REDUCIBILITY IN THE FOUR-COLOR THEOREM

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ABSTRACT

We give details of the "reducibility" part of our proof of the Four Color theorem [1].

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1. CONFIGURATIONS

We assume familiarity with [1]. The purpose of this manuscript is to provide more details about the proof of [1, theorem (3.2)]. As a first step we need to explain how configurations are stored.

Let R be a circuit with vertices $1, 2, \ldots, r$ in order, let $e_1, e_2, \ldots, e_r$ be the edges of R, in order and such that for $i = 2, 3, \ldots, r$, the ends of e_i are i and $i-1$. Let $\kappa, \lambda : E(R) \to \{-1, 0, 1\}$ be two colorings of R. We say that κ and λ are *similar* if $\{\kappa^{-1}(-1), \kappa^{-1}(0), \kappa^{-1}(1)\} = \{\lambda^{-1}(-1), \lambda^{-1}(0), \lambda^{-1}(1)\}$. We say that κ is *canonical* if either $\kappa(e) = 0$ for every edge e of R, or there exists an integer k such that $1 \le k < r$, $\kappa(e_r) = \kappa(e_{r-1}) = \cdots = \kappa(e_{k+1}) = 0$ and $\kappa(e_k) = 1$. Clearly every coloring of R is similar to a unique canonical coloring. The *code* of a coloring $\kappa : E(R) \to \{-1, 0, 1\}$ is $\sum_{i=1}^{r} \kappa'(e_i) 3^{i-1}$, where κ' is the canonical coloring similar to κ.

Let K be a configuration, let G be the free completion of K with ring R, and let either X be empty or a contract for K. Let the vertices of G be $1, 2, \ldots, n$, where $1, 2, \ldots, r$ are the vertices of R in clockwise order around the infinite region of G. Let $\mathcal{C}(K)$ be the set of all restrictions to $E(R)$ of tri-colorings of G, and let $\mathcal{C}'(K)$ be the maximal consistent subset of $\mathcal{C}^* - \mathcal{C}(K)$, where $\mathcal{C}^*$ is the set of all mappings of $E(R) \to \{-1, 0, 1\}$. A matrix $A = (a_{i,j})$ is a *configuration matrix of K* if

(i) for $i = 1, 2, \ldots, n$, $a_{i,0}$ is the degree of vertex i of G, and $a_{i,1}, a_{i,2}, \ldots, a_{i,a_{i,0}}$ are the neighbors of i listed in clockwise order as they appear around i; moreover, if $i \le r$ then $a_{i,1}$ and $a_{i,a_{i,0}}$ belong to $V(R)$,

(ii) $a_{0,0} = n$ and $a_{0,1} = r$,

(iii) $a_{0,2}$ and $a_{0,3}$ are the numbers of canonical colorings in $\mathcal{C}(K)$ and $\mathcal{C}'(K)$, respectively,

(iv) $a_{0,4} = |X|$, and $X = \{f_1, f_2, \ldots, f_k\}$, where $k = |X|$ and for $i = 1, 2, \ldots, k$, f_i has ends $a_{0,2i+3}$ and $a_{0,2i+4}$.

Each of the 633 good configurations is presented in terms of a configuration matrix, and so we need to verify that an input matrix is indeed a configuration matrix of some configuration. Let r and n be integers, let $A = (a_{i,j})$ be an integer matrix with rows corresponding to $i = 0, 1, \ldots, n$, and let us consider the following conditions, where for notational convenience we put $d_i = a_{i,0}$.

(1) $2 \le r < n$,

(2) $3 \le d_i \le n-1$ for all $i = 1, 2, \ldots, r$, and $5 \le d_i \le n-1$ for all $i = r+1, r+2, \ldots, n$.

(3) $1 \le a_{i,j} \le n$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, d_i$,

(4) if $i = 1, 2, \ldots, r$, then $a_{i,1} = i+1$ (or 1 if $i = r$), $a_{i,d_i} = i-1$ (or r if $i = 1$), and $r+1 \le a_{i,j} \le n$ for $j = 2, 3 \ldots, d_i - 1$,

(5) $d_1 + d_2 + \cdots + d_n = 6n - 6 - 2r$,

(6) for every $i = r+1, r+2, \ldots, n$ there exist at most two integers j such that $a_{i,j} > r$ and $a_{i,j+1} \le r$, and if there are two then $a_{i,j+2} > r$ for both such integers (where a_{i,d_i+1} and a_{i,d_i+2} mean $a_{i,1}$ and $a_{i,2}$, respectively), and

(7) let $i = 1, 2, \ldots, n$, let $j = 1, 2, \ldots, d_i - 1$ if $i \le r$ and $j = 1, 2, \ldots, d_i$ if $i > r$, and let $k = a_{i,j}$. Then there exists an integer p such that $a_{i,j+1} = a_{k,p}$ (or $a_{i,1} = a_{k,p}$ if $j = d_i$), and $i = a_{k,p+1}$ (or $i = a_{k,1}$ if $p = d_k$, in which case $k > r$).

(1.1) *Let r, n and $A = (a_{i,j})$ satisfy (1)–(7). Then there exist a configuration K and a free completion G of K with ring R such that (i) and (ii) hold.*

Proof. It is straightforward to construct a graph G with vertex-set $\{1, 2, \ldots, n\}$ such that for $i = 1, 2, \ldots, n$, the neighbors of i are $a_{i,1}, a_{i,2}, \ldots, a_{i,d_i}$. From (7) we deduce that the cyclic orderings $a_{i,1}, a_{i,2}, \ldots, a_{i,d_i}$ of the neighbors of i for $i = 1, 2, \ldots, n$ define an embedding of G into a surface Σ such that every face is a triangle, except one bounded by a circuit R. By (4) the vertices of R are $1, 2, \dots, r$ in order. From (5) we deduce by Euler's formula that Σ is the sphere, and so G may be regarded as a near-triangulation. Let K be defined by $G(K) = G \backslash V(C)$, and for $v \in V(K)$ let $\gamma_K(v) = d_G(v)$. We claim that K is a configuration. By (4) R is an induced circuit of G, and hence $G(K)$ is connected. Thus $G(K)$ is a near-triangulation, and we must verify conditions (i), (ii) and (iii) in the definition of a configuration. Condition (i) follows from (6), condition (ii) follows from (2), because every vertex on the infinite region of $G(K)$ is adjacent to a vertex of R, and (iii) follows from (1), because r is the ring-size of K. $\square$

2. EXTENDABLE COLORINGS

The objective of this section is to explain how we compute $\mathcal{C}(K)$. We compute all tri-colorings of G and record their restrictions to $E(R)$. We first number the edges of G as $e_1, e_2, \dots, e_m$, where $m = 3(n-1) - r$ and the edges of R are $e_1, e_2, \dots, e_r$. All that matters for the correctness is that e_m and e_{m-1} are on a common triangle. Using the algorithm below we compute all mappings $c : E(G) - E(R) \to \{1, 2, 4\}$ such that $c(e_m) = 1$, $c(e_{m-1}) = 2$ and $c(e) \neq c(f)$ if e and f are on a common triangle. During the course of the algorithm we maintain a variable F_i defined for $i < m-1$ as the set of all $c(e_j)$ such that $j > i$ and e_i and e_j are on a common triangle. At the begining we set $c(e_m) = 1$, $c(e_{m-1}) = 2$, $F_{m-1} = \{1, 4\}$ and $j = m-1$, and keep repeating steps 1, 2, 3 below.

Step 1. While $c(e_j) \in F_j$ we keep repeating steps (i) and (ii) below.

(i) Double $c(e_j)$, and

(ii) while $c(e_j) = 8$ repeat the following steps:

  (a) if $j \geq m-1$ terminate computation,

  (b) increase j by one and double $c(e_j)$.

Step 2. If $j = r+1$ then a tri-coloring of G can be read off from c. Record the code of its restriction to $E(R)$. Double $c(e_j)$ and while $c(e_j) = 8$ repeat steps (a) and (b) above.

Step 3. If $j > r+1$ decrease j by one, set $c(e_j) = 1$, and compute F_j.

We record the codes of restrictions of tri-colorings of G to $E(R)$ in an array called "live", so that for $i = 0, 1, \ldots, (3^{r-1}-1)/2$, live$[i] = 0$ if some (and hence every) coloring of $E(R)$ with code i is the restriction to $E(R)$ of a tri-coloring of G, and live$[i] = 1$ otherwise.

3. CONSISTENT SETS

We now explain how we compute $\mathcal{C}'(K)$. Let $K, G, R, \mathcal{C}^*$ be as in Section 1. We say that a coloring κ of R is *balanced* if $|\kappa^{-1}(-1)|$, $|\kappa^{-1}(0)|$, $|\kappa^{-1}(1)|$ and r all have the same parity. It is easy to see that every member of $\mathcal{C}'(K)$ is balanced. Let $M = \{(m_1, \mu_1), (m_2, \mu_2), \ldots, (m_k, \mu_k)\}$ be a signed matching in R. We say that M is *balanced* if $r - \sum_{i=1}^{k} (\mu_i - 1)/2$ is even. Let $\mathcal{C}_0 = \mathcal{C}^* - \mathcal{C}(K)$, and let $\mathcal{M}_0$ be the set of all balanced signed matchings in R. Let $i \geq 0$ be an integer, and assume that $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_i$ and $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_i$ have already been defined. Define $\mathcal{M}_{i+1}$ to be the set of all signed matchings $M \in \mathcal{M}_i$ such that $\mathcal{C}_i$ contains every coloring κ of R that θ–fits M for some $\theta \in \{-1, 0, 1\}$, and let $\mathcal{C}_{i+1}$ be the set of all colorings $\kappa \in \mathcal{C}_i$ such that for every $\theta \in \{-1, 0, 1\}$ there is a signed matching $M \in \mathcal{M}_{i+1}$ such that κ θ–fits M. We need the following proposition.

(3.1) *If $\mathcal{C}_i = \mathcal{C}_{i+1}$, then $\mathcal{C}'(K) = \mathcal{C}_i$.*

Proof. We first show that $\mathcal{C}_i$ is consistent. To this end we notice that $\mathcal{M}_{i+1} = \mathcal{M}_{i+2}$. Let $\kappa \in \mathcal{C}_i$, and let $\theta \in \{-1, 0, 1\}$. Since $\kappa \in \mathcal{C}_{i+1}$ we deduce that there exists a signed matching $M \in \mathcal{M}_{i+1}$ such that κ θ–fits M. Since $M \in \mathcal{M}_{i+2}$ by the above observation, if a coloring κ' θ–fits M, then $\kappa' \in \mathcal{C}_{i+1} = \mathcal{C}_i$, as desired.

To complete the proof we must show that $\mathcal{C}'(K) \subseteq \mathcal{C}_j$ for all $j = 0, 1, \ldots$. We proceed by induction. Clearly $\mathcal{C}'(K) \subseteq \mathcal{C}_0$. Assume now that $\mathcal{C}'(K) \subseteq \mathcal{C}_j$ for some integer $j \geq 0$; we wish to show that if $\kappa \in \mathcal{C}_j - \mathcal{C}_{j+1}$, then $\kappa \notin \mathcal{C}'(K)$. Let κ be as stated. Then there exists $\theta \in \{-1, 0, 1\}$ such that κ θ–fits no $M \in \mathcal{M}_{j+1}$. If κ θ–fits no signed matching in R then $\kappa \notin \mathcal{C}'(K)$, and so we may assume that κ θ–fits a signed matching M. If $M \notin \mathcal{M}_0$, then κ is unbalanced, and hence $\kappa \notin \mathcal{C}'(K)$. We may therefore assume that $M \in \mathcal{M}_0$. Then $M \in \mathcal{M}_k - \mathcal{M}_{k+1}$ for some integer k with $0 \leq k \leq j$. Hence there exists $\theta' \in \{-1, 0, 1\}$ and a coloring $\kappa' \notin \mathcal{C}_k$ θ'-fitting M. By replacing κ' by a similar coloring if necessary we may assume that $\theta = \theta'$. Since $\kappa' \notin \mathcal{C}'(K)$ by the induction hypothesis, we deduce that $\kappa \notin \mathcal{C}'(K)$, as desired. □

To compute $\mathcal{C}'(K)$ we iteratively compute $\mathcal{M}_i$ and $\mathcal{C}_i$ until $\mathcal{C}_i = \mathcal{C}_{i+1}$. Instead of $\mathcal{C}_i$ we compute the codes of members of $\mathcal{C}_i$, and store this information by updating the array "live". To complete the description we need to explain how we store and compute $\mathcal{M}_i$. To this end we need the following definitions. Let $M = \{(m_1, \mu_1), (m_2, \mu_2), \ldots, (m_k, \mu_k)\}$ be a signed matching in R, where for $i = 1, 2, \ldots, k$, $m_i = \{a_i, b_i\}$, $b_i < a_i$ and $a_1 = \max\{a_1, a_2, \ldots, a_k\}$. We define the *code* of M to be $\sum_{i=1}^{k} \left(3^{a_i-1} + \mu_i 3^{b_i-1}\right)$ if $a_1 < r$, and $(3^r - 1)/2 - \sum_{i=1}^{k} \left(3^{a_i-1} + (3-\mu_i)3^{b_i-1}/2\right)$ otherwise. We define $(h_2, h_3, \ldots, h_k)$, the *choice sequence of* M, by $h_i = 2(3^{a_i-1} + \mu_i 3^{b_i-1})$ if $a_1 < r$ and $h_i = 3^{a_i-1} + \mu_i 3^{b_i-1}$ otherwise. The following is straightforward to verify.

(3.2) *Let M be a signed matching with code c and choice sequence $(h_2, h_3, \ldots, h_k)$. Then $\left\{\left|c + \sum_{i=2}^{k} \epsilon_i h_i\right| : \epsilon_i \in \{0, 1\}\right\}$ is the set of codes of all colorings of R that θ–fit M for some $\theta \in \{-1, 0, 1\}$. Moreover, let a_1 be as above, and let κ be a canonical coloring θ–fitting M for some $\theta \in \{-1, 0, 1\}$. Let the code of κ be $d = c + \sum_{i=2}^{k} \epsilon_i h_i$, where $\epsilon_i \in \{0, 1\}$ for $i = 2, 3, \ldots, k$. If $a_1 < r$ then $\theta = 0$. If $a_1 = r$ then $\theta = 1$ if $d < 0$, and $\theta = -1$ if $d > 0$.*

The function "augment" generates all the members of $\mathcal{M}_0$ always in the same order, say $M_1, M_2, \dots, M_p$. We use the bits of the array "real" to store $\mathcal{M}_i$; that is, after the ith iteration the jth bit of "real" is 1 if and only if $M_j \in \mathcal{M}_i$. To update "real" we run through all bits of "real" that are currently set to 1, generate all the codes of colorings as in (3.2), and if for some of them the corresponding entry in "live" is zero, we set the current bit of "real" to zero. Also, if none of the corresponding entries of "live" are zero, we mark each such entry by θ, where θ is as in the second half of (3.2). To update "live" (that is, to compute $\mathcal{C}_{i+1}$ from $\mathcal{C}_i$) we run through all nonzero entries of "live" and set to zero all those that were not marked by every $\theta \in \{-1, 0, 1\}$ (except live[0], which is exceptional).

4. CONTRACTS

It remains to explain how we verify that a proposed contract is indeed a contract. Let K, G, R be as in Section 1, and let X be a contract for K as specified by condition (iv) in the definition of a configuration matrix. Most of the conditions in the definition of a contract are straightforward to verify, and so we only explain how we check that no coloring in $\mathcal{C}'(K)$ is the restriction to $E(R)$ of a tri-coloring of G modulo X. We do that by computing all tri-colorings of G modulo X using an algorithm similar to the one described in Section 2. More precisely, the algorithm runs as follows. Let $e_1, e_2, \dots, e_m$ be the numbering of $E(G)$ described in Section 2. We compute a mapping $c : E(G) - X \to \{1, 2, 4\}$ such that c' defined by $c'(e) = \lfloor c(e)/2 \rfloor - 1$ is a tri-coloring of G modulo X. For an integer i with $1 \le i \le m$ and $e_i \notin X$ let D_i be the set of all e_j, where $i < j \le m$ and e_i, e_j belong to a triangle none of whose edges belong to X. Let S_i be the set of all e_j, where $i < j \le m$, $e_j \notin X$, and e_i, e_j belong to a triangle whose third edge belongs to X. Let s be the maximum integer with $s \le m$ and $e_s \notin X$, and let s' be the maximum integer with $s' < s$ and $e_{s'} \notin X$. During the course of the algorithm we maintain a variable F_i defined for $i < s'$ as $\{c(f) \mid f \in D_i\} \cup \bigcup \{\{1, 2, 4\} - \{c(f)\} \mid f \in S_i\}$. At the begining we set $c(e_s) = 1$,

$c(e_{s'}) = 1$, $F_{s'} = \{4\} \cup \{c(f) \mid f \in D_{s'}\} \cup \bigcup \{\{1,2,4\} - \{c(f)\} \mid f \in S_{s'}\}$ and $j = s-1$, and keep repeating steps 1, 2, 3 below.

Step 1. While $c(e_j) \in F_j$ we keep repeating steps (i) and (ii) below.

(i) Double $c(e_j)$, and

(ii) while $c(e_j) = 8$ repeat the following steps:

  (a) set j to the smallest j' with $s \geq j' > j$ and $e_{j'} \notin X$ (or $s+1$ if no such j' exists),

  (b) if $j \geq s$ terminate computation; otherwise double $c(e_j)$.

Step 2. If $j = 1$ then c can be converted to a tri-coloring of G modulo X. Verify that its restriction to $E(R)$ does not belong to $\mathcal{C}'(K)$. Double $c(e_j)$ and while $c(e_j) = 8$ repeat steps (a) and (b) above.

Step 3. If $j > 1$ decrease j by one, set $c(e_j) = 1$, and compute F_j.

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