DISCHARGING CARTWHEELS

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ABSTRACT

We give details of the "discharging" part of our proof of the Four Color theorem [1].

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1. AXLES

We assume familiarity with [1]. The purpose of this manuscript is to give a description of the program that we used to establish [1, Theorem (7.1)].

We begin by showing that it suffices to prove the equivalent of [1, Theorem (7.1)] for parts (K, a, b), where $a, b : V(K) \to \{5, 6, \ldots, 12\}$. While this is not really necessary, it makes the computer programs slightly simpler and more elegant. We say that a part (K, a, b) is *limited* if $a(v) \le 12$ for every $v \in V(K)$. A *trivial limited part* is a part (K, a, b) such that $b(v) = 5$ and $a(v) = 12$ for every vertex v of K except the hub. It is unique up to isomorphism.

(1.1) *Let* $d = 7, 8, 9, 10, 11$. *If the trivial limited part with hub of degree* d *is successful, then so is the trivial part with hub of degree* d.

Proof. Let W be a cartwheel with hub of degree d. Let G' be obtained from $G(W)$ by replacing every fan over a vertex of valency > 12 by an 8-edge path, and let W' be the cartwheel with $G(W') = G'$ and $\gamma_{W'}(v) = \min\{\gamma(v), 12\}$ if $v \in V(G(W)) \cap V(G')$ and $\gamma_{W'}(v) = 12$ if $v \in V(G') - V(G(W))$. Then the trivial limited part of degree d fits W', and hence either a good configuration appears in W', or $N_{\mathcal{P}}(W') \le 0$. Since every good configuration K satisfies $\gamma_K(v) \le 11$ for every $v \in V(K)$, and $N_{\mathcal{P}}(W') = N_{\mathcal{P}}(W)$ by condition (iv) in the definition of a rule and the fact that $\delta(v) \in \{5, 6, 7, 8, \infty\}$ for every rule $(G, \beta, \delta, r, s, t)$ in [1, Figure 5] and every $v \in V(G)$, we deduce that either a good configuration appears in W, or $N_{\mathcal{P}}(W) \le 0$, as required. $\square$

If (K, a, b) is a part, then K is, up to isomorphism, determined by the mappings a, b. We now make this precise. Let d be an integer. An *axle of degree* d is a pair $A = (l, u)$, where $l, u : \{1, 2, \ldots, 5d\} \to \{5, 6, \ldots, 12\}$ such that

(A1) $l(i) \le u(i)$ for every $i = 1, 2 \ldots, 5d$, (A2) $l(i) \in \{5,6,7,8,9\}$ and $u(i) \in \{5,6,7,8,12\}$ for all $i = 1,2,\ldots,5d$, and

(A3) for $i = 1,2,\ldots,d$, if $l(i) \neq u(i)$, then $(l(j),u(j)) = (5,12)$ for $j = 2d+i, 3d+i, 4d+i$.

We write $l_A = l$ and $u_A = u$, and put $l_A(0) = u_A(0) = d$. Let A be an axle of degree d, and let (K,a,b) be a part such that

(P1) the hub of K is 0,

(P2) the spokes of K are $1,2,\ldots,d$ in order,

(P3) the hats of K are $d+1, d+2, \ldots, 2d$ in order (so that for $i = 1,2,\ldots,d-1$, $d+i$ is adjacent to i and $i+1$, and $2d$ is adjacent to 1 and d),

(P4) for $i = 1,2,\ldots,d$, if $k = l(i) = u(i)$, then ($5 \leq k \leq 8$ by (A2) and) the vertices of the fan over i are $2d+i$, $3d+i, \ldots, (k-4)d+i$ in order (so that $2d+i, 3d+i, \ldots, (k-4)d+i, d+i$ form a path in K in order; if $k = 5$ there are no fan vertices), and

(P5) b, a are the restrictions of l, u to $V(K)$, respectively.

In these circumstances we say that (K,a,b) is the part derived from A. It is unique up to isomorphism.

A *condition* is a pair (n,m), where $n \in \{1,2,\ldots,5d\}$ and $m \in \{-8,-7,-6,-5,6,7,8,9\}$. We say that a condition (n,m) is *compatible* with an axle A if

(C1) $l_A(n) \leq -m < u_A(n)$ if $m < 0$,

(C2) $l_A(n) < m \leq u_A(n)$ if $m > 0$, and

(C3) either $n \leq 2d$, or $n = jd+i$, where $j \in \{2,3,4\}$, $i \in \{1,2,\ldots,d\}$ and $l_A(i) = u_A(i) \geq j+4$.

If (n,m) is a condition we define $\neg(n,m)$ to be the condition $(n,1-m)$. It follows immediately that (n,m) is compatible with an axle if and only if $\neg(n,m)$ is.

Let A be an axle, and let $c = (n,m)$ be a condition compatible with A. We define

(l', u') by

$$l'(i) = \begin{cases} l_A(i) & \text{if } i \neq n \text{ or } m < 0 \\ m & \text{otherwise} \end{cases}$$

$$u'(i) = \begin{cases} u_A(i) & \text{if } i \neq n \text{ or } m > 0 \\ -m & \text{otherwise.} \end{cases}$$

It follows that (l', u') is an axle; we put $A \wedge c = (l', u')$. It follows immediately that if A is an axle and c is a condition compatible with A, then the parts derived from $A \wedge c$ and $A \wedge (\neg c)$ are a complementary pair of refinements of the part derived from A. We say that an axle is *successful* if the part derived from it is successful. By (1.1) we can restate [1, Theorem (7.1)] as follows. An axle A of degree d is *trivial* if $(l_A(i), u_A(i)) = (5, 12)$ for all $i = 1, 2, \ldots, 5d$. We denote the trivial axle of degree d by Ω_d.

(1.2) *For $d = 7, 8, 9, 10, 11$ the trivial axle of degree d is successful.*

Let W be a cartwheel and A an axle, both of degree d. We say that W is *compatible* with A if the part derived from A fits W. It is easy to see that W is compatible with A if and only if W satisfies (P1), (P2), (P3), (P4), and $l_A(n) \leq \gamma_W(n) \leq u_A(n)$ for all $n \in \{0, 1, \ldots, 2d\}$ and all n of the form $n = jd + i$, where $i \in \{1, 2, \ldots, d\}$, $l_A(i) = u_A(i)$ and $j = 2, 3, \ldots, l_A(i) - 4$. We say that an axle A is *reducible* if for every cartwheel W compatible with A a good configuration appears in W.

2. OUTLETS

Recall that a pass P obeys a rule $R = (G, \beta, \delta, r, s, t)$ if P is isomorphic to some (K, r, s, t) where $G(K) = G$ and $\beta(v) \leq \gamma_K(v) \leq \delta(v)$ for every vertex $v \in V(G)$. Let h be the corresponding homeomorphism of Σ mapping $G(K(P))$ to G and $\gamma_{K(P)}$ to γ_K. If h is orientation-preserving we say that P *orientation-obeys* R; otherwise we say that P *anti-orientation-obeys* R. If $\mathcal{R}$ is a set of rules we write $P \approx \mathcal{R}$ to denote that P orientation-obeys a member of $\mathcal{R}$.

We say that a rule $R = (G, \beta, \delta, r, s, t)$ is *coherent* if

(i) for every cartwheel W and every pass P obeying R, if P appears in W in such a way that either $s(P)$ or $t(P)$ is the hub of W, if $v \in V(P)$ is a fan of W and $u \in V(G(W))$ is the unique spoke of W adjacent to v in $G(W)$, then $u \in V(P)$ and $\beta(u') = \delta(u')$ for the corresponding vertex u' of G, and

(ii) if there exists a pass that both orientation-obeys R and anti-orientation-obeys R, then every pass that orientation-obeys R also anti-orientation-obeys R.

Rules 4, 10 and 31 in [1, Figure 4] are not coherent, but each can be split into two coherent rules with the same net effect. Let $\mathcal{R}'$ be the set of rules obtained this way. We say that a coherent rule R is *symmetric* if some (and hence every) pass that orientation-obeys R also anti-orientation-obeys R. Let $\mathcal{R}''$ be obtained from $\mathcal{R}'$ by adding, for every non-symmetric rule $(G, \beta, \delta, r, s, t) \in \mathcal{R}'$ the rule $(G^*, \beta, \delta, r, s, t)$, where G^* is isomorphic to G as an abstract graph and as a drawing is a "mirror image" of G. Let $\mathcal{R}'''$ be obtained from $\mathcal{R}''$ by replacing the first rule by two identical rules of value one. Finally, let $\mathcal{R}$ be the set of all rules $(G, \beta, \delta', r, s, t)$ such that $(G, \beta, \delta, r, s, t) \in \mathcal{R}'''$ and $\delta'(v) = \min\{\delta(v), 12\}$ for every $v \in V(G)$. Notice that a pass may obey more than one rule in $\mathcal{R}$, and that if $(G, \beta, \delta, r, s, t) \in \mathcal{R}$, then $5 \leq \beta \leq \delta \leq 12$ and $r = 1$. The following holds.

(2.1) *Let $d \geq 5$ be an integer, let W be a cartwheel compatible with Ω_d, let w be the hub of W and let s be a spoke of W. Then*

$$\begin{aligned}&\sum(r(P) : P \approx \mathcal{R}, P \text{ appears in } W, t(P) = w, s(P) = s)\\ &\quad = \sum(r(P) : P \sim \mathcal{P}, P \text{ appears in } W, t(P) = w, s(P) = s)\end{aligned}$$

and

$$\begin{aligned}&\sum(r(P) : P \approx \mathcal{R}, P \text{ appears in } W, s(P) = w, t(P) = s)\\ &\quad = \sum(r(P) : P \sim \mathcal{P}, P \text{ appears in } W, s(P) = w, t(P) = s).\end{aligned}$$

Now if W is a cartwheel with hub w and a spoke s and $R \in \mathcal{R}$, then a pass $P \approx \{R\}$ appears in W with $s(P) = s$, $t(P) = w$ if and only if for some vertices v of $G(W)$, $\gamma_W(v)$ is within certain bounds determined by R. This motivates the following definition. An *outlet of degree* d is a pair $T = (M, r)$, where r is a non-zero integer, called the *value of* T, and M is a set $\{(p_1, l_1, u_1), (p_2, l_2, u_2), \ldots, (p_n, l_n, u_n)\}$ such that

(T1) $p_1, p_2, \ldots, p_n$ are integers with $1 \leq p_i \leq 5d$,

(T2) $l_1, l_2, \ldots, l_n, u_1, u_2, \ldots, u_n$ are integers with $l_i \leq u_i$,

(T3) $l_i \in \{5, 6, 7, 8, 9\}$ and $u_i \in \{5, 6, 7, 8, 12\}$ for every $i = 1, 2, \ldots, n$, and

(T4) if $p_k = jd + i$ for some $k \in \{1, 2, \ldots, n\}$, $j \in \{2, 3, 4\}$ and $i \in \{1, 2, \ldots, d\}$, then there exists a $t \in \{1, 2, \ldots, n\}$ such that $p_t = i$ and $l_t = u_t \geq j + 4$.

We say that T is *reduced* if $p_1, p_2, \ldots, p_n$ are pairwise distinct and $(l_i, u_i) = (5, 12)$ for no $i = 1, 2, \ldots, n$. We write $r(T) = r$ and $M(T) = M$. If i is an integer and $x \in \{0, 1, \ldots, d\}$ we define

$$i \oplus_d x = \begin{cases} i + x & \text{if } x + (i-1) \bmod d < d \\ i + x - d & \text{otherwise.} \end{cases}$$

A *positioned outlet* of degree d is a pair (T, x), where T is an outlet of degree d, and $x \in \{1, 2, \ldots, d\}$. Let A be an axle of degree d, and let (T, x) be a positioned outlet of degree d, where $x \in \{1, 2, \ldots, d\}$ and $M(T) = \{(p_1, l_1, u_1), (p_2, l_2, u_2), \ldots, (p_n, l_n, u_n)\}$. We say that (T, x) is *enforced* by A if

$$l_i \leq l_A(p_i \oplus_d (x-1)) \leq u_A(p_i \oplus_d (x-1)) \leq u_i$$

for all $i = 1, 2, \ldots, n$. We say that (T, x) is *permitted by* A if

$$u_i \geq l_A(p_i \oplus_d (x-1)) \quad \text{and} \quad u_A(p_i \oplus_d (x-1)) \geq l_i$$

for all $i = 1, 2, \ldots, n$.

(2.2) *For every integer $d = 7, 8, 9, 10, 11$ and for every rule $R = (G, \beta, \delta, r, s, t) \in \mathcal{R}$ there exist unique reduced outlets T and T' such that $r(T) = -r(T') = r$ and for every axle A and every integer $x \in \{1, 2, \ldots, d\}$,*

(i) (T, x) is enforced by A if and only if for every cartwheel W compatible with A there exists a pass $P \approx \{R\}$ appearing in W with $s(P) = x$ and $t(P) = 0$,

(ii) (T', x) is enforced by A if and only if for every cartwheel W compatible with A there exists a pass $P \approx \{R\}$ appearing in W with $s(P) = 0$ and $t(P) = x$,

(iii) (T, x) is permitted by A if and only if there exist a cartwheel W compatible with A and a pass $P \approx \{R\}$ appearing in W with $s(P) = x$ and $t(P) = 0$,

(iv) (T', x) is permitted by A if and only if there exist a cartwheel W compatible with A and a pass $P \approx \{R\}$ appearing in W with $s(P) = 0$ and $t(P) = x$.

For $d = 7, 8, 9, 10, 11$ let $\mathcal{T}_d$ be the set of all outlets T, T' corresponding to rules $R \in \mathcal{R}$ as in (2.2).

Let A be an axle, let (T, x) be a positioned outlet, and let

$$M(T) = \{(p_1, l_1, u_1), (p_2, l_2, u_2), \ldots, (p_n, l_n, u_n)\}.$$

We define $A \wedge (T, x)$ to be the pair $A' = (l, u)$, where for $i = 1, 2, \ldots, 5d$, $l(i)$ is the least integer $l' \geq l_A(i)$ such that $l' \geq l_j$ for all $j \in \{1, 2, \ldots, n\}$ with $p_j = i$, and $u(i)$ is the largest integer $u' \leq u_A(i)$ such that $u' \leq u_j$ for all $j \in \{1, 2, \ldots, n\}$ with $p_j = i$. The following is straightforward.

(2.3) *Let A be an axle of degree d, and let (T, x) be a positioned outlet of degree d. Then $A \wedge (T, x)$ is an axle if and only if (T, x) is permitted by A.*

It should be noted that while $\mathcal{R}$ does not depend on d, the corresponding outlets do. We therefore input $\mathcal{R}$ in the form of a file (same for every d), and compute the corresponding outlets of degree d at the beginning of the computation. It is not necessary to check correctness of this part of the program; the reader can alternatively verify by inspection that the set $\mathcal{T}_d$ was computed correctly.

The members of $\mathcal{R}$ are stored as follows. Let $(G, \beta, \delta, r, s, t) \in \mathcal{R}$. Let us assume for convenience that $-1 \notin V(G)$. We define a sequence $v_0, v_1, \ldots, v_{16}$ such that $v_i \in V(G) \cup \{-1\}$ and every vertex of G occurs in the sequence exactly once. If u, v are adjacent vertices of G, let $T(u, v)$ be the vertex w of G such that u, v, w form a triangle in G in clockwise order, and let $T(u, v) = -1$ if no such vertex w exists. We define $v_0 = s$, $v_1 = t$, $v_2 = T(v_0, v_1)$, $v_3 = T(v_1, v_0)$, $v_4 = T(v_0, v_2)$, $v_5 = T(v_3, v_0)$, $v_6 = T(v_2, v_1)$, $v_7 = T(v_1, v_3)$, $v_8 = T(v_4, v_2)$, $v_9 = T(v_3, v_5)$, $v_{10} = T(v_8, v_2)$, $v_{11} = T(v_3, v_9)$, $v_{12} = T(v_0, v_4)$, $v_{13} = T(v_0, v_{12})$, $v_{14} = T(v_5, v_0)$, $v_{15} = T(v_6, v_1)$, $v_{16} = T(v_{15}, v_1)$. To input a rule we list $\beta(v_0)$, $\delta(v_0)$, $\beta(v_1)$, $\delta(v_1)$ and all triples $(i, \beta(v_i), \delta(v_i))$ such that $2 \leq i \leq 16$ and $v_i \in V(G)$.

3. HUBCAPS

Let $d \geq 5$ be an integer. A *hubcap of degree* d is a collection $((x_1, y_1, v_1), (x_2, y_2, v_2), \ldots, (x_n, y_n, v_n))$ of triples of integers such that every integer $i = 1, 2, \ldots d$ appears in the list $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ exactly twice. Let us make two remarks here. First, this definition differs somewhat from the one given in [1]. Second, in the actual program we use the convention that if a triple of integers appears in a hubcap twice, it is only listed once. Hubcaps will be used to obtain upper bounds on $N_{\mathcal{P}}(W)$. We now explain how.

Let A be an axle of degree d, and let $x, y \in \{1, 2, \ldots, d\}$. We put

$$L_d(A, x, y) = \max \sum_{(T,k)} r(T),$$

where the max is taken over all sets $\{(T_1, k_1), (T_2, k_2), \ldots, (T_n, k_n)\}$ of positioned outlets with $T_i \in \mathcal{T}_d$, $k_i \in \{x, y\}$ $(i = 1, 2, \ldots, d)$ and such that $A' = A \wedge (T_1, k_1) \wedge (T_2, k_2) \wedge \cdots \wedge$

(T_n, k_n) is a non-reducible axle, and the sum is over all pairs (T, k) such that $T \in \mathcal{T}_d$, $k \in \{x, y\}$ and (T, k) is enforced by A'. We say that a hubcap $H = \{(x_1, y_1, v_1), (x_2, y_2, v_2), \ldots, (x_n, y_n, v_n)\}$ is a *hubcap* for A (and that A *has a hubcap*) if

(H1) for all $i = 1, 2, \ldots, n$, $L_d(A, x_i, y_i) \le v_i$, and

(H2) $10(6-d) + \left\lfloor \frac{1}{2} \sum_{i=1}^{n} v_i \right\rfloor \le 0$.

(3.1) *Let A be an axle of degree d that has a hubcap. Then A is successful.*

Proof. Let A be an axle of degree d, and let $H = \{(x_1, y_1, v_1), (x_2, y_2, v_2), \ldots, (x_n, y_n, v_n)\}$ be a hubcap for A. Let W be a cartwheel compatible with A, and assume that no good configuration appears in W. Then

$$\begin{aligned}
N_{\mathcal{P}}(W) = 10(6-d) &+ \sum (r(P) : P \sim \mathcal{P}, P \text{ appears in } W, t(P) = 0) \\
&- \sum (r(P) : P \sim \mathcal{P}, P \text{ appears in } W, s(P) = 0) \\
= 10(6-d) &+ \sum (r(P) : P \approx \mathcal{R}, P \text{ appears in } W, t(P) = 0) \\
&- \sum (r(P) : P \approx \mathcal{R}, P \text{ appears in } W, s(P) = 0) \\
= 10(6-d) &+ \frac{1}{2} \sum_{i=1}^{n} \Big(\sum (r(P) : P \approx \mathcal{R}, P \text{ appears in } W, t(P) = 0, s(P) \in \{x_i, y_i\}) \\
&- \sum (r(P) : P \approx \mathcal{R}, P \text{ appears in } W, s(P) = 0, t(P) \in \{x_i, y_i\}) \Big) \\
\le 10(6-d) &+ \frac{1}{2} \sum_{i=1}^{n} L_d(A, x_i, y_i) \le 10(6-d) + \frac{1}{2} \sum_{i=1}^{n} v_i.
\end{aligned}$$

Since $N_{\mathcal{P}}(W)$ is an integer, we deduce that

$$N_{\mathcal{P}}(W) \le 10(6-d) + \left\lfloor \frac{1}{2} \sum_{i=1}^{n} v_i \right\rfloor \le 0,$$

as desired. □

We need an algorithm that given an axle A and a hubcap H verifies that H is a hubcap for A. Most of that is reasonably straightforward, except for verifying that

$L_d(A, x, y) \leq v$. That is accomplished by a function "CheckBound" which we now describe. Let $(T_0, z_0), (T_1, z_1), \ldots, (T_{n-1}, z_{n-1})$ be all the positioned outlets with $T_i \in \mathcal{T}_d$ and $z_i \in \{x, y\}$; the parameters of "CheckBound" are integers $p \in \{0, 1, \ldots, n-1\}$, $s_i \in \{-1, 0, 1\}$ $(i = 0, 1, \ldots, n-1)$, v and an axle A. If

(i) for $i = 0, 1, \ldots, n-1$, if $s_i = 1$ then (T_i, z_i) is enforced by A, and

(ii) for $i = p, p+1, \ldots, n-1$ if $s_i = -1$ then (T_i, z_i) is not permitted by A,

then the function "CheckBound" verifies that either A is reducible, or

$$\max_S \sum_j r(T_j) \leq v,$$

where the max is taken over all sets S such that

$$\{i : 0 \leq i < n,\ s_i = 1\} \subseteq S \subseteq \{i : 0 \leq i < n,\ s_i \neq -1\},$$

and $A' = A \wedge \bigwedge_{i \in S}(T_i, z_i)$ is an axle, and the sum is over all j such that $0 \leq j < n$ and (T_j, z_j) is enforced by A'. (Equivalently, the max can be taken over all set S such that in addition $r(T_i) > 0$, (T_i, z_i) is permitted by A for every $i \in S$, and if (T_i, z_i) is enforced by A then $i \in S$.) Thus a call to "CheckBound" with parameters $p = 0$, $s_i = 0$ $(i = 0, 1, \ldots, n-1)$, v and A verifies that $L_d(A, x, y) \leq v$. The function "CheckBound" proceeds in the following steps.

(1) For every $i = p, p+1, \ldots, n-1$ with $s_i = 0$, if (T_i, z_i) is enforced by A then set $s_i = 1$, and if (T_i, z_i) is not permitted by A then set $s_i = -1$.

(2) Compute $f = \sum(r(T_i) : 0 \leq i < n, s_i = 1)$ and $a = \sum(r(T_i) : 0 \leq i < n, s_i = 0, r(T_i) > 0)$.

(3) If $a + f \leq v$ then the inequality holds. Return.

(4) If $f > v$ test if A is reducible. If it is return, otherwise display an error message and stop.

(5) For all $q = p, p+1, \ldots, n-1$ with $s_q = 0$ and $r(T_q) > 0$ repeat steps (6)–(10).

(6) Set $s'_i = s_i$ for $i \in \{0, 1, \ldots, n-1\} - \{q\}$, $s'_q = 1$, and $A' = A \wedge (T_q, z_q)$.

(7) If for some $i \in \{0, 1, \ldots, p-1\}$ with $s_i = -1$ the positioned outlet (T_i, z_i) is forced by A', then go to step (9), otherwise go to step (8).

(8) Call "CheckBound" recursively with arguments q, s'_i, v, A'.

(9) Set $s_q = -1$ and $a = a - r(T_q)$.

(10) If $a + f \leq v$ then the inequality holds. Return.

4. ASSERTIONS

Let $d \geq 5$ be an integer, and let A be an axle of degree d. We say that A is *fan-free* if $(l_A(i), u_A(i)) = (5, 12)$ for all $i = 2d+1, 2d+2, \ldots, 5d$. If A is fan-free we define τA to be the axle (l', u'), where $(l'(i \oplus_d 1), u'(i \oplus_d 1)) = (l(i), u(i))$ for $i = 1, 2, \ldots, 2d$ and $(l'(i), u'(i)) = (5, 12)$ for $i = 2d+1, 2d+2, \ldots, 5d$, and we define σA to be the axle (l'', u''), where

$$(l''(i), u''(i)) = \begin{cases} (l(d+1-i), u(d+1-i)) & \text{for } i = 1, 2, \ldots, d \\ (l(3d-i), u(3d-i)) & \text{for } i = d+1, d+2, \ldots, 2d-1 \\ (l(i), u(i)) & \text{for } i = 2d, 2d+1, \ldots, 5d. \end{cases}$$

Thus τA is the axle obtained from A by rotating by one unit, and σA is the axle obtained from A by reflecting. A *disposition* D is either $\emptyset$ (regarded as a formal symbol informing us that a certain axle is reducible), or a hubcap or a triple (k, ϵ, M), where k is an integer, $\epsilon \in \{0, 1\}$ and M is a fan-free axle. In the first case we say that D is a *reducibility disposition*, in the second case we say that D is a *hubcap disposition*, and in the third case we say that D is a *symmetry disposition*. Let A be an axle, let $\mathcal{M}$ be a set of axles, and let D be a disposition. We say that D *disposes of A relative to $\mathcal{M}$* if the following conditions hold.

(i) If D is a reducibility disposition, then A is reducible.

(ii) If D is a hubcap disposition, then D is a hubcap for A.

(iii) If $D = (k, \epsilon, M)$ is a symmetry disposition, then $M \in \mathcal{M}$ and every cartwheel compatible with A is compatible with $\tau^k \sigma^\epsilon M$.

The following is straightforward.

(4.1) *Let A be an axle, let $\mathcal{M}$ be a set of axles such that every member of $\mathcal{M}$ is successful, and let D be a disposition. If D disposes of A relative to $\mathcal{M}$, then A is successful.*

Let $t \geq 0$ be an integer. An *assertion of depth at most* t is a sequence $(c_1, S_1, c_2, S_2, \ldots, c_n, S_n, D)$, where $n \geq 0$ is an integer, $c_1, c_2, \ldots, c_n$ are conditions, $S_1, S_2, \ldots, S_n$ are assertions of depth at most $t-1$, and D is a disposition. (Thus if $t = 0$, then $n = 0$.) An *assertion* is an assertion of depth at most t for some integer $t \geq 0$. A *history* is a set of conditions. Let $d \geq 5$ be an integer, let A be an axle of degree d, let $\mathcal{M}$ be a set of fan-free axles of degree d, and let H be a history. We say that an assertion $S = (c_1, S_1, c_2, S_2, \ldots, c_n, S_n, D)$ *holds* for $(A, \mathcal{M}, H)$ if the following two conditions are satisfied.

(S1) D disposes of $A \wedge \bigwedge_{i=1}^{n} (\neg c_i)$ relative to $\mathcal{M} \cup \mathcal{M}''$, where $\mathcal{M}''$ is the set of fan-free axles of the form $\Omega_d \wedge \bigwedge_{c \in H} c \wedge c_i$, where $i \in \{1, 2, \ldots, n\}$.

(S2) For all $i = 1, 2, \ldots, n$, S_i holds for $(A', \mathcal{M}', H')$, where $A' = A \wedge \bigwedge_{j=1}^{i-1} (\neg c_j) \wedge c_i$, $H' = H \cup \{c_i\}$, and $\mathcal{M}'$ consists of all members of $\mathcal{M}$ and all fan-free axles of the form $\Omega_d \wedge \bigwedge_{c \in H} c \wedge c_j$ for $j = 1, 2, \ldots, i-1$.

An assertion S is a *presentation of degree* d if S holds for $(\Omega_d, \emptyset, \emptyset)$. Our proof of (1.2) is based on the following.

(4.2) *If there exists a presentation of degree d, then Ω_d is successful.*

To deduce (1.2) from (4.2) suffices to exhibit a presentation of degree d for every $d = 7, 8, 9, 10, 11$. Theorem (4.2) itself follows from the following more general statement.

(4.3) *Let $d \geq 5$ be an integer, let A be an axle of degree d, let $\mathcal{M}$ be a set of fan-free axles of degree d, let H be a history, and let S be an assertion that holds for $(A, \mathcal{M}, H)$. Assume that*

(i) every member of $\mathcal{M}$ is successful, and

(ii) for every cartwheel W compatible with $\Omega_d \wedge \bigwedge_{c \in H} c$ but not with A such that $N_{\mathcal{P}}(W) > 0$, a good configuration appears in W.

Then A is successful.

Proof. Let $A, H, \mathcal{M}, S$ be as stated. We proceed by induction on the depth of S. Let S be of depth at most t, and assume that the theorem holds for all assertions of depth at most $t-1$. Let $S = (c_1, S_1, c_2, S_2, \ldots, c_n, S_n, D)$ and for $i = 1, 2, \ldots, n$ let

$$A_i = A \wedge \bigwedge_{j<i} (\neg c_j) \wedge c_i,$$
$$H_i = H \cup \{c_i\},$$
$$\mathcal{M}_i = \mathcal{M} \cup \left\{ \Omega_d \wedge \bigwedge_{c \in H_1} c, \Omega_d \wedge \bigwedge_{c \in H_2} c, \ldots, \Omega_d \wedge \bigwedge_{c \in H_{i-1}} c \right\}.$$

We first prove the following.

(1) *Let $i = 1, 2, \ldots, n$. Then the following statements hold.*

(a) *For $j = 1, 2, \ldots, i-1$, if a cartwheel W is compatible with $\Omega_d \wedge \bigwedge_{c \in H_j} c$ but not with A_j, then either $N_{\mathcal{P}}(W) \leq 0$ or a good configuration appears in W.*

(b) *A_j is successful for $j = 1, 2, \ldots, i-1$.*

(c) *Every member of $\mathcal{M}_i$ is successful.*

We prove (1) by induction on i. Let $i = 1, 2, \ldots, n$, and assume that (a), (b) and (c) hold for every $i' < i$.

To prove (a) we may assume that $i > 1$, for otherwise (a) is vacuously true. It suffices to prove the conclusion for $j = i - 1$. To this end let W be compatible with $\Omega_d \wedge \bigwedge_{c \in H_{i-1}} c$

but not with A_{i-1}. If W is not compatible with A then (a) follows from (ii), and so we may assume that W is compatible with A. Moreover, W is compatible with $A \wedge c_{i-1}$, and so we deduce that W is compatible with one of $A_1, A_2, \ldots, A_{i-2}$, and hence either $N_{\mathcal{P}}(W) \leq 0$ or a good configuration appears in W by the induction hypothesis that (b) holds for $i-1$. This proves (a).

To prove (b) it is enough to establish that A_{i-1} is successful. Since S_{i-1} holds for $(A_{i-1}, \mathcal{M}_{i-1}, H_{i-1})$, since every member of $\mathcal{M}_{i-1}$ is successful by the induction hypothesis that (c) holds for every $i' < i$, and since every cartwheel compatible with $\Omega_d \wedge \bigwedge_{c \in H_{i-1}} c$ but not with A_{i-1} satisfies (ii) by (a) above, we deduce from the induction hypothesis that (4.3) holds for all assertions of depth at most $t-1$ that A_{i-1} is successful, as required for (b).

To prove (c) let W be compatible with a member of $\mathcal{M}_i$. By (i) we may assume that W is compatible with $\Omega_d \wedge \bigwedge_{c \in H_j} c$ for some $j = 1, 2, \ldots, i-1$. If W is not compatible with A_j the conclusion follows from (a); otherwise it follows from (b). This completes the proof of (1).

We are now ready to complete the proof of (4.3). Every cartwheel compatible with A is either compatible with A_j for some $j = 1, 2, \ldots, n$, or with $A' = A \wedge \bigwedge_{i=1}^{n} (\neg c_i)$. Since each A_j is successful by (1b), it suffices to show that A' is successful. Since S holds for $(A, \mathcal{M}, H)$ it follows that D disposes of A' relative to $\mathcal{M} \cup \mathcal{M}'$, where $\mathcal{M}'$ is the set of fan-free axles of the form $\Omega_d \wedge \bigwedge_{c \in H} \wedge c_i$, where $i \in \{1, 2, \ldots, n\}$. Every member of $\mathcal{M}$ is successful by (i) and every member of $\mathcal{M}'$ is successful by (a) and (b), and hence A' is successful by (4.1), as required. □

5. PRESENTATIONS

A presentation is described by means of a file, which in turn is described as a sequence of lines. Each line has a *level* associated with itself. The lines are of two types – *condition lines* describing conditions and *disposition lines* describing dispositions. The lines are numbered consecutively, starting from 2 (the first actual line of a file describes the degree) so that line 2 has level 0.

Let σ be a finite sequence of lines. Let l be the lowest level of a line in σ, and let $\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{n+1}$ (in this order) be all the lines of level l in σ. If

(i) λ_1, λ_{n+1} are the first and last lines in σ, respectively,

(ii) $\lambda_1, \lambda_2, \ldots, \lambda_n$ are condition lines describing conditions $c_1, c_2, \ldots, c_n$, respectively, and λ_{n+1} is a disposition line describing a disposition D,

(iii) for all $i = 1, 2, \ldots, n$, the sequence of lines of σ strictly between λ_i and λ_{i+1} describes an assertion S_i, and

(iv) the levels of any two consecutive lines in σ differ by exactly 1,

then we say that σ *describes the assertion* $S = (c_1, S_1, c_2, S_2, \ldots, c_n, S_n, D)$. We have created five files "present7", "present8", "present9", "present10" and "present11" that describe assertions $\mathcal{P}_7, \mathcal{P}_8, \mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11}$, respectively. Our computer program verifies that they are presentations of appropriate degrees.

Before proceeding further we need to explain a close relationship between axles and outlets. Let T be an outlet of degree d with $r(T) = 1$ and

$$M(T) = \{(p_1, l_1, u_1), (p_2, l_2, u_2), \ldots, (p_n, l_n, u_n)\}.$$

Let A be the axle such that $l_i \leq l_A(p_i) \leq u_A(p_i) \leq u_i$ for all $i = 1, 2, \ldots, n$, and, subject to that, $l_A(j) \in \{5, 6, 7, 8, 9\}$ is minimum and $u_A(j) \in \{5, 6, 7, 8, 12\}$ is maximum for all $j = 1, 2, \ldots, 5d$. We say that A is the *axle corresponding* to T, and that T is an *outlet corresponding* to A. The *null condition* is the pair $(0, 0)$. If A is an axle, then $A \wedge (0, 0)$ is undefined, and hence, in particular, is not a fan-free axle.

The program reads and processes lines of the presentation file in order. During execution it maintains variables l, A_i, c_i, $(i = 0, 1, \ldots, l)$, t, $T_0, T_1, \ldots, T_{t-1}$, where t, l are integers, A_i are axles, c_i are conditions or null conditions, and $T_0, T_1, \ldots, T_{t-1}$ are outlets. At the beginning we set $l = t = 0$, $A_0 = \Omega_d$, $c_0 = (0, 0)$, and keep reading lines from the input file until all lines are exhausted. After reading a condition line at level l describing a condition c the program verifies that c is a condition and that it is compatible with A_l, sets $A_{l+1} = A_l \wedge c$ and $A_l = A_l \wedge (\neg c)$. If $B = \Omega_d \wedge \bigwedge_{i=1}^{l} c_i$ is a fan-free axle it sets T_t to be the outlet corresponding to B and increases t by 1. It sets $c_l = c$, $c_{l+1} = (0, 0)$, and increases l by 1.

After reading a disposition line at level l describing a disposition D it verifies that D disposes of A_l relative to the set of axles corresponding to $T_0, T_1, \ldots, T_{t-1}$, and sets t to be the largest integer t' such that either $t' = 0$ or $T_{t'-1}$ was added while executing a line of level $< l$.

If λ is an input line, let $l^{[\lambda]}, t^{[\lambda]}, A_i^{[\lambda]}, c_i^{[\lambda]}, T_i^{[\lambda]}$ denote the values of the variables l, t, A_i, c_i, T_i immediately prior to reading line λ. Let $\sigma^{[\lambda]}$ be the sequence of lines consisting of λ and the lines of level $\geq l^{[\lambda]}$ immediately following λ, and let $S^{[\lambda]}$ be the assertion described by $\sigma^{[\lambda]}$. Let $H^{[\lambda]}$ consist of all $c_i^{[\lambda]}$ $(i = 0, 1, \ldots, l^{[\lambda]} - 1)$, and let $\mathcal{M}^{[\lambda]}$ be the set of all axles corresponding to $T_0^{[\lambda]}, T_1^{[\lambda]}, \ldots, T_{t^{[\lambda]}-1}^{[\lambda]}$. The following is an immediate consequence of the description of the algorithm, and implies that the program correctly verifies that an input file describes a presentation.

(5.1) *For every input line λ the program verifies that $S^{[\lambda]}$ holds for $(A^{[\lambda]}, \mathcal{M}^{[\lambda]}, H^{[\lambda]})$.*

To complete the description we must explain how the program verifies disposition. Hubcap dispositions were discussed in Section 3, reducibility dispositions are addressed in the next section, and so it remains to explain how we verify symmetry dispositions.

A line λ describing a symmetry disposition contains four integers k, ϵ, l, m, where $k \in \{0, 1, \ldots, d-1\}$, $\epsilon \in \{0, 1\}$, m is such that the line number m has level l, and during its processing an outlet T was added to the list $T_1, \ldots, T_t$ such that if M denotes the axle corresponding to T, then every cartwheel compatible with $A^{[\lambda]}_{l[\lambda]}$ is compatible with $\tau^k \sigma^\epsilon M$. For $\epsilon = 0$ the latter is equivalent to the fact that $(T, k+1)$ is enforced by $A^{[\lambda]}_{l[\lambda]}$, which is what we actually test for. We use a similar test for $\epsilon = 1$.

6. TESTING APPEARANCE

We need to be able to test whether a given axle is reducible, and the purpose of this section is to describe such test. Let A be an axle of degree d, and let B be the axle defined for $i = 0, 1, \ldots, 5d$ by $(l_B(i), u_B(i)) = (u_A(i), u_A(i))$ if $1 \leq i \leq d$ and $u_A(i) \leq 8$, and $(l_B(i), u_B(i)) = (l_A(i), u_A(i))$ otherwise. Let (K, a, b) be the part derived from B, and let L be the configuration with $G(L) = K$ and $\gamma_L(v) = a(v)$ for all $v \in V(G(L))$. We say that L is the *skeleton of* A. We say that L is a *skeleton* if it is a skeleton of some axle. The *hub*, *spokes*, *hats* and *fans* of a skeleton are defined in the obvious way. Let K, L be configurations. We say that K is a *subconfiguration* of L if $G(K)$ is a subdrawing of $G(L)$ and γ_K is the restriction of γ_L to $V(G(K))$. An *induced subconfiguration* is defined analogously. Thus a configuration K appears in a cartwheel W if and only if K is an induced subconfiguration of W. Let L be a subconfiguration of a skeleton K. We say that L is *well-positioned* in K if for every spoke $v \in V(G(K)) - V(G(L))$, at least one of the two hats adjacent to v does not belong to $V(G(L))$. We say that an axle A is *semi-reducible* if a good configuration is a well-positioned induced subconfiguration of its skeleton. If K is such a good configuration, then it follows that K appears in every cartwheel W compatible with A such that $\gamma_W(v) = \gamma_K(v)$ for every $v \in V(G(K))$.

Later in this section we describe how we test semi-reducibility, but now, with that as a subroutine, let us explain how we test whether A is reducible. (Actually, we only test for a sufficient condition for reducibility, but it suffices for our purposes.) We start by putting A on a stack, and keep repeating the following steps.

(1) If the stack is empty, then A is reducible and we stop. Otherwise pop an axle, say B, from the stack.

(2) Test if B is semi-reducible. If not then the test failed; we display an error message and stop. Otherwise let L be a good configuration that is a well-positioned induced subconfiguration of the skeleton of B.

(3) For every vertex v of $G(L)$ such that $l_B(v) < u_B(v)$ do the following:

   (a) Let $u'_B(v) = u_B(v) - 1$ and $u'_B(u) = u_B(u)$ for $u \neq v$. Then (l_B, u'_B) is an axle.

   (b) Put (l_B, u'_B) on the stack.

We now explain how we test semi-reducibility, but before we do that we should point out that verifying this part of the program is not necessary, for there is an independent function "CheckIso" which (rather crudely) verifies from first principles that a mapping produced by the semi-reducibility routine gives an isomorphism onto an induced subconfiguration. The semi-reducibility algorithm itself is very simple; however, its justification requires some effort.

We say that a configuration K has *radius at most two* if there exists a vertex $v \in V(G(K))$ such that for every vertex $u \in V(G(K))$ there is a path P in $G(K)$ with ends u, v and $|E(P)| \leq 2$. The vertex v is called a *center* of K. The following is easy to check by inspection, and is also verified by our computer program.

(6.1) *Every good configuration has radius at most two.*

Our semi-reducibility test is based on the following theorem. Let L be a good configuration, and let L_0 be its free completion with ring R. If $G(L)$ is 2-connected let $J = G(L)$;

otherwise there is a unique vertex $v \in V(G(L))$ such that $G(L)\backslash v$ is disconnected. Choose a neighbor $v' \in V(R)$ of v in L_0, and let J be the subdrawing of $G(L_0)$ induced by $V(G(L)) \cup \{v'\}$. Then J is a 2-connected near-triangulation. In either case we say that J is an *enhancement* of L.

(6.2) *Let L, K be configurations, let L be good, let J be an enhancement of L, let J' be a 2-connected near-triangulation with $V(J') = V(J)$ such that J' is a subdrawing of J and there exists a 1-1 mapping $f : V(J') \to V(G(K))$ such that if $u, v, w \in V(J')$ form a triangle in J' in the clockwise order, then $f(u), f(v), f(w)$ form a triangle in $G(K)$ in the clockwise order. Assume further that $\gamma_L(v) = \gamma_K(f(v))$ for every $v \in V(G(L))$. Then a configuration K_0 isomorphic to L is a subconfiguration of K. Moreover, if L has radius at most two, if K is a skeleton of an axle of degree at least six, and if K_0 is well-positioned in K, then K_0 is an induced subconfiguration of K.*

Proof. There exist near-triangulations $J_0 = J', J_1, \ldots, J_n = J$, all with vertex-set $V(J)$ such that for $i = 1, 2, \ldots, n$, J_i is obtained from J_{i-1} by adding an edge e_i with ends u_i and v_i. Then e_i is incident with exactly one finite triangle, say T_i, of J_i. Let $w_i \notin \{u_i, v_i\}$ be the third vertex incident with T_i, and assume that the notation for u_i and v_i is chosen so that u_i, v_i, w_i form a triangle in this clockwise order. We claim the following.

(1) *For $i = 1, 2, \ldots, n$, $f(u_i)$ is adjacent to $f(v_i)$ in $G(K)$, and the vertices $f(u_i), f(v_i)$, $f(w_i)$ form a triangle in $G(K)$ in clockwise order.*

We prove (1) by induction on i. Let $z_1 = v_i, z_2, \ldots, z_m = u_i$ be all the neighbors of w_i in J_{i-1} listed in the clockwise order in which they appear around w_i. Since J_{i-1} is a 2-connected near-triangulation we deduce that z_j, z_{j+1}, w_i form a triangle for every $j = 1, 2, \ldots, m-1$, and hence $f(z_j), f(z_{j+1}), f(w_i)$ form a triangle in $G(K)$ by the assumptions of (6.2) and the induction hypothesis. Since u_i, v_i are adjacent in J_i we deduce that w_i

is not incident with the infinite region of J, and hence $w_i \in V(G(L))$. If $f(v_i)$ and $f(u_i)$ are not adjacent in $G(K)$, then $\gamma_K(f(w_i)) > d_{J_i}(w_i)$, and hence $\gamma_K(f(w_i)) > d_{J_i}(w_i) = d_J(w_i) = \gamma_L(w_i)$, a contradiction. Thus $f(v_i)$ and $f(u_i)$ are adjacent in $G(K)$, and claim (1) follows.

Next we claim

(2) *If u, v are adjacent in J, then $f(u), f(v)$ are adjacent in $G(K)$.*

To prove (2) let u, v be adjacent in J, and let P be a path in J with vertex-set $u_0 = u, u_1, \ldots, u_k = v$ in order such that

(i) $f(u_i)$ is adjacent to $f(u_{i-1})$ in $G(K)$ for every $i = 1, 2, \ldots, k$,

and, subject to that,

(ii) k is minimum.

Such a path exists, because every path P with $E(P) \subseteq E(J')$ satisfies (i). We claim that $k = 1$. To prove this we first notice that if $k = 2$, then $f(u), f(v)$ are adjacent by (1), and so we may assume that $k > 2$. Let C be the circuit obtained from P by adding the edge u, v. Since J is a near-triangulation we deduce that some pair of vertices u', v' of P other than the two ends are adjacent in J. Regardless of whether $f(u'), f(v')$ are adjacent in $G(K)$ or not we obtain a contradiction to the minimality of k. This proves our claim that $k = 1$, and hence completes the proof of (2).

From (1) and (2) we deduce the first part of (6.2). For the second part let L have radius at most two, and let K be the skeleton of an axle of degree at least six. By the first part we may assume that L is a well-positioned subconfiguration of K. If L is not an induced subconfiguration then some two vertices $u, v \in V(G(L))$ are adjacent in $G(K)$, but not in $G(L)$. Since L has radius at most two there exists a path P in $G(L)$ with ends u, v and $|E(P)| \leq 4$. Let us choose such a path P with $|E(P)|$ minimum. Let

C be the circuit of $G(K)$ obtained from P by adding the edge uv. Let Δ be the disk bounded by C that is disjoint from the infinite region of $G(K)$. Since $|V(C)| \le 5$, L is well-positioned in K and J is isomorphic to a 2-connected subdrawing of $G(K)$, we deduce that Δ contains no vertex of $G(K)$ in its interior. Since $G(K)$ is a near-triangulation we deduce by the minimality of $|E(P)|$ that $|V(P)| = 3$; let w be the interior vertex of P. Since u, v are adjacent in $G(K)$, but not in $G(L)$, and $G(L)$ is an induced subdrawing of J, we deduce that w is incident with the infinite region of J. Since J is 2-connected, w is not incident with the infinite region of $G(K)$, and $d_{G(K)}(w) = d_J(w)$. It follows that $\gamma_K(w) = d_{G(K)}(w) = d_J(w) < \gamma_L(w)$, a contradiction which proves (6.2). $\square$

Let J be an enhancement of a good configuration L. For $v \in V(J)$ let $\xi(v) = \gamma_L(v)$ if $v \in V(G(L))$ and $\xi(v) = 0$ otherwise. A *query* for L is a quadruple $(u, v, z, \xi(z))$, where u, v, z are vertices of J forming a triangle in the clockwise order. A *question* for L is a sequence $Q = (Q_0, Q_1, \ldots, Q_n)$ such that for $i = 0, 1, \ldots, n$, $Q_i = (u_i, v_i, z_i, \xi(z_i))$ is a query for L such that $z_0, z_1, \ldots, z_n$ are pairwise distinct and make up $V(G(L))$, z_0 and z_1 are adjacent in $G(L)$, and for $i = 2, 3, \ldots, n$, $u_i, v_i \in \{z_0, z_1, \ldots, z_{i-1}\}$. If Q is a question as above we denote by $J(Q)$ the subdrawing of J consisting of all vertices of J, and those edges of J that belong to at least one of the triangles u_i, v_i, z_i. Let L, J, Q be as above, and let K be a configuration. We say that Q has a positive answer for K if there exists a 1-1 mapping $f : V(J) \to V(G(K))$ such that

(Q1) $f(z_0)$ is adjacent to $f(z_1)$,

(Q2) $\gamma_K(f(z_i)) = \xi(z_i)$ for all $i = 0, 1, \ldots, n$ with $\xi(z_i) > 0$,

(Q3) $f(u_i), f(v_i), f(z_i)$ form a triangle in $G(K)$ in the clockwise order for all $i = 2, 3, \ldots, n$.

From (6.2) we deduce

(6.3) *Let L be a good configuration, let Q be a question for L, and let A be an axle with* *skeleton K. If Q has a positive answer for K, then a configuration K_0 isomorphic to L is a subconfiguration of K. If K_0 is well-positioned, then K_0 is an induced subconfiguration of K, and hence A is semi-reducible.*

Let L be a good configuration, and let $Q = (u, v, z, d)$ be a query for L. We define Q^* to be (v, u, z, d). If $Q = (Q_0, Q_1, \ldots, Q_n)$ is a question for L we define Q^*, its *reflection*, to be $(Q_0, Q_1, Q_2^*, Q_3^*, \ldots, Q_n^*)$. Theorem (6.3) does have a converse, the following. We omit a proof, because the result is not needed.

(6.4) *Let L be a good configuration, let Q be a question for L, let A be an axle with skeleton K, and assume that L appears in K. Then Q or Q^* has a positive answer for K.*

To test semi-reducibility we first compute, for every good configuration L, a question for L. Then given an axle A we check if Q or Q^* has a positive answer for the skeleton K of A. If not, then we stop. Otherwise we compute K_0 as in (6.3) and check whether it is well-positioned in K.

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