_{17} \leq 1$ and $R_{14} + R_{19} \leq 1$. Finally, if $R_8 = 2$, then $\gamma(v_3) = \gamma(v_6) = 5$ which forces $R_{16} = R_{18} = 0$, so $R_8 + R_{16} + R_{18} \leq 2$ whence $R = (R_8 + R_{16} + R_{18}) + R_9 + (R_{15} + R_{17}) + (R_{14} + R_{19}) \leq 5$ as desired. This completes the proof when $\{\gamma(u_1), \gamma(u_2)\} = \{5, 6\}$.

Assume then that $\gamma(v_2) = \gamma(v_7) = 6$. Thus $R = R_8 + R_9 + R_{20} + R_{21} + R_{22}$. Now $R_{20} \leq 1$ because otherwise $\gamma(v_3)$ can't equal two different values, and similarly $R_{22} \leq 1$. Moreover, $R_{20} + R_{22} \leq 1$ because a pass obeying rule 20 must have $\gamma(v_4) = \gamma(v_5)$ and a pass obeying rule 22 must have $\gamma(v_4) \neq \gamma(v_5)$. It will now be established that $R_8 + R_{21} \leq 2$. If $R_8 = 2$, then $R_{21} = 0$ or else conf(1,7,2) appears on v, v_3, v_4, v_5, v_6. If $R_{21} = 2$, then we may assume that $\{\gamma(v_3), \gamma(v_6)\} = \{5, 6\}$ or else $R_8 = 0$ or conf(1,7,2) appears in W. However, now conf(4,10,6) appears in W on $w, v, v_2, v_3, v_4, v_5, v_6, v_7, t_W(\{w, v_7\}, v)$. $t = t_W(\{v_7, w\}, v)$ and u is either v_3 or v_6. Thus $R_8 + R_{21} \leq 2$ and $R = (R_8 + R_{21}) + (R_{20} + R_{22}) + R_9 \leq 5$, as desired. This completes the proof of the case $\gamma(v_2) = \gamma(v_6) = 6$.

The next case is $\{\gamma(v_2), \gamma(v_7)\} = \{5, 7\}$, so $R = R_8 + R_9 + R_{18} + R_{19} + R_{23} + R_{24} + R_{25}$. We may assume by symmetry that $\gamma(v_2) = 5$ and $\gamma(v_7) = 7$. This implies that $R_8, R_9, R_{18}, R_{19}, R_{23}, R_{24}, R_{25} \leq 1$. Also, $R_{18} + R_{19} \leq 1$, because $R_{18} > 0$ implies $\gamma(w) = 7$ and $R_{19} > 0$ implies $\gamma(w) \geq 8$. It will now be shown that $R_{23} + R_{24} + R_{25} \leq 1$. If $R_{25} > 0$ then $\gamma(v_6) = 5$ and if $R_{24} > 0$ then $\gamma(v_6) = 6$, so $R_{24} + R_{25}$. If $R_{23} = 1$ then $\gamma(v_6) = 5$ and $\gamma(t) = 5$ so $R_{24} = R_{25} = 0$. Thus $R_{23} + R_{24} + R_{25} \leq 1$. This shows $R \leq 4$.

The last case for $\gamma(v) = 7$ is $\gamma(v_2) \geq 6$ and $\gamma(v_7) \geq 7$ and it follows that $R = R_8 + R_9 + R_{21} + R_{22} + R_{26} + R_{27}$. Since $\gamma(v_2) \geq 7$, $R_8, R_9, R_{21}, R_{22}, R_{26}, R_{27} \leq 1$. Also, $R_{21} > 0$ implies $\gamma(v_4) = \gamma(v_5) = 5$ which forces $R_{22} = 0$. Thus $R \leq 5$.

Now we consider the $\gamma(v) = 8$ case and let the counter-clockwise listing of vertices of v be $w = v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$. The only rules with $\gamma(v) = 8$ are $28, 29, 30, 31, 32$ and therefore $R = R_{28} + R_{29} + R_{30} + R_{31} + R_{32}$. By inspection, $R_{30}, R_{31}, R_{32} \leq 1$ and, if fact, $R_{30} + R_{31} + R_{32} \leq 1$ which implies $R \leq 5$.

If $\gamma(v) \geq 9$ then $R = 0$. This completes the proof of Lemma 7.3.1.

Theorem 7.3.1 *Let W be a cartwheel with $N_{\mathcal{P}}(W) > 0$ and with hub w having degree at least 12. Some configuration isomorphic to one in Appendix A appears in W.*

Proof: Suppose that no configuration isomorphic to one in Appendix A appears in W. Let X be the set of neighbors of w and for $x \in X$, let $c(x)$ denote the sum of $r(P)$ over all passes $P \in \mathcal{P}$ that appear in W with source x and sink w. By Lemma 7.1.2, $\sum_{x \in X} c(x) \leq 5|X| = 5\gamma(w)$. Therefore $\mathcal{N}_{\mathcal{P}}(W) = 10(6 - \gamma(w)) + \sum_{x \in X} c(x) \leq 60 - 5\gamma(w) \leq 0$ since $\gamma(w) \geq 12$. This contradicts the hypothesis that $N_{\mathcal{P}}(W) > 0$.

7.4 Computer Aided Cases for Unavoidability

Theorem 7.2.1 and Theorem 7.3.1 dispose of the cases when the hub has degree 5, 6 or some integer that is at least 12. We invoke Theorem 7.4.1 to take care of the other cases.

Theorem 7.4.1 *If W is a cartwheel with hub w, with $N_{\mathcal{P}}(W) > 0$ and $7 \leq \gamma(w) \leq 11$, then some configuration isomorphic to one in Appendix A appears in W.*

Proof: The formal proof of this was generated by computer. Each individual step of this proof can be checked by hand and so it is in principle possible to check the entire proof by hand. However, this computer generated proof is very long, and so a hand check is not very practicable. Instead, the formal proof can be checked using the same computer program [44] that Robertson et. al. used in [8] to prove the Four Color Theorem.

Theorems 7.2.1, 7.3.1 and 7.4.1 together constitute a proof of Theorem 1.2.7, and thus conclude the Unavoidability part of Theorem 1.2.3.

Appendix A

The Unavoidable Set

This Appendix contains pictures of the 942 configurations that comprise the unavoidable set used to prove Theorem 1.2.6 and Theorem 1.2.7. The value in $\gamma_K(v) \in \{5, 6, \ldots\}$ that is associated with every vertex v in each configuration's K's graph is represented pictorially according to Figure 4.1 in Section 4.2.

We now give some motivation for the definition of configurations that is found in Section 4.2. When considering configurations, we must remember that they basically represent subgraphs whose removal will allow a coloring to be found by induction. As such, they will be "contained" in some large triangulation, say T. The purpose then of assigning a value in $\{5, 6, \ldots\}$ to each vertex of a configuration is to specify what the degree of that vertex is in T. This explains the condition that $\gamma_K(v) = d_T(v)$ in the Section 4.2 definition of what it means for a configuration K to appears in a triangulation T.

We mention again the convention that the configuration in row y and collumn z of page x of this Appendix A will be referred to by conf(x, y, z).