Three-coloring triangle-free graphs on surfaces II. 4-critical graphs in a disk*

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Abstract

Let G be a plane graph of girth at least five. We show that if there exists a 3-coloring ϕ of a cycle C of G that does not extend to a 3-coloring of G, then G has a subgraph H on $O(|C|)$ vertices that also has no 3-coloring extending ϕ. This is asymptotically best possible and improves a previous bound of Thomassen. In the next paper of the series we will use this result and the attendant theory to prove a generalization to graphs on surfaces with several precolored cycles.

1 Introduction

This paper is a part of a series aimed at studying the 3-colorability of graphs on a fixed surface that are either triangle-free, or have their triangles restricted in some way. Historically the first result in this direction is the following classical theorem of Grötzsch [6].

Theorem 1.1. *Every triangle-free planar graph is* 3*-colorable.*

Thomassen [10, 11, 13] found three reasonably simple proofs. Recently, two of us, in joint work with Kawarabayashi [3] were able to design a linear-time algorithm to 3-color triangle-free planar graphs, and as a by-product found perhaps a yet simpler proof of Theorem 1.1. Another significantly different proof was given by Kostochka and Yancey [8].

The statement of Theorem 1.1 cannot be directly extended to any surface other than the sphere. In fact, for every non-planar surface Σ there are infinitely

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many 4-critical triangle-free graphs that can be drawn in Σ. (A graph is 4-*critical* if it is not 3-colorable, but every proper subgraph is.) For instance, the graphs obtained from an odd cycle of length five or more by applying Mycielski's construction [1, Section 8.5] have that property. Thus an algorithm for testing 3-colorability of triangle-free graphs on a fixed surface will have to involve more than just testing the presence of finitely many obstructions.

The situation is different for graphs of girth at least five by another deep theorem of Thomassen [12], the following.

Theorem 1.2. *For every surface Σ there are only finitely many 4-critical graphs of girth at least five that can be drawn in Σ.*

Thus the 3-colorability problem on a fixed surface has a polynomial-time algorithm for graphs of girth at least five, but the presence of cycles of length four complicates matters. Let us remark that there are no 4-critical graphs of girth at least five on the projective plane and the torus [10] and on the Klein bottle [9].

The only non-planar surface for which the 3-colorability problem for triangle-free graphs is fully characterized is the projective plane. Building on earlier work of Youngs [15], Gimbel and Thomassen [5] obtained the following elegant characterization. A graph drawn in a surface is a *quadrangulation* if every face is bounded by a cycle of length four.

Theorem 1.3. *A triangle-free graph drawn in the projective plane is 3-colorable if and only if it has no subgraph isomorphic to a non-bipartite quadrangulation of the projective plane.*

For other surfaces there does not seem to be a similarly nice characterization. Gimbel and Thomassen [5, Problem 3] asked whether there is a polynomial-time algorithm to test the 3-colorability of triangle-free graphs embeddable in a fixed surface. In a later paper of this series we will resolve this question in the affirmative. The algorithm naturally breaks into two steps. The first is when the graph is a quadrangulation, except perhaps for a bounded number of larger faces of bounded size, which will be allowed to be precolored. In this case there is a simple topological obstruction to the existence of a coloring extension based on the so-called "winding number" of the precoloring. Conversely, if the obstruction is not present and the graph is highly "locally planar", then we can show that the precoloring can be extended to a 3-coloring of the entire graph. This can be exploited to design a polynomial-time algorithm. With additional effort the algorithm can be made to run in linear time.

The second step covers the remaining case, when the graph has either many faces of size at least five, or one large face, and the same holds for every subgraph. In that case we show that the graph is 3-colorable. That is a consequence of the following theorem, which will form the cornerstone of this series.

Theorem 1.4. *There exists an absolute constant K with the following property. Let G be a graph drawn in a surface Σ of Euler genus γ with no separating cycles of length at most four, and let t be the number triangles in G. If G is 4-critical,* *then* $\sum |f| \leq K(t+\gamma)$*, where the summation is over all faces* f *of* G *of length at least five.*

If G has girth at least five, then $t = 0$ and every face has length at least five. Thus Theorem 1.4 implies Theorem 1.2, and, in fact, improves the bound given by the proof of Theorem 1.2 in [12]. The fact that our bound in Theorem 1.4 is linear in the number of triangles is needed in our solution [4] of a problem of Havel [7], as follows.

Theorem 1.5. *There exists an absolute constant* d *such that if* G *is a planar graph and every two distinct triangles in* G *are at distance at least* d*, then* G *is* 3*-colorable.*

Our technique is a refinement of the standard method of reducible configurations. We show that every sufficiently generic graph G (i.e., a graph that is large enough and cannot be decomposed to smaller pieces along cuts simplifying the problem) embedded in a surface contains one of a fixed list of subgraphs. Each such configuration enables us to obtain a smaller 4-critical graph G' with the property that every 3-coloring of G' corresponds to a 3-coloring of G. Furthermore, we perform the reduction in such a way that a properly defined weight of G' is greater or equal to the weight of G. A standard inductive argument then shows that the weight of every 4-critical graph is bounded, which also restricts its size. Unfortunately, this brief exposition hides a large number of technical details that need to be dealt with.

In this paper, we introduce this basic technique and apply it to prove the following special case of Theorem 1.4.

Theorem 1.6. *Let* G *be a graph of girth at least five drawn in the plane, let* C *be a cycle in* G*, and let* ϕ *be a* 3*-coloring of* C *that does not extend to a* 3*-coloring of* G*. Then there exists a subgraph* H *of* G *such that* C *is a subgraph of* H*,* $|V(H)| \leq 1715|V(C)|$ *and* H *has no* 3*-coloring extending* ϕ*.*

After we obtained a proof of Theorem 1.6, but before we wrote it down and made it public, the first author and Kawarabayashi [2] generalized Theorem 1.6 to list-coloring. Their proof is about as long as ours, but has the added advantage that it replaces 1715 by a much smaller constant. However, we are proceeding with publication of our paper, because we need the theory it develops for the proof of Theorem 1.4 for graphs of girth at least five, which will appear in the next paper of our series. Whether this result (for graphs of girth at least five) can be extended to list-coloring remains a very interesting open problem. An affirmative answer would be implied by the following conjecture.

Conjecture 1.7. *There exists a constant* K *with the following property. Let* G *be a planar graph of girth at least five, let* C_1, C_2 *be two cycles in* G*, and for every* $v \in V(G)$ *let* $L(v)$ *be a non-empty set such that* $|L(v)| \geq 3$ *for every* $v \in V(G) - V(C_1) - V(C_2)$*. If there exists no proper coloring* ϕ *of* G *such that* $\phi(v) \in L(v)$ *for every* $v \in V(G)$*, then* G *has a subgraph* H *on at most* K *vertices such that* C_1 *and* C_2 *are subgraphs of* H *and there exists no proper coloring* ψ *of* H *such that* $\phi(v) \in L(v)$ *for every* $v \in V(H)$*.*

In order to avoid duplication of work in the next paper of the series we state many of the auxiliary results in this paper in the more general setting of graphs on surfaces. For this purpose, we require some definitions introduced in the following section. In Section 3, we describe more precisely what we mean by a reducible configuration, its appearance in the considered graph and its reduction. In Section 4, we show that the reductions preserve 3-colorings. In Section 5, we give the discharging argument used to show the existence of a reducible configuration. In Section 6, we argue that the reductions preserve the assumptions of the theorem. In Section 7, we analyze the change of the weights during the reduction. Finally, in Section 8, we combine the results to prove Theorem 1.6.

2 Definitions

All graphs in this paper are finite and simple, with no loops or parallel edges.

A *surface* is a compact connected 2-manifold with (possibly null) boundary. Each component of the boundary is homeomorphic to the circle, and we call it a *cuff*. For non-negative integers a, b and c, let $\Sigma(a, b, c)$ denote the surface obtained from the sphere by adding a handles, b crosscaps and removing the interiors of c pairwise disjoint closed discs. A standard result in topology shows that every connected surface is homeomorphic to $\Sigma(a, b, c)$ for some choice of a, b and c. Note that $\Sigma(0, 0, 0)$ is a sphere, $\Sigma(0, 0, 1)$ is a closed disk, $\Sigma(0, 0, 2)$ is a cylinder, $\Sigma(1, 0, 0)$ is a torus, $\Sigma(0, 1, 0)$ is a projective plane and $\Sigma(0, 2, 0)$ is a Klein bottle. The *Euler genus* $g(\Sigma)$ of the surface $\Sigma = \Sigma(a, b, c)$ is defined as $2a + b$. For a cuff C of Σ, let $\hat{C}$ denote an open disk with boundary C disjoint from Σ, and let $\Sigma + \hat{C}$ be the surface obtained by gluing Σ and $\hat{C}$ together, that is, by closing C with a patch. Let $\hat{\Sigma} = \Sigma + \hat{C}_1 + \ldots + \hat{C}_c$, where $C_1, \ldots, C_c$ are the cuffs of Σ, be the surface without boundary obtained from Σ by patching all the cuffs.

Consider a graph G embedded in the surface Σ; when useful, we identify G with the topological space consisting of the points corresponding to the vertices of G and the simple curves corresponding to the edges of G. A *face* f of G is a maximal arcwise-connected subset of $\hat{\Sigma} - G$. The boundary of a face is equal to a union of closed walks of G, which we call the *boundary walks* of f.

A *ring* in a graph G is a subgraph of G that either is a cycle or has one vertex and no edges. If $v \in V(G)$, then by the ring v we will mean the ring with vertex-set $\{v\}$. An embedding of G in Σ is *normal* if each cuff C that intersects G either does so in exactly one vertex v or is equal to a cycle B in G. In the former case, we call v a *vertex ring* and the face of G that contains C the *cuff face* of v. In the latter case, note that B is the boundary walk of the face $\hat{C}$ of G; we say that B is a *facial ring*. A face of G is a *ring face* if it is equal $\hat{C}$ for some ring C, and *internal* otherwise. Thus every cuff face is internal. We write $F(G)$ for the set of internal faces of G. A vertex v of G is a *ring vertex* if v is incident with a ring (i.e., it is drawn in the boundary of Σ), and *internal* otherwise. A cycle K in G is *separating* or *separates the surface* if $\hat{\Sigma} - K$ has at least two components, and K is *non-separating* otherwise. A cycle K is *contractible* if there exists a closed disk $\Delta \subseteq \Sigma$ with boundary equal to K. A cycle K *surrounds the cuff* C if K is not contractible in Σ, but it is contractible in $\Sigma + \hat{C}$. We say that K *surrounds a ring* R if K surrounds the cuff incident with R.

Let G be a graph embedded in a surface Σ, let the embedding be normal, and let $\mathcal{R}$ be the set of vertex rings and facial rings of this embedding. In those circumstances we say that G is a *graph in* Σ *with rings* $\mathcal{R}$. Furthermore, some vertex rings are designated as *weak vertex rings*. At this point, let us remark that weak vertex rings are a technical device designed to deal with cutvertices in Theorem 1.4. They will not play any role in this paper, but we need to introduce them in order to be able to formulate the lemmas in this paper in such a way that they can be applied in the proof of Theorem 1.4.

The length $|R|$ of a facial ring is the length of the corresponding face. For a vertex ring R, we define $|R| = 0$ if R is weak and $|R| = 1$ otherwise. For an internal face f, by $|f|$ we mean the sum of the lengths of the boundary walks of f (in particular, if an edge appears twice in the boundary walks, it contributes 2 to $|f|$); if a boundary walk consists just of a vertex ring R (in which case R is an isolated vertex), it contributes $|R|$ to $|f|$.

Let G be a graph with rings $\mathcal{R}$. A *precoloring* ψ of $\mathcal{R}$ is a 3-coloring of the graph $H = \bigcup \mathcal{R}$. Note that H is a (not necessarily induced) subgraph of G. A precoloring of $\mathcal{R}$ *extends to a* 3*-coloring of* G if there exists a 3-coloring ϕ of G such that $\phi(v) \neq \psi(v)$ for every weak vertex ring v and $\phi(v) = \psi(v)$ for every other vertex v incident with one of the rings. The graph G is $\mathcal{R}$*-critical* if $G \neq H$ and for every proper subgraph G' of G that contains $\mathcal{R}$, there exists a precoloring of $\mathcal{R}$ that extends to a 3-coloring of G', but not to a 3-coloring of G. For a precoloring ϕ of the rings, the graph G is ϕ*-critical* if ϕ does not extend to a 3-coloring of G, but it extends to a 3-coloring of every proper subgraph of G that contains $\mathcal{R}$. Let us remark that if G is ϕ-critical for some ϕ, then it is $\mathcal{R}$-critical, but the converse is not true (for example, consider a graph consisting of a single facial ring with two chords). On the other hand, if ϕ is a precoloring of the rings of G that does not extend to a 3-coloring of G, then G contains a (not necessarily unique) ϕ-critical subgraph.

3 Reducible configurations

By a plane graph we mean a graph G drawn in the plane with no crossings. Thus G has exactly one unbounded face, called the *infinite face*; all the other faces are called *finite*. An *isomorphism* of plane graphs maps finite faces to finite faces and the infinite face to the infinite face.

A *configuration* is a quintuple $\gamma = (G, \mathcal{F}, d, \mathcal{I}, \mathcal{A})$, where

- G is a plane graph,
- $\mathcal{F}$ is a set of finite faces of G,
- d is a function that maps a set $\mathrm{dom}(d) \subseteq V(G)$ to $\{3, 4, \ldots\}$,
- $\mathcal{A}$ is a subset of $V(G)$ of size zero or two, and

- $\mathcal{I}$ is a subset of $V(G)$.

If γ is a configuration, then we define $G_\gamma := G$, $\mathcal{F}_\gamma := \mathcal{F}$, $d_\gamma := d$, $\mathcal{I}_\gamma := \mathcal{I}$ and $\mathcal{A}_\gamma := \mathcal{A}$.

Two configurations γ and γ' are *isomorphic* if there exists an isomorphism ϕ of the plane graphs G_γ and $G_{\gamma'}$ that maps $\mathcal{F}_\gamma$ to $\mathcal{F}_{\gamma'}$, $\mathcal{I}_\gamma$ to $\mathcal{I}_{\gamma'}$, $\mathcal{A}_\gamma$ to $\mathcal{A}_{\gamma'}$, $\mathrm{dom}(d_\gamma)$ to $\mathrm{dom}(d_{\gamma'})$ and $d_\gamma(v) = d_{\gamma'}(\phi(v))$ for every $v \in \mathrm{dom}(d_\gamma)$. Figure 1 contains the depictions of several configurations, using the following conventions. The graph G_γ is drawn in the figure (ignoring the "half-edges" and dashed edges for a moment); $\mathcal{F}_\gamma$ consists of all the finite faces of G_γ that do not include any half-edges in their interior; the elements of $\mathcal{I}_\gamma$ are indicated by $\mathcal{I}$ next to them; if $\mathcal{A}_\gamma$ is non-empty, then the two vertices of $\mathcal{A}_\gamma$ are joined by a dashed edge; the set $\mathrm{dom}(d_\gamma)$ consists of vertices drawn by empty circles; and the value $d_\gamma(v)$ is equal to the number of edges and half-edges incident with v in the figure. A configuration is *good* if it is isomorphic to one of the configurations depicted in Figure 1.

Let γ be a good configuration and either let $H = G_\gamma$, or let H be a plane graph obtained from G_γ by identifying two vertices of $V(G_\gamma) \setminus \mathrm{dom}(d_\gamma)$ that are at distance at least five in G_γ. (The latter is only possible when γ is R7 or R7$''$.) In those circumstances we say that H is an *imprint* of γ. It follows that every face in $\mathcal{F}_\gamma$ may be regarded as a face of H, and that $\mathrm{dom}(d_\gamma) \subseteq V(H)$.

Let G be a graph in a surface Σ with rings $\mathcal{R}$. We say that a configuration γ *faintly appears* in G if

- some imprint H of γ is a subgraph of G,
- every face in $\mathcal{F}_\gamma$ is an internal face of G,
- $\mathrm{dom}(d_\gamma) \cap V(\mathcal{R}) = \emptyset$,
- if $v \in \mathrm{dom}(d_\gamma)$, then $\deg_G(v) = d_\gamma(v)$,
- at most one vertex of $\mathcal{I}_\gamma$ belongs to $V(\mathcal{R})$, and
- no cuff face of a vertex ring belongs to $\mathcal{F}_\gamma$.

If a configuration γ faintly appears in G, then we say that a subgraph J of G *touches* γ if an edge of J is incident with a face in $\mathcal{F}_\gamma$. We say that γ *weakly appears* in G if it faintly appears and

- no cycle of length at most four distinct from facial rings touches γ and if γ is R7, then $x_3 \neq x_7$ or $x_1 \neq x_6$,
- if $u, v \in \mathrm{dom}(d_\gamma)$ are adjacent in G, then u, v are adjacent in G_γ,
- if γ is isomorphic to R4 and the vertices corresponding to x_4 and x_5 both

belong to $\mathcal{R}$, then the vertex corresponding to v_2 does not belong to $\mathcal{R}$.

Let a good configuration γ weakly appear in G. We wish to define a new graph G' in Σ with rings $\mathcal{R}$. For the definition we need to distinguish several cases. Assume first that γ is not isomorphic to R4. Let the graph G' be obtained from $G \backslash \mathrm{dom}(d_\gamma)$ by adding an edge joining the vertices in $\mathcal{A}_\gamma$ if $\mathcal{A}_\gamma \neq \emptyset$, and by identifying the vertices in $\mathcal{I}_\gamma$. If parallel edges are created, remove all edges but one from each bunch of parallel edges, so that each edge of G' corresponds to a unique edge of G. Since no cycle of length at most four touches γ and if γ is R7, then $x_3 \neq x_7$ or $x_1 \neq x_6$, it follows that G' has no loops. It also follows that $\mathcal{R}$ is a set of rings for G'. We will refer to the added edge as the *new edge* and to the vertex that resulted from the identification of vertices as the *new vertex*.

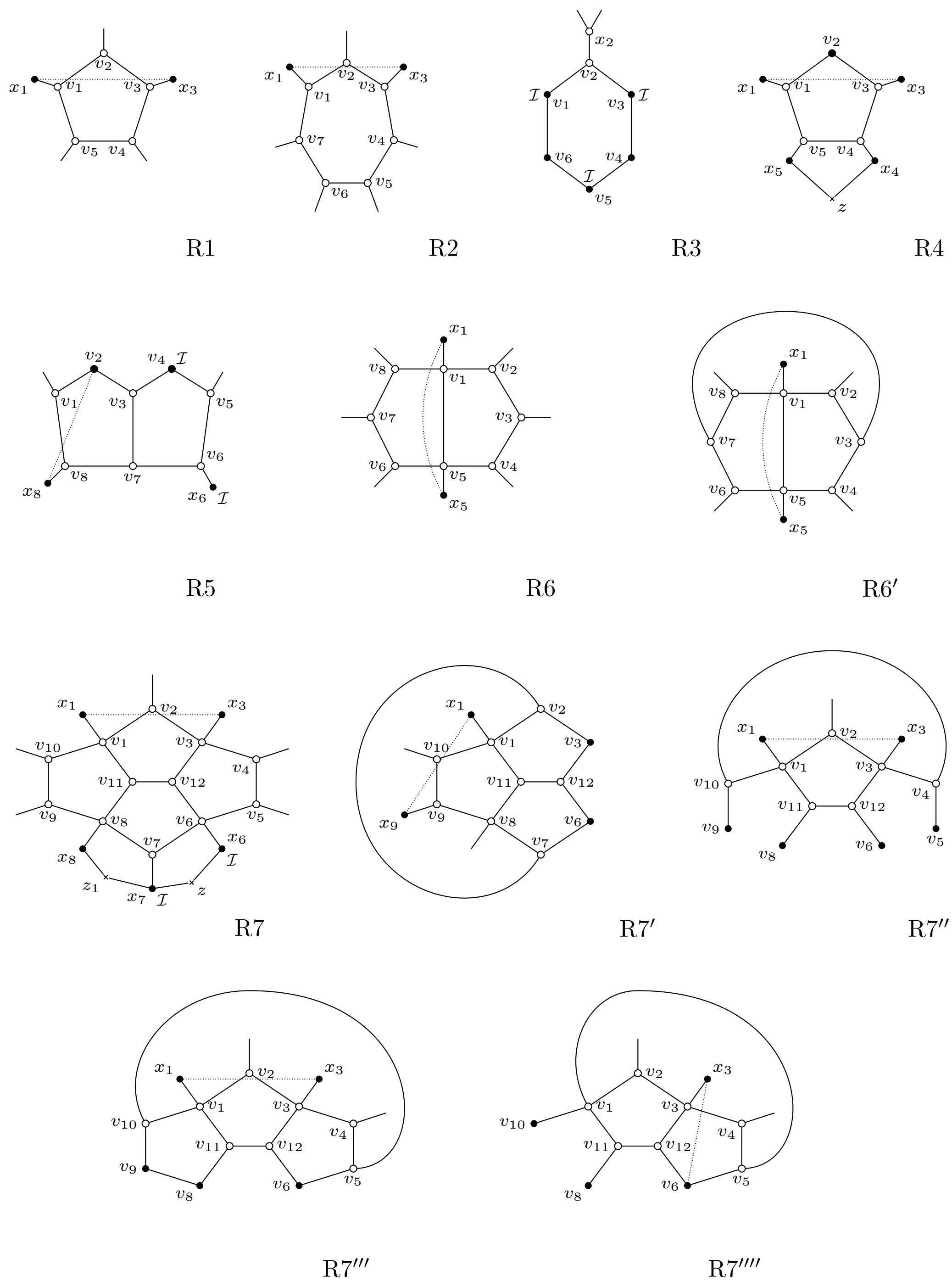

Figure 1: Reducible configurations.

If two vertices $u, v \in \mathcal{I}_\gamma$ have a common neighbor $x \in V(G_\gamma) \setminus \mathrm{dom}(d_\gamma)$ and w is the new vertex arising by identification of u and v, then we call the edge wx *squashed*.

We also need to specify an embedding of G' in Σ. There is a unique natural way to make the edge additions and vertex identifications inside the faces of $\mathcal{F}_\gamma$, and that is how the embedding of G' will be defined. Formally, for every pair $u, v \in \mathcal{A}_\gamma$ and every pair $u, v \in \mathcal{I}_\gamma$ of distinct vertices we define the *replacement* u, v*-path* as the shortest path from u to v in G_γ. It follows by inspecting all the good configurations that the replacement path is unique. Now we identify u and v or join them by an edge along the replacement u, v-path P, with the proviso that if P includes a vertex $v \in V(G_\gamma) \setminus \mathrm{dom}(d_\gamma)$ (specifically, vertex v_4 or v_6 of R3 or vertex z of R7), then prior to making the edge addition or vertex identification we shift P slightly into the unique face f of $\mathcal{F}_\gamma$ incident with v. By the conditions of weak appearance, f is not a cuff face of v, and hence P stays in Σ and its homotopy does not change by such a shift. This completes the definition of G' when γ is not R4.

Now let γ be R4. If not both x_4 and x_5 belong to $\mathcal{R}$, then we proceed as above, treating the configuration as if $\{x_4, x_5\}$ belonged to $\mathcal{I}_\gamma$; that is, identifying those vertices. We may therefore assume that both x_4, x_5 belong to $\mathcal{R}$. Let ϕ be a 3-coloring of $\mathcal{R}$; the definition of G' will now depend on ϕ. If $\phi(x_4) = \phi(x_5)$, then we define G' exactly as in the previous two paragraphs; in particular, we do not identify x_4 and x_5. If $\phi(x_4) \neq \phi(x_5)$, then we let G' be obtained from $G \setminus \{v_1, v_3, v_4, v_5\}$ by identifying v_2 and x_5 along the "replacement path" $v_2 v_1 v_5 x_5$ (we do not add the edge between $x_1, x_3 \in \mathcal{A}_\gamma$). Let us remark that the last condition in the definition of weak appearance guarantees that in this case v_2 does not belong to $\mathcal{R}$. Then G' is a graph in Σ with rings $\mathcal{R}$, and we say that it is the γ*-reduction of* G. When we wish to emphasize the dependence on ϕ we will say that G' is the γ*-reduction of* G *with respect to* ϕ.

4 Colorings

In this section, we show that each 3-coloring of the γ-reduction of a graph G extends to a 3-coloring of G. Most of the reductions were used earlier [6, 10], but R5, R7 and their variants seem to be new. For the sake of completeness we include proofs of extendability for all good configurations.

Lemma 4.1. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, let γ be a good configuration that weakly appears in G, let ϕ_0 be a 3-coloring of $\mathcal{R}$, and let G_1 be the γ-reduction of G with respect to ϕ_0. If ϕ_0 extends to a 3-coloring of G_1, then it extends to a 3-coloring of G.*

Proof. Let γ be as stated, and let the vertices of G_γ be labeled as in Figure 1. Let ϕ be a 3-coloring of G_1 that extends the coloring ϕ_0. Then ϕ can be regarded as a 3-coloring of $G \setminus \mathrm{dom}(d_\gamma)$, and our objective is to extend it to a 3-coloring of G. For each vertex $v_i \in \mathrm{dom}(d_\gamma)$ that has a unique neighbor outside of the configuration, let this neighbor be denoted by x_i. We will use the following easy observations:

(1) *Suppose that $u_1, u_2 \in V(G)$ are adjacent vertices of degree three, w_1 and w_2 are the neighbors of u_1 distinct from u_2 and w_3 and w_4 are the neighbors of u_2 distinct from u_1. A 3-coloring ψ of $w_1, \ldots, w_4$ extends to u_1 and u_2, unless $\psi(w_1) = \psi(w_3) \neq \psi(w_2) = \psi(w_4)$ or $\psi(w_1) = \psi(w_4) \neq \psi(w_2) = \psi(w_3)$.*

(2) *Let $P = u_1u_2 \ldots u_k$ be a path in G and $L_1, \ldots, L_k$ lists of colors of size two, such that $L_i \neq L_j$ for some $1 \leq i < j \leq k$. Then there exist colorings ψ_1, ψ_2 and ψ_3 of P such that $\psi_i(v_j) \in L_j$ for $1 \leq i \leq 3$ and $1 \leq j \leq k$, and for each $1 \leq i < j \leq 3$ either $\psi_i(u_1) \neq \psi_j(u_1)$ or $\psi_i(u_k) \neq \psi_j(u_k)$.*

Let us now consider each configuration separately.

Configurations R1 **and** R2**.** Each of the vertices of the cycle $v_1v_2 \ldots v_k$ (where $k = 5$ for the configuration R1 and $k = 7$ for R2) has a list of two available colors, and the lists of v_1 and v_3 are not the same. By (2), there exists a coloring of the path $v_1 \ldots v_k$ from these lists such that the colors of v_1 and v_k are not the same, giving a coloring of G, as desired.

Configuration R3**.** The vertices v_1, v_3 and v_5 inherit the color of the new vertex. Then we can color the vertices x_2 and v_2 in order, because at the time each of those vertices is colored it is adjacent to vertices of at most two different colors.

Configuration R4**.** Suppose first that at least one of x_4 and x_5 is internal, or that both belong to $\mathcal{R}$ and $\phi_0(x_4) = \phi_0(x_5)$. If $\phi(x_1) = \phi(v_2)$, then color the vertices in the order v_3, v_4, v_5 and v_1 (each of them has neighbors of at most two different colors when it is being colored). The case that $\phi(x_3) = \phi(v_2)$ is symmetric. Therefore, we may assume that $\phi(x_1) = 1$, $\phi(v_2) = 2$ and $\phi(x_3) = 3$. Set $\phi(v_1) = 3$ and $\phi(v_3) = 1$ and extend the coloring to v_4 and v_5 by (1). Then ϕ is a desired 3-coloring of G.

We may therefore assume that both x_4 and x_5 belong to $\mathcal{R}$ and $\phi_0(x_4) \neq \phi_0(x_5)$. In this case, the definition of γ-reduction ensures that $\phi(v_2) = \phi(x_5)$. We may assume that $\phi(v_2) = \phi(x_5) = 1$ and $\phi(x_4) = 2$. Let us set $\phi(v_4) = 1$ and color v_3, v_1 and v_5 in this order.

Configuration R5**.** The reduction ensures that $\phi(v_2) \neq \phi(x_8)$ and $\phi(v_4) = \phi(x_6)$. If $\phi(v_2) = \phi(v_4)$, then ϕ extends—color the vertices in the order v_1, v_8, v_5, v_6, v_7 and v_3, and observe that for each of these vertices, at most two different colors appear on already colored neighbors. Thus we may assume that $\phi(v_2) = 1$ and $\phi(v_4) = \phi(x_6) = 2$. We set $\phi(v_3) = 3$ and $\phi(v_7) = 2$, and color the vertices v_5 and v_6 by (1). As $\phi(x_8) \neq \phi(v_2) \neq \phi(v_7)$, (1) implies that the coloring extends to v_1 and v_8.

Configurations R6 **and** R6′**.** In both cases, the reduction ensures that $\phi(x_1) \neq \phi(x_5)$, say $\phi(x_1) = 1$ and $\phi(x_5) = 2$. If $\phi(x_6) = 1$, then set $\phi(v_5) = 1$, and color the vertices in order v_4, v_3, v_2, v_1, v_8, v_7 and v_6. Therefore, we may assume that this is not the case. By symmetry, we may also assume that $\phi(x_4) \neq 1$ and $\phi(x_2), \phi(x_8) \neq 2$. If $\phi(x_2) = \phi(x_8) = 3$, then set $\phi(v_1) = 3$, $\phi(v_5) = 1$ and color v_6, v_7, v_8, v_4, v_3 and v_2 in this order. Otherwise, by symmetry we may assume that $\phi(x_2) = 1$. If v_3 and v_7 are adjacent, or if $\phi(x_3) \neq 1$, then set $\phi(v_3) = \phi(v_5) = 1$ and color v_4, v_6, v_7, v_8, v_1 and v_2 in this order. Therefore, assume that v_3 and v_7 are not adjacent and $\phi(x_3) = 1$.

If $\phi(x_6) = 3$, then set $\phi(v_4) = 1$, $\phi(v_1) = \phi(v_3) = 2$ and $\phi(v_2) = \phi(v_5) = 3$ and color v_8, v_7 and v_6 in this order. Thus, assume that $\phi(x_6) = 2$. By the argument symmetrical to the one used for x_3, we conclude that ϕ extends unless $\phi(x_7) = 2$. If $\phi(x_8) = 3$, then set $\phi(v_4) = \phi(v_6) = \phi(v_8) = 1$, $\phi(v_1) = \phi(v_3) = 2$ and $\phi(v_2) = \phi(v_5) = \phi(v_7) = 3$. Thus assume that $\phi(x_8) = 1$ and by symmetry, $\phi(x_4) = 2$. In this case, set $\phi(v_5) = \phi(v_7) = 1$, $\phi(v_1) = \phi(v_3) = 2$ and $\phi(v_2) = \phi(v_4) = \phi(v_6) = \phi(v_8) = 3$.

Configuration R7**.** The reduction ensures that $\phi(x_1) \neq \phi(x_3)$, say $\phi(x_1) = 1$ and $\phi(x_3) = 2$. To preserve the symmetry of the configuration, let us for a while ignore the identification of x_6 and x_7.

Suppose first that $\phi(x_8) = 1$. By (2), there exists a coloring ψ of the path $v_1 v_2 \ldots v_8$ such that $\psi(v_1) = \psi(v_8) \in \{2, 3\}$. We can extend ψ to v_{12} and v_{11}. By (1), if $\phi(x_9) \neq \phi(x_{10})$, then ψ extends to v_9 and v_{10} as well. Consider the case that $\phi(x_9) = \phi(x_{10}) = c$. Set $\phi(v_{11}) = 1$. If $\phi(x_2) = 1$, then color v_3 by 1, and color the vertices v_4, v_5, $\ldots$, v_{10}, v_1, v_2, v_{12} in this order. If $\phi(x_6) = 2$, then color v_{12} by 2 and extend the coloring to the 10-cycle $v_1 \ldots v_{10}$. Therefore, assume that $\phi(x_2) \neq 1$ and $d = \phi(x_6) \neq 2$. Let us distinguish several cases:

- $d = 3$, $\phi(x_4) = 1$ and $\phi(x_5) = 3$: In this case, set $\phi(v_{12}) = 3$, $\phi(v_3) = 1$ and color v_2, v_1, v_{10}, v_9, $\ldots$, v_4 in order.
- $d = 1$ and $\phi(x_4) = \phi(x_5)$: Set $\phi(v_3) = 1$ and color the vertices v_2, v_1, v_{10}, v_9, $\ldots$, v_6, v_{12} in order. Note that $\phi(v_3) = 1 \neq \phi(v_6)$, thus ϕ extends the coloring to v_4 and v_5 by (1).
- Otherwise, set $\phi(v_2) = 1$, $\phi(v_3) = 3$, $\phi(v_{12}) = 2$, $\phi(v_6) = 4 - d$, and color vertices v_7, $\ldots$, v_{10}, v_1 in order. By (1), this coloring extends to v_4 and v_5.

We conclude that if ϕ does not extend to the empty-circle vertices, then $\phi(x_8) = c_1 \neq 1$, and by the symmetry, $\phi(x_6) = c_2 \neq 2$.

There are four possible colorings of v_1 and v_8 (two choices of colors for each of these vertices, so that the color of v_1 is not 1 and the color of v_8 is not c_1). By (1), out of these four colorings, all but at most one extend to v_9 and v_{10}; if such a coloring of v_1 and v_8 exists, let it be denoted by ω_1; otherwise, set $\omega_1(v_1) = 1$ and $\omega_1(v_8) = c_1$. Symmetrically, let ω_2 be the unique coloring of v_3 and v_6 such that $\omega_2(v_3) \neq 2$, $\omega_2(v_6) \neq c_2$ and ω_2 does not extend to v_4 and v_5, if such a coloring exists, and $\omega_2(v_3) = 2$ and $\omega_2(v_6) = c_2$ otherwise.

If $\phi(x_2) = 2$, then let $a = 2$, otherwise let $a = 3$. Note that any color $c \neq 2 = \phi(x_3)$ satisfies $|\{a, c, \phi(x_2)\}| = 2$. In the following cases, we can extend ϕ to a coloring ψ of the path $v_1v_{10}v_9v_8v_7v_6$ such that $\psi(v_1) = a$ and $b = \psi(v_6) \neq \omega_2(v_6)$:

- $\omega_1(v_1) \neq a$: choose $b \notin \{\phi(x_6), \omega_2(v_6)\}$, color v_7 and v_8, and note that we can extend this coloring to v_9 and v_{10} by the definition of ω_1.
- $\omega_2(v_6) = c_2$: color the vertices v_{10}, v_9, …, v_6 in this order.
- $\phi(x_7) \notin \{c_1, \omega_1(v_8)\} \cap \{c_2, \omega_2(v_6)\}$ or $\{c_1, \omega_1(v_8)\} = \{c_2, \omega_2(v_6)\}$: excluding the previous two cases, we may assume that $c_1 \neq \omega_1(v_8)$ and $c_2 \neq \omega_2(v_6)$. Color v_8 by the color $d \notin \{c_1, \omega_1(v_8)\}$ and v_6 by the color $b \notin \{c_2, \omega_2(v_6)\}$, extend the coloring to v_9 and v_{10} by the definition of ω_1, and observe that $|\{\phi(x_7), b, d\}| \leq 2$, thus v_7 can be colored as well.

If such a coloring ψ exists, then choose a color $c \neq \phi(x_3)$ such that $c = b$ or $\{b, c\} \neq \{a, \psi(v_8)\}$; this ensures that the coloring extends to v_{11} and v_{12} by (1). Since $b \neq \omega_2(v_6)$, this coloring extends to v_4 and v_5 as well. Finally, the choice of a ensures that $|\{a, c, \phi(x_2)\}| = 2$, hence the coloring extends to v_2. Therefore, we may assume that such the coloring ψ does not exist, i.e., $\omega_1(v_1) = a$, $\omega_2(v_6) \neq c_2$, $\{c_1, \omega_1(v_8)\} \neq \{c_2, \omega_2(v_6)\}$ and $\phi(x_7) \in \{c_1, \omega_1(v_8)\} \cap \{c_2, \omega_2(v_6)\}$.

Let us now distinguish two cases:

- $\phi(x_9) \neq \phi(x_{10})$: By (1), $a = \omega_1(v_1) = \phi(x_9)$. If $c_1 \neq a$, then set $\phi(v_1) = \phi(v_8) = a$ and color v_{10}, v_9, v_7, v_6, …, v_2 in this order (v_2 can be colored by the choice of a), and color v_{12} and v_{11}; hence, assume that $c_1 = a$.

  If $\phi(x_{10}) = 5 - a$, then set $\phi(v_1) = \phi(v_8) = 5 - a$, $\phi(v_{10}) = a$, and $\phi(v_9) = 1$. Note that $\phi(x_7) \in \{c_1, \omega_1(v_8)\} = \{a, 5-a\}$ and $\{c_2, \omega_2(v_6)\} = \{1, \phi(x_7)\}$. Set $\phi(v_7) = 1$ and choose $\phi(v_6) \notin \{c_2, \omega_2(v_6)\}$, i.e., $\phi(v_6) = 5 - \phi(x_7)$. Extend the coloring to v_2, v_3, v_{12} and v_{11} in this order. As$\phi(v_6) \neq \omega_2(v_6)$, this coloring extends to v_4 and v_5, giving a coloring of the whole configuration.

  Therefore, assume that $\phi(x_{10}) = 1$. Then $\omega_1(v_8) = 1$ and $\phi(x_7) \in \{1, a\}$. Let us set $\phi(v_1) = \phi(v_7) = \phi(v_9) = 5 - a$, $\phi(v_{10}) = a$ and $\phi(v_8) = 1$. Let us choose color $\phi(v_6) \notin \{c_2, \omega_2(v_6)\}$; note that $\phi(v_6) \neq 5 - a$, since $\{c_2, \omega_2(v_6)\} \neq \{c_1, \omega_1(v_8)\} = \{1, a\}$. Color v_2 and v_3, and extend the coloring to v_4 and v_5 (this is possible, since $\phi(v_6) \neq \omega_2(v_6)$). We may assume that this coloring does not extend to v_{11} and v_{12}, i.e., $\{\phi(v_3), \phi(v_6)\} = \{1, 5-a\}$, hence $\phi(v_3) = 5 - a$ and $\phi(v_6) = 1$. As $\phi(v_6) \notin \{c_2, \omega_2(v_6)\}$, we get $\{c_2, \omega_2(v_6)\} = \{a, 5-a\}$ and $\phi(x_7) = a$. Since $c_2 \neq 2$, we have $c_2 = 3$ and$\omega_2(v_6) = 2$. As $\omega_2(v_3) \neq 2$, it follows that $\phi(x_4) = 2$ and $\phi(x_5) \neq 2$.

  Consider the coloring ψ with $\psi(v_8) = 5 - a$, $\psi(v_7) = \psi(v_9) = 1$, $\psi(v_6) = 2$, $\psi(v_3) = \psi(v_5) = 4 - \phi(x_5)$ and $\psi(v_4) = \phi(v_5)$, and assume that this coloring does not extend to the coloring of the whole configuration. On one hand, we may color v_1 by a and v_{10} by $5 - a$; then ψ extends to v_2 by the definition of a, and since it does not extend to v_{11} and v_{12}, we have $\{a, 5-a\} = \{2, 4 - \phi(x_5)\}$, and $\phi(x_5) = 1$. On the other hand, we may color v_1 by $5 - a$, v_{12} by 1 and v_{10} and v_{11} by a. Since this coloring does not extend to v_2, we have $|\{5-a, 3, \phi(x_2)\}| = 3$, and $a = 3$ and $\phi(x_2) = 1$. In that case, we can color the configuration by setting $\phi(v_3) = \phi(v_6) = \phi(v_8) = 1$, $\phi(v_1) = \phi(v_5) = \phi(v_7) = \phi(v_9) = \phi(v_{12}) = 2$ and $\phi(v_2) = \phi(v_4) = \phi(v_{10}) = \phi(v_{11}) = 3$.

- $\phi(x_9) = \phi(x_{10})$: By symmetry, we may also assume that $\phi(x_4) = \phi(x_5)$. At this point, we use the second relation guaranteed by the reduction, $\phi(x_7) = c_2$. If $c_2 \neq 3$, then set $\phi(v_7) = 3$, $\phi(v_8) = 1$ and $\phi(v_6) = 2$, color the 5-cycle $v_1v_2v_3v_{12}v_{11}$, and extend the coloring to v_4, v_5, v_9 and v_{10} by (1). Thus, assume that $c_2 = 3$.

  If $\phi(x_2) \neq 1$, then set $\phi(v_2) = \phi(v_6) = \phi(v_8) = 1$, $\phi(v_1) = \phi(v_7) = \phi(v_{12}) = 2$ and $\phi(v_3) = \phi(v_{11}) = 3$, and extend the coloring to v_4, v_5, v_9 and v_{10} by (1).

  Finally, if $\phi(x_2) = 1$, then set $\phi(v_2) = \phi(v_8) = 5 - c_1$, $\phi(v_1) = c_1$, $\phi(v_3) = \phi(v_7) = \phi(v_{11}) = 1$, $\phi(v_6) = 2$ and $\phi(v_{12}) = 3$, and extend the coloring to v_4, v_5, v_9 and v_{10} by (1).

Configuration R7′**.** If $\phi(v_3) = \phi(v_6)$, then first color the 6-cycle $v_2v_1v_{10}v_9v_8v_7$ (this is possible, as each of the vertices has at most one colored neighbor), and then color v_{11} and v_{12}. Thus, assume that $\phi(v_3) = 1$, $\phi(v_6) = 2$ and $\phi(v_{12}) = 3$. Color the 5-cycle $v_1v_{11}v_8v_9v_{10}$ (this is possible, as $\phi(x_1) \neq \phi(x_9)$). Note that in this coloring, $\phi(v_1) \neq 2$ or $\phi(v_8) \neq 1$, as $\phi(v_{11}) \neq \phi(v_{12}) = 3$. Therefore, the coloring extends to v_2 and v_7 by (1).

Configuration R7″**.** The reduction ensures that $\phi(x_1) \neq \phi(x_3)$, say $\phi(x_1) = 1$ and $\phi(x_3) = 2$. Also, by symmetry, we may assume that $c = \phi(x_2) \neq 1$. Suppose first that $\phi(v_8) \neq 1$. Then try coloring v_{11} and v_3 by 1 and v_1 by c. By (1), this coloring extends unless $\phi(v_9) = 1$ and $\phi(v_5) = c$. If $\phi(v_6) \neq 2$, then set the color of v_3 to 3, instead, and observe that the coloring extends. Otherwise, $\phi(v_6) = 2$, and set $\phi(v_{12}) = \phi(v_2) = 1$, $\phi(v_3) = 3$, and color v_{11} and v_1. The coloring extends to v_{10} and v_4 by (1).

Therefore, we may assume that $\phi(v_8) = 1$. Suppose that $\phi(v_6) \neq c$. Then try coloringv_1 and v_{12} by c, v_{11} and v_2 by $5 - c$ and v_3 by 1. By (1), this coloring extends to v_4 and v_{10} unless $\phi(v_5) = c$ and $\phi(v_9) = 1$. In that case, set $\phi(v_2) = 1$, $\phi(v_3) = 3$, color v_{12}, v_{11} and v_1 in this order, and extend the coloring to v_4 and v_{10} by (1). Thus, we may assume that $\phi(v_6) = c$.

If $c \neq 2$, then set $\phi(v_3) = c$ and color v_4, v_{10}, v_1, v_2, v_{11} and v_{12} in this order; hence, assume that $c = 2$. Consider the coloring that assigns 1 to v_2 and v_{12}, 3 to v_{11} and v_3 and 2 to v_1. If this coloring does not extend to v_4 and v_{10}, then (1) implies that $\phi(v_5) = 2$ and $\phi(v_9) = 3$. In that case, set $\phi(v_2) = \phi(v_4) = \phi(v_{12}) = 1$, $\phi(v_{10}) = \phi(v_{11}) = 2$ and $\phi(v_1) = \phi(v_3) = 3$.

Configuration R7‴**.** The reduction ensures that $\phi(x_1) \neq \phi(x_3)$, say $\phi(x_1) = 1$ and $\phi(x_3) = 2$. If $\phi(v_8) \neq 1$ and $\phi(v_6) \neq 2$, then color v_{11} by 1, v_{12} by 2 and extend the coloring to the 6-cycle $v_{10}v_1v_2v_3v_4v_5$.

Assume now that $\phi(v_8) = 1$ or $\phi(v_6) = 2$. Suppose first that $\phi(v_6) \neq 2$, and thus $\phi(v_8) = 1$. Then try setting the color of v_1, v_5 and v_{12} to 2 and coloring v_{11}

and v_{10}. If $\phi(x_2) = 2$ or $\phi(x_4) = 2$ or $\phi(x_2) = \phi(x_4)$, then the coloring extends to v_2, v_3 and v_4, thus assume that $\{\phi(x_2), \phi(x_4)\} = \{1, 3\}$. If $\phi(v_9) \neq 2$ or $\phi(v_6) \neq 3$, then set $\phi(v_2) = \phi(v_4) = \phi(v_{11}) = 2$, $\phi(v_1) = 3$, color v_{12} and v_3 and extend the coloring to v_5 and v_{10} by (1). Otherwise, $\phi(v_9) = 2$ and $\phi(v_6) = 3$ and we set $\phi(v_5) = 1$, $\phi(v_1) = \phi(v_4) = \phi(v_{12}) = 2$, $\phi(v_{10}) = \phi(v_{11}) = 3$, $\phi(v_2) = \phi(x_4)$ and $\phi(v_3) = \phi(v_2)$.

Therefore, it suffices to consider the case that $\phi(v_6) = 2$. If $\phi(x_4) \neq 2$, then set $\phi(v_4) = 2$, color the 5-cycle $v_1v_2v_3v_{12}v_{11}$, and color v_{10} and v_5. So we have $\phi(x_4) = 2$. Suppose that $\phi(x_2) \neq 2$. Then set $\phi(v_2) = 2$ and $\phi(v_1) = 3$. If $\phi(v_8) \neq 2$, then color v_{11} by 2 and color v_{10}, v_5, v_4, v_3 and v_{12} in this order. On the other hand, if $\phi(v_8) = 2$, then note that $\phi(v_9) \neq 2$, and set $\phi(v_{10}) = 2$, $\phi(v_3) = \phi(v_5) = \phi(v_{11}) = 1$ and $\phi(v_4) = \phi(v_{12}) = 3$. Thus, assume that $\phi(x_2) = 2$.

Try setting $\phi(v_2) = \phi(v_4) = \phi(v_{12}) = 1$ and $\phi(v_3) = \phi(v_5) = 3$. If $\phi(v_9) \neq 1$, then set $\phi(v_{10}) = 1$ and color v_{11} and v_1; thus assume that$\phi(v_9) = 1$. If $\phi(v_8) \neq 2$, then set $\phi(v_{10}) = \phi(v_{11}) = 2$ and $\phi(v_1) = 3$.

Finally, consider the case that $\phi(v_9) = 1$ and $\phi(v_8) = 2$. Then, we set $\phi(v_3) = \phi(v_5) = \phi(v_{11}) = 1$, $\phi(v_1) = 2$ and $\phi(v_2) = \phi(v_4) = \phi(v_{10}) = \phi(v_{12}) = 3$.

Configuration R7''''**.** The reduction ensures that $\phi(x_3) \neq \phi(v_6)$, say $\phi(v_6) = 1$ and $\phi(x_3) = 2$. Suppose first that $\phi(v_8) \neq \phi(v_{10})$. If $\phi(v_{10}) \neq 2$, then let $\phi(v_{12}) = 2$, $\phi(v_{11}) = \phi(v_{10})$ and extend the coloring to the 5-cycle $v_1v_2v_3v_4v_5$; thus assume that $\phi(v_{10}) = 2$. If $\phi(x_2) \neq 2$, then set $\phi(v_2) = 2$, $\phi(v_3) = 1$, and color v_4, v_5, v_1, v_{11} and v_{12} in this order. If $\phi(x_2) = 2$, then set $\phi(v_1) = \phi(v_3) = 1$, $\phi(v_2) = 3$, and color v_{11}, v_{12}, v_4 and v_5, in this order.

Therefore, assume that $\phi(v_8) = \phi(v_{10}) = c$. If $c = 2$, then color v_{12} by 2, extend the coloring to the 5-cycle $v_1 \dots v_5$, and color v_{11}. If $c = 3$, then set $\phi(v_1) = \phi(v_3) = 1$, $\phi(v_{11}) = 2$, $\phi(v_{12}) = 3$, and color v_2, v_4 and v_5 in this order. Thus, assume that $c = 1$. Try setting $\phi(v_1) = \phi(v_{12}) = 2$ and $\phi(v_{11}) = \phi(v_5) = 3$. If $\phi(x_4) \neq 2$, then set $\phi(v_4) = 2$ and color v_2 and v_3. If $\phi(x_4) = 2$ and $\phi(x_2) \neq 1$, then set $\phi(v_2) = \phi(v_4) = 1$ and $\phi(v_3) = 3$.

Finally, consider the case that $\phi(x_2) = 1$ and $\phi(x_4) = 2$. Then, set $\phi(v_3) = 1$, $\phi(v_2) = \phi(v_5) = \phi(v_{11}) = 2$ and $\phi(v_1) = \phi(v_4) = \phi(v_{12}) = 3$. □

5 Discharging

Let G be a graph in a surface Σ with rings $\mathcal{R}$. A face is *open* 2*-cell* if it is homeomorphic to an open disk. A face is *closed* 2*-cell* if it is open 2-cell and bounded by a cycle. A face f is *omnipresent* if it is not open 2-cell and each of its boundary walks is either a vertex ring or a cycle bounding a closed disk $\Delta \subseteq \hat{\Sigma} \setminus f$ containing exactly one ring. We say that G has an *internal* 2*-cut* if there exist sets $A, B \subseteq V(G)$ such that $A \cup B = V(G)$, $|A \cap B| = 2$, $A - B \neq \emptyset \neq B - A$, A includes all vertices of $\mathcal{R}$, and no edge of G has one end in $A - B$ and the other in $B - A$.

We wish to consider the following conditions that the triple $G, \Sigma, \mathcal{R}$ may or may not satisfy:

(I0) every internal vertex of G has degree at least three,

(I1) G has no even cycle consisting of internal vertices of degree three,

(I2) G has no cycle C consisting of internal vertices of degree three, and two distinct adjacent vertices $u, v \in V(G) - V(C)$ such that both u and v have a neighbor in C,

(I3) every internal face of G is closed 2-cell and has length at least 5,

(I4) if a path of length at most two has both ends in $\mathcal{R}$, then it is a subgraph of $\mathcal{R}$,

(I5) no two vertices of degree two in G are adjacent,

(I6) if Σ is the sphere and $|\mathcal{R}| = 1$, or if G has an omnipresent face, then G does not contain an internal 2-cut,

(I7) the distance between every two distinct members of $\mathcal{R}$ is at least four,

(I8) every cycle in G that does not separate the surface has length at least seven,

(I9) if a cycle C of length at most 9 in G bounds an open disk Δ in $\hat{\Sigma}$ such that Δ is disjoint from all rings, then Δ is a face, a union of a 5-face and a $(|C| - 5)$-face, or C is a 9-cycle and Δ consists of three 5-faces intersecting in a vertex of degree three.

Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying (I3). We say that a good configuration γ *appears* in $(G, \mathcal{R})$ if it faintly appears and the following conditions hold:

- no vertex ring belongs to $\mathcal{I}_\gamma$,
- if γ is isomorphic to R3, then either $\mathcal{I}$ contains a vertex of $\mathcal{R}$ or there exists a vertex $v \in \mathcal{I}$ such that v and all its neighbors are internal,
- if γ is isomorphic to R4, then the vertex that corresponds to v_2 is internal and has degree at least 4, and neither x_4 nor x_5 is a vertex ring,
- if γ is isomorphic to R5, then v_4 is an internal vertex and the face whose boundary contains the path corresponding to $v_6 v_7 v_8$ has length at least seven,
- if γ is isomorphic to R6 or R6$'$, then both vertices of $\mathcal{A}_\gamma$ are internal, and all neighbors of at least one of them are internal,
- if γ is isomorphic to one of R7, R7$'$, R7$''$, R7$'''$, R7$''''$, then all vertices in $\mathcal{A}_\gamma \cup \mathcal{I}_\gamma$ and all their neighbors are internal, and
- if γ is isomorphic to R7, then the vertex corresponding to x_8 and all its neighbors are internal.

Let G be a graph in a surface Σ with rings $\mathcal{R}$, and let M be a subgraph of G. We define the *initial charge* of the triple $(G, \Sigma, \mathcal{R})$ as follows. The faces bounded by facial rings get charge 0. Every internal face f gets charge $|f| - 4$. A vertex of degree two belonging to a facial ring gets charge $-1/3$, a vertex of degree d forming a vertex ring gets charge d, any other vertex of $\mathcal{R}$ of degree d gets charge $d - 3$, and all internal vertices get charge $d - 4$. Finally, we increase the charge of each face incident with an edge of M by 5/3.

Lemma 5.1. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, let g be the Euler genus of Σ, let n_2 be the number of vertices of degree two in facial rings, and let M be a subgraph of G. Then the sum of initial charges of all vertices and faces of G is at most $4g + 4|\mathcal{R}| + 2n_2/3 + 10|E(M)|/3 - 8$.*

Proof. Let n_1 be the number of vertex rings, let n_3 be the number of vertices of degree at least three belonging to facial rings, and let F be the set of ring faces of G. Let us recall that, by definition, a face of G is a maximal arcwise-connected component of $\hat{\Sigma} - G$. Thus $F \cup \mathcal{F}(G)$ is the set of all faces of G. We have $n_3 \le \sum_{f \in F} |f| \le 4|F| + \sum_{f \in F}(|f| - 4)$, and hence $4n_1 + n_3 \le 4|\mathcal{R}| + \sum_{f \in F}(|f| - 4)$. By Euler's formula, the sum of the initial charges of all vertices and faces is at most

$$\begin{aligned}\sum_{f \in F(G)}(|f| - 4) + \sum_{v \in V(G)}(\deg(v) - 4) + 4n_1 + 2n_2/3 + n_3 + 10|E(M)|/3 \\ \le 4g + 4|\mathcal{R}| + 2n_2/3 + 10|E(M)|/3 - 8,\end{aligned}$$

as desired. □

A 5-face f is *k-dangerous* if f is not incident with an edge of M and f is incident with exactly k internal vertices of degree three. Let $f_1 = uvawb$ be a 4-dangerous face, where w is the unique incident vertex that is not internal of degree three. Let f_2 be the face incident with uv distinct from f_1. We say that f_2 is *linked to f_1 (through the edge uv)*. Let xy be an edge such that y has degree three, and let g_1, g_2, g_3 be the faces incident with y such that xy is incident with q_1 and g_2. Then the face g_3 is *opposite* to x. A 4-dangerous face f is *extremely* 4*-dangerous* if it is neither incident with a vertex of $\mathcal{R}$ nor opposite to any vertex ring.

Let us apply the following *primary discharging rules*, resulting in the *primary charge*:

Rule 1: Every internal face f sends $1/3$ to each incident vertex v such that $\deg(v) = 2$ and v belongs to a facial ring, or $\deg(v) = 3$ and v is internal.

Rule 2: Every vertex v belonging to a facial ring sends $1/3$ to each 4-dangerous face incident with v. Every vertex ring sends $1/3$ to each incident face that is not a cuff face, $8/9$ to its cuff face and $1/3$ to each opposite face.

Rule 3: Let f be a face linked to an extremely 4-dangerous face f' through an edge uv. If f has length at least 6, or f is incident with an edge of M, or f is the cuff face of a vertex ring, then f sends $1/3$ to f' across the edge uv.

Rule 4: Let $v_1v_2v_3v_4$ be a subwalk of the boundary walk of an internal face f' of length at least seven, such that f' is linked to extremely 4-dangerous faces through both v_1v_2 and v_3v_4. Let f be the other face incident with the edge v_2v_3. If f has length at least six, then f sends $1/9$ to f' across the edge v_2v_3.

Lemma 5.2. *Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying* (I0) *and* (I3) *and let M be a subgraph of G. Then the primary charge of each vertex is non-negative, the primary charge of a vertex of degree d forming a vertex ring is at least $d/9$, and the primary charge of a vertex of degree $d \geq 3$ incident with a facial ring is at least $2(d-3)/3$. Moreover, the primary charge of a vertex of degree $d \geq 4$ that does not belong to $\mathcal{R}$ is exactly $d-4$.*

Proof. By Rule 1, the internal vertices of degree three have primary charge 0. The charge of internal vertices of degree $d \geq 4$ is unchanged, i.e., $d - 4 \geq 0$. A vertex forming a vertex ring has initial charge d and sends at most $(6d+5)/9$ by Rule 2. By (I3), we have $d \geq 2$; thus its primary charge is at least $(3d-5)/9 \geq d/9$. Consider now a vertex v of degree d that belongs to a facial ring. If $d = 2$, then the initial charge of v is $-1/3$ and v receives $1/3$ by Rule 1. Observe that v sends nothing by Rule 2, since both neighbors of v belong to $\mathcal{R}$; thus the primary charge of v is 0. If $d \geq 3$, then v sends charge by Rule 2 to at most $d-3$ faces, and hence its primary charge is at least $d-3-(d-3)/3 = 2(d-3)/3$, as desired. □

The primary charge of a face corresponding to a facial ring R is zero, as it is equal to its initial charge. Let us now estimate the primary charge of internal faces. A subgraph $M \subseteq G$ *captures* (≤ 4)*-cycles* if M contains all cycles of G of length at most 4 and furthermore, M is either null or has minimum degree at least two.

Lemma 5.3. *Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying* (I0), (I1), (I3), (I4), (I5) *and* (I7)*, let M be a subgraph of G that captures (≤ 4)-cycles and assume that if a configuration isomorphic to one of* R1, R2, ..., R5 *appears in G, then it touches M. If f is an internal face of G, then the primary charge of f is non-negative. Furthermore, if the primary charge of f is zero, then f has length exactly five, it is not incident with an edge of M, it is not a cuff face, and*

(a) *f is 3-dangerous, or*
(b) *f is incident with a vertex of $\mathcal{R}$, or*
(c) *f is 4-dangerous and a face of length at least 6 is linked to f, or*
(d) *f is 4-dangerous, the face h linked to f has length five and either h is incident with an edge of M or h is the cuff face of a vertex ring, or*
(e) *f is 4-dangerous and is opposite to a vertex ring.*

Otherwise, the primary charge of f is least $2/9$, and if $|f| \geq 8$, then the primary charge of f is at least $5|f|/9-4$. Also, if f is a 6-face incident with a vertex of degree two belonging to a facial ring, then f has primary charge at least $2/3$.

Proof. Suppose first that f has length exactly five. Let us consider the case that f is incident with an edge of M. Then f may send charge by Rules 1 and 3. If f sends charge across an edge uv by Rule 3 to a face f', then both u and v have degree three and no edge of f' belongs to M. Since M has minimum degree at least two, it follows that no edge incident with u or v belongs to M;

hence f sends charge by Rule 3 to at most two faces. The primary charge of f is at least $1 + 5/3 - 5/3 - 2/3 = 1/3 > 2/9$.

Therefore, we may assume that f is not incident with any edge of M, and in particular, f does not share an edge with any cycle of length at most 4. Suppose that f is the cuff face of a vertex ring v. Then f sends charge by Rules 1 and 3; however, f is linked to at most one extremely 4-dangerous face and f receives $8/9$ from v. The primary charge of f is at least $1 + 8/9 - 4/3 - 1/3 = 2/9$, as desired.

If f is not the cuff face of a vertex ring, then f sends charge only by Rule 1. Let us distinguish several cases according to the number of internal vertices of degree three incident with f.

- *All vertices incident with f are internal and have degree three.* Then f and its incident vertices form a configuration isomorphic to R1 that appears in G, which is a contradiction.

- *The face f is incident with exactly four internal vertices of degree three.* Let $f = v_1v_2v_3v_4v_5$ and suppose that all these vertices except for v_2 are internal and have degree three. If v_2 is not internal, then either v_2 is a vertex ring or it has degree at least four, since v_1 and v_3 are internal. The charge of f after applying Rule 1 is $-1/3$.

The face f is incident with no edge of M, hence f is 4-dangerous. If v_2 belongs to a ring, then f receives $1/3$ by Rule 2, making its charge zero, and hence f satisfies (b). Thus we may assume that v_2 is internal and of degree at least 4. Similarly, if f is opposite to a vertex ring, then f receives $1/3$ by Rule 2 and f satisfies (e), hence it suffices to consider the case that f is extremely 4-dangerous.

If the face h with that f shares the edge v_4v_5 has length five, then the faces f and h form an imprint of R4 (v_2 is distinct from the vertices incident with h, since f does not share an edge with a cycle of length at most 4). If h is the cuff face of a vertex ring, then f receives $1/3$ from h by Rule 3, the primary charge of f is zero and f satisfies (d). Otherwise, a configuration isomorphic to R4 appears in G. By hypothesis the face h is incident with an edge of M.

We conclude that h either has length at least 6 or is incident with an edge of M. In both cases, h sends $1/3$ to f by Rule 3. Thus the primary charge of f is zero, and f satisfies (c) or (d).

- *The face f is incident with exactly three internal vertices of degree three.* In this case f sends $1/3$ to each of the three incident internal vertices of degree three by Rule 1, making its charge zero. (The face f is not incident with a vertex of $\mathcal{R}$ of degree two belonging to a facial ring, since both neighbors of such a vertex belong to $\mathcal{R}$). Since f does not share an edge with M, f is 3-dangerous and satisfies (a).

- *The face f is incident with exactly two internal vertices of degree three.* Then f sends $1/3$ to each of them, and at most $1/3$ to a vertex of $\mathcal{R}$ of degree two by Rule 1, making its charge non-negative. Furthermore, if the charge is zero, then f satisfies (b); otherwise the charge is at least $1/3$, as desired.

• *The face f is incident with at most one internal vertex of degree three.* Then f sends at most 2/3 by Rule 1 and (I5), and its primary charge is at least 1/3, as desired.

Thus we have proved the lemma when f has length five. Let us now consider the case that f has length six, and let $f = v_1v_2v_3v_4v_5v_6$. By (I1) not all vertices incident with f are internal and of degree three. Thus f sends at most 5/3 by Rule 1 and at most 4/3 by Rules 3 and 4. If f is incident with an edge of M, then its primary charge is at least $2 + 5/3 - 5/3 - 4/3 = 2/3$, as desired, and so we may assume that f is incident with no edge of M. If f does not send charge using Rules 3 or 4, then its primary charge is at least $2 - 5/3 = 1/3$. Furthermore, if some vertex incident with f, say v_2, has degree two and belongs to a facial ring R, then by (I5), v_1 and v_3 belong to R and have degree at least three, thus the primary charge of f is at least $2 - 4/3 = 2/3$.

If, say, v_1 is a vertex ring, then the faces incident with the edges v_1v_2, v_1v_6, v_2v_3 and v_5v_6 are not extremely 4-dangerous; hence, f sends at most 2/3 by Rule 3 and nothing by Rule 4. Furthermore, f receives at least 1/3 from v_1 by Rule 2, thus the primary charge of f is at least $2 - 5/3 - 2/3 + 1/3 = 0$. If f sends less than 5/3 by Rule 1 or less than 2/3 by Rule 3, or if f is the cuff face of v_1, then the primary charge is at least 1/3, as desired. Otherwise, f forms an appearance of $\gamma =$ R3, with $\mathcal{I}_\gamma = \{v_2, v_4, v_6\}$, contradicting the hypothesis of the lemma.

We show that the situation that f is incident neither with a vertex ring nor with an edge of M and sends charge by Rule 3 or 4 cannot occur. Suppose that f sends charge across v_2v_3 by Rule 3 or 4. It follows that v_2 and v_3 are internal and of degree three. Let x_2 be the neighbor of v_2 other than v_1 and v_3, and let x_3 be defined analogously. Then both x_2 and x_3 are internal vertices of degree three. If v_1 and v_5 both belong to $\mathcal{R}$, then by (I4) v_6 is a vertex of degree two, and by (I4) and (I5) v_4 is an internal vertex, implying that $\gamma =$ R3 appears in G (with $\mathcal{I}_\gamma = \{v_2, v_4, v_6\}$). This contradicts the hypothesis; hence, assume that at least one of v_1 and v_5 is internal, and symmetrically, at least one of v_4 and v_6 is internal.

If both v_1 and v_5 are internal, then by (I4) and (I7) at least one of v_4 and v_6 is internal as well. Since neither v_4 nor v_6 is a vertex ring, $\gamma =$ R3 appears in G with $\mathcal{I}_\gamma = \{v_2, v_4, v_6\}$. This is a contradiction; hence exactly one of v_1 and v_5 belongs to $\mathcal{R}$. Therefore, $\gamma =$ R3 appears in G with $\mathcal{I}_\gamma = \{v_1, v_3, v_5\}$. This is a contradiction, finishing the case that $|f| = 6$.

Finally, we consider the case that $|f| \geq 7$. Let us estimate the amount of charge sent from f and received by f using Rules 3 a 4. If $v_1v_2v_3v_4$ is a subwalk of the boundary walk of f and f sends 1/3 across v_2v_3 by Rule 3, then assign 1/9 of this charge to each of v_1v_2, v_2v_3 and v_3v_4. If f sends 1/9 across v_2v_3 by Rule 4, then add 1/9 to the charge assigned to v_2v_3; if f receives 1/9 across v_2v_3, then remove 1/9 from the charge assigned to v_2v_3. We claim that each edge has at most 1/9 assigned to it, and hence that f sends at most $|f|/9$ by Rules 3 and 4.

Suppose for a contradiction that more than 1/9 is assigned to the edge v_2v_3.

By symmetry, we can assume that f sends charge by Rule 3 to the face f_{12} across v_1v_2. Let $f_{23} \neq f$ be the face incident with the edge v_2v_3. If f sends charge across v_2v_3 by Rule 3, then the faces f_{12} and f_{23} form an appearance of a configuration isomorphic to R5. It follows that f_{12} or f_{23} is incident with an edge of M. This is a contradiction, because Rule 3 sends charge to 4-dangerous faces only. Furthermore, f does not send charge across v_2v_3 by Rule 4, because f is linked to f_{12} through v_1v_2.

Since more than 1/9 is assigned to v_2v_3, it follows that f sends charge across v_3v_4 by Rule 3 and does not receive charge by Rule 4 across v_2v_3. Therefore, f_{23} has length five and f_{12} and f_{23} form an appearance of a configuration isomorphic to R5 as before. Since f_{12} is 4-dangerous, some edge of M is incident with f_{23} but not with f_{12}. Since all neighbors of v_2 and v_3 have degree three and M has minimum degree at least two, it follows that some edge of M is incident with the face $f_{34} \neq f$ that is incident with v_3v_4. This is a contradiction, because f sends charge to f_{34} by Rule 3.

We can now bound the primary charge of f. If f has length at least eight, then f sends at most $|f|/3$ by Rule 1 and at most $|f|/9$ by Rules 3 and 4; thus its primary charge is at least $|f| - 4 - |f|/3 - |f|/9 = 5|f|/9 - 4 > 2/9$, as desired.

Finally, assume that f has length exactly seven. If f is incident with an edge of M, then f sends at most 7/3 by Rule 1, making the primary charge of f at least $3 + 5/3 - 7/3 - 7/9 = 14/9$. If f is incident with no edge of M, then f and its incident vertices do not form an appearance of a configuration isomorphic to R2, and that in turn implies that f is incident with no more than six internal vertices of degree three. Thus f sends at most 2 by Rule 1, and hence the primary charge of f is at least $3 - 2 - 7/9 = 2/9$, as desired. □

We now modify the primary charges using three additional rules into what we will call "final charges". A vertex is *safe* if its degree is at least five, or if it belongs to $\mathcal{R}$, or if it is incident with a face with strictly positive primary charge. A face f is *k-reachable* from a vertex v if there exists a path P of length at most k (P may have length zero), joining v to a vertex incident with f, such that no vertex of $P \backslash v$ is safe. In particular, every vertex of $P \backslash v$ is internal and has degree at most four, and all faces incident with them have length 5, which implies that the number of faces that are 3-reachable from a vertex of degree d is bounded by $20d$. Furthermore, if v is incident to a face f with strictly positive primary charge, then two of the neighbors of v are safe, and we conclude that at most $20(d-3)+26$ faces distinct from f are 3-reachable from v.

Let $\epsilon > 0$ be a real number, to be specified later. Starting from the primary charges we now apply the following three rules, resulting in the *final charge*:

Rule 5: The charge of each vertex of degree three that belongs to a facial ring is increased by 26ϵ,

Rule 6: each face of strictly positive primary charge sends 46ϵ units of charge to each incident vertex,

Rule 7: if v is either a vertex ring or a safe vertex of degree at least three, then v sends a charge of ϵ to each internal face of zero primary charge that is 3-reachable from v.

Lemma 5.4. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, let g be the Euler genus of Σ, let n_2 and n_3 be the number of vertices of degree two and three, respectively, incident with facial rings, let $\epsilon > 0$, and let M be a subgraph of G. Then the sum of final charges of all vertices and faces of G is at most $4g + 4|\mathcal{R}| + 26\epsilon n_3 + 2n_2/3 + 10|E(M)|/3 - 8$.*

Proof. This follows from Lemma 5.1 and the description of the discharging rules. □

Lemma 5.5. *Let $G, \Sigma, \mathcal{R}, M$ be as in Lemma 5.3, and let $\epsilon \leq 1/180$. Then the final charge of every vertex is non-negative and the final charge of every vertex of degree $d \geq 4$ belonging to a facial ring is at least $(2/3 - 20\epsilon)(d-3) - 26\epsilon$.*

Proof. Let v be a vertex of G of degree d. Lemma 5.2 tells us that the primary charge of v is non-negative. If v is safe, then it sends at most $20\epsilon d$ units of charge by Rule 7; otherwise it sends nothing using Rules 5–7. Assume first that v is not in $\mathcal{R}$. If $d \geq 5$, then the primary charge of v is $d-4$, and its final charge is at least $d - 4 - 20\epsilon d$, which is non-negative by the choice of ϵ. If $d \leq 4$ and v is not incident with a face of positive primary charge, then its final charge is the same as its primary charge, and so the conclusion follows from Lemma 5.2. If $d \leq 4$ and v is incident with a face of positive primary charge, then it receives at least 46ϵ units of charge using Rule 6 and sends at most 46ϵ units using Rule 7. Thus v has non-negative final charge.

Let us now assume that v belongs to $\mathcal{R}$. If v is a vertex ring, then v has primary charge at least $d/9$, making its final charge at least $(1/9 - 20\epsilon)d \geq 0$. Suppose that v is incident with a facial ring. If $d = 2$, then v sends no charge by Rules 5–7 and its final charge is zero. If $d = 3$, then v receives 26ϵ units using Rule 5, and sends at most 26ϵ units using Rule 7. Finally, if $d \geq 4$, then v has primary charge at least $2(d-3)/3$ by Lemma 5.2, and it sends at most $20(d-3)\epsilon + 26\epsilon$ units of charge, and hence its final charge is at least $(2/3 - 20\epsilon)(d-3) - 26\epsilon$, which is non-negative by the choice of ϵ. □

Lemma 5.6. *Let $G, \Sigma, \mathcal{R}, M$ be as in Lemma 5.3, and let $\epsilon > 0$ be arbitrary. Then the final charge of every internal face of length six or seven is at least $2/9 - 322\epsilon$, and the final charge of every internal face of length $l \geq 8$ is at least $(5/9 - 46\epsilon)l - 4$.*

Proof. Lemma 5.3 gives a lower bound on the primary charge of a face f, and f sends at most $46\epsilon|f|$ units of charge using Rule 6. □

Lemma 5.7. *Let $G, \Sigma, \mathcal{R}, M$ be as in Lemma* 5.3*, satisfying additionally* (I8)*, and assume that if a configuration isomorphic to one of* R1, R2, ..., R6 *or* R7 *appears in G, then it touches M. Then every internal face of zero primary charge is* 3*-reachable from some safe vertex.*

Proof. Let f be an internal face of zero primary charge. Lemma 5.3 implies that f is a 5-face, and unless f is 1-reachable from a safe vertex, we have that f is 3-dangerous and all vertices incident with f are internal and have degree at most four. Let $f = w_1w_2w_3w_4w_5$, and suppose first that w_1 and w_5 have degree four. In this case, we prove the following stronger claim: both w_1 and w_5 are at distance at most two from a safe vertex.

Let f' be the other face incident with the edge w_1w_5. To prove the claim we may assume that no vertex incident with f or f' is safe, for otherwise the claim holds. Then f' has primary charge zero, because no vertex incident with f is safe. Since w_1 and w_5 have degree at least four, Lemma 5.3 implies that f' is 3-dangerous. Since f and f' have zero primary charge, they do not share an edge with M, and in particular, they do not share an edge with any cycle of length at most four. We deduce that the faces f and f' and their incident vertices form a faint appearance of a configuration isomorphic to R6. Since f and f' are incident with no edge of M, this is not an appearance; hence either w_1 or w_5 has a neighbor in $\mathcal{R}$, or the distance from both w_1 and w_5 to a vertex of $\mathcal{R}$ is most two. In both cases, w_1 and w_5 are at distance at most two from a safe vertex, as desired. This concludes the case when w_1 and w_5 have degree four.

We may therefore assume that w_1 and w_3 have degree four. Let f_1, f_2, f_3, f_4 and f_5 be the other faces incident with the edges w_1w_2, w_2w_3, w_3w_4, w_4w_5 and w_5w_1, respectively. Similarly as before we may assume that f_1, f_2, f_3, f_4 and f_5 are all 3-dangerous 5-faces and vertices incident with them have degree at most four, for otherwise f is 3-reachable from a safe vertex. If any of those faces contained two consecutive vertices x and y of degree four, then by the previous paragraph, both x and y would be at distance at most two from a safe vertex, and hence f would be 3-reachable from such a safe vertex. We may therefore assume that this is not the case. Since no cycle of length at most 4 shares an edge with f or f_i for $1 \leq i \leq 5$, we deduce that the faces $f, f_1, f_2, f_3, f_4, f_5$ and their incident vertices and edges form a faint appearance of a configuration γ isomorphic to R7, unless f_3 and f_5 are incident with a common vertex, i.e., unless v_4 is identified with v_9, or v_5 is identified with v_{10} in the depiction of R7 in Figure 1. Suppose that say $v_4 = v_9$. Since this vertex has degree three, we conclude that $\{v_3, v_5\} \cap \{v_8, v_{10}\} \neq \emptyset$. As f does not share an edge with M, we have $v_3 \neq v_8$, $v_3 \neq v_{10}$ and $v_5 \neq v_8$. However, if $v_5 = v_{10}$, then the cycle $v_5v_6v_{12}v_{11}v_1$ does not separate the surface, contrary to (I8).

It follows that R7 faintly appears, but does not appear, in G. Thus, using the labeling of the vertices as in Figure 1, one of x_1, x_3, x_6, x_7, x_8 or one of their neighbors belongs to $\mathcal{R}$. Therefore, f is 3-reachable from a safe vertex, as desired. □

Lemma 5.8. *Let $G, \Sigma, \mathcal{R}, M$ be as in Lemma* 5.7*, let $\epsilon \le 2/2079$, and assume that if a good configuration appears in G, then it touches M. Then the final charge of every internal face of length five is at least ϵ.*

Let us remark that $2079 = 9(5 \cdot 46 + 1)$.

Proof. Let f be an internal face of length five. If f has positive primary charge, then by Lemma 5.3 it has primary charge at least 2/9. It sends 46ϵ units of charge to each incident vertex by Rule 6, and hence f has final charge at least $2/9 - 5 \cdot 46\epsilon \ge \epsilon$.

We may therefore assume that f has primary charge zero. By Lemma 5.7, f is 3-reachable from some safe vertex, and hence has final charge at least ϵ because of Rule 7, as desired. □

Let $s : \{5, 6, \ldots\} \to \mathbb{R}$ be a function (that we specify later) satisfying
(S1) $s(5) = 2\epsilon$,
(S2) $s(7) \le 4/9 - 644\epsilon$,
(S3) $s(l) \le (10/9 - 92\epsilon)l - 8$ for every integer $l \ge 8$.
Suppose that we are given such a function and a graph G in Σ with rings $\mathcal{R}$. For an internal face f of G, we define $w(f) = s(|f|)$ if f is open 2-cell and $|f| \ge 5$, and $w(f) = |f|$ otherwise. We define $w(G, \mathcal{R})$ as the sum of $w(f)$ over all internal faces f of G.

Lemma 5.9. *Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying* (I0)–(I8)*, let M be a subgraph of G that captures (≤ 4)-cycles and assume that if a configuration isomorphic to one of* R1, R2, …, R7 *appears in G, then it touches M. Let ϵ be a real number satisfying $0 < \epsilon < 2/2079$, and let $s : \{5, 6, \ldots\} \to \mathbb{R}$ be a function satisfying* (S1)–(S3)*. Then the final charge of every vertex is non-negative, the final charge of every face bounded by a ring is zero, and the final charge of every internal face f is at least $s(|f|)/2$.*

Proof. The final charge of every face bounded by a ring is clearly zero. The remaining assertions follow from Lemmas 5.5, 5.6 and 5.8 using conditions (S1)–(S3). □

Lemma 5.10. *Let $G, \Sigma, \mathcal{R}, M, \epsilon, s$ be as in Lemma* 5.9*, and let g, n_2, n_3 be as in Lemma 5.4. Then $w(G, \mathcal{R}) \le 8g + 8|\mathcal{R}| + 52\epsilon n_3 + 4n_2/3 + 20|E(M)|/3 - 16$.*

Proof. By Lemma 5.9 the quantity $w(G, \mathcal{R})$ is at most twice the sum of the final charges of all vertices and faces of G, and hence the lemma follows from Lemma 5.4. □

We need the following refinement of the previous lemma.

Lemma 5.11. *Let $G, \Sigma, \mathcal{R}, M, \epsilon, s$ be as in Lemma* 5.9*. Then $w(G, \mathcal{R}) \le 8g + 8|\mathcal{R}| + 52\epsilon n_3 + 4n_2/3 + 20|E(M)|/3 - 8b/9 - 16$, where b is the number of internal 6-faces of G incident with a vertex of degree two contained in a facial ring, plus the number of vertices of degree at least four incident with a facial ring.*

Proof. This follows similarly as Lemma 5.10, since according to Lemma 5.3, each 6-face incident with a vertex of degree two contained in a facial ring has charge by at least 4/9 higher than the bound used to derive Lemma 5.10, and since the final charge of a vertex of degree at least four incident with a facial ring is at least $2/3 - 46\epsilon > 4/9$. □

6 Reductions

In this section, we argue that subject to a few assumptions, reducing a good configuration does not create cycles of length at most four.

Let G be a graph in a surface Σ with rings $\mathcal{R}$, and let P be a path of length at most four with ends $u, v \in V(\mathcal{R})$ and otherwise disjoint from $\mathcal{R}$. We say that P is *allowable* if

- u, v belong to the same ring of $\mathcal{R}$, say R,
- P has length at least three,
- there exists a subpath Q of R with ends u, v such that $P \cup Q$ is a cycle of length at most eight that bounds an open disk $\Delta \subset \Sigma$,
- if P has length three, then $P \cup Q$ has length five and Δ is a face of G, and
- if P has length four, then Δ includes at most one edge of G, and if it includes one, then that edge joins the middle vertex of P to the middle vertex of Q.

We say that G is *well-behaved* if every path P of length at least one and at most four with ends $u, v \in V(\mathcal{R})$ and otherwise disjoint from $\mathcal{R}$ is allowable.

We say that a configuration γ *strongly appears in* G if it both appears and weakly appears in G and

- if $u, v \in \mathcal{A}_\gamma$ are distinct, then at least one of u, v is internal,
- if $u, v \in \mathcal{I}_\gamma$ are distinct, $u \in V(\mathcal{R})$, and $w \in V(\mathcal{R})$ is a neighbor of v, then u and w are adjacent and $uw, wv \in E(G_\gamma)$, and
- if γ is isomorphic to R7, then the vertices corresponding to v_2 and z are distinct, non-adjacent and have no common neighbor distinct from v_1, v_3, x_6 and x_7.

Lemma 6.1. *Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying* (I0)*,* (I2) *and* (I8)*, and assume that G is well-behaved. If a configuration isomorphic to one of* R1*,* R2*, …,* R7 *appears in G and no cycle in G of length four or less touches it, then either a good configuration strongly appears in G, or Σ is a disk, $\mathcal{R} = \{R\}$, R has length $2s$ for some $s \in \{5, 7\}$, $V(G) = V(R) \cup V(C)$ for a cycle C of length s, and each vertex of C is internal of degree three and has one neighbor in R.*

Proof. Let γ be a good configuration appearing in G, such that no cycle in G of length four or less touches γ. If possible, we choose γ so that it is equal to one of R1, R6′, R7′, R7″, R7‴ or R7⁗. We claim that, possibly after relabeling the vertices of G_γ, γ strongly appears in G. To prove that we first notice that the first condition of weak appearance holds by hypothesis and (I8)—if $x_3 = x_7$, then $x_3v_3v_{12}v_6v_7$ is a 5-cycle separating x_1 from x_6. The third condition is implied by appearance. The second condition of weak appearance follows from our choice of γ and the fact that no cycle of length at most four touches γ. For example, if γ is R7, then v_2 and v_7 are not adjacent, because R7′ does not appear in G by the choice of γ. Additionally, when γ is R5, we use (I2) to show that v_1 is not adjacent to v_5.

It remains to prove that γ satisfies the conditions of strong appearance. Let us discuss the configurations separately. If γ is R1 or R2, it suffices to show that we can choose the labels of the vertices of γ so that x_1 is internal. If that is not possible, then each vertex of γ is adjacent to a vertex belonging to $\mathcal{R}$. Since G is well-behaved it follows that there exists a ring $R \in \mathcal{R}$ that satisfies the conclusion of the lemma for $s = 5$ if γ is R1 and for $s = 7$ when γ is R2.

If γ is R3, we only need to prove the second condition of strong appearance. Suppose that say $v_3 \in V(\mathcal{R})$ and v_5 has a neighbor x_5 in $\mathcal{R}$ other than v_4. Since G is well-behaved, v_4 is an internal vertex and $v_3v_4v_5x_5$ together with a path in $\mathcal{R}$ bound a 5-face, implying that v_4 has degree two. This contradicts (I0).

If γ is R4, then note that the path $x_1v_1v_2v_3x_3$ is not allowable, since by the definition of appearance, v_2 has degree at least four. Therefore, at least one of x_1 and x_3 is internal, and γ strongly appears.

If γ is R5, we need to prove the first and the second condition of strong appearance. For the first one, observe that the path $v_2v_1v_8x_8$ is not allowable, since v_1 has degree at least three. For the second condition, since γ appears in G, we have that v_4 is internal; thus it suffices to consider the case that x_6 and a neighbor x_4 of v_4 belongs to $\mathcal{R}$. Since $v_3v_4v_5v_6v_7$ is not an appearance of R1 in G, v_4 has degree at least four, and thus the paths $v_2v_3v_4x_4$ and $x_4v_4v_5v_6x_6$ cannot both be allowable. It follows that v_2 is internal, and similarly all neighbors of v_2 are internal. However, then we can relabel the vertices of γ, switching v_2 with v_4, v_6 with v_8, etc., and obtain a strong appearance of R5 in G.

For the configurations R6, …, R7′′′′, the first two conditions follow from the definition of appearance. Therefore, suppose that γ is R7 and let us now consider the last condition in the definition of strong appearance. Again, we we use symmetry: if the condition does not hold for γ we swap v_1 and v_3, v_6 and v_8, and so on. The vertex v_2 cannot be equal to or adjacent to both z and z_1, since $v_2 \neq x_7$ (otherwise, R7′ would appear in G), x_7 has degree at least three and no cycle of length at most four touches γ. Unless the condition holds, we can assume that $z_1 \neq v_2$, z_1 is not adjacent to v_2 and that z_1 and v_2 have a common neighbor x_2 distinct from v_1, v_3, x_7 and x_8. Since no cycle of length at most four touches γ, we have $z \notin \{v_2, v_3, x_2\}$. If $z = v_1$, then the cycle $K = v_1v_{11}v_8v_7x_7$ separates z_1 from v_2 by (I8), and thus $x_2 \in V(K)$. This is a contradiction, since then a cycle of length at most four touches γ. Therefore, z is distinct from and non-adjacent to v_2. Furthermore, z is not adjacent to x_2, as otherwise $x_2zx_7z_1$ touches γ. □

Let G be a graph in a surface Σ with rings $\mathcal{R}$, let γ be a good configuration that weakly appears in G, let G' be the γ-reduction of G, and let C' be a cycle in G'. If C is a cycle in G such that either $C = C'$ or C' is obtained from C by replacing a squashed edge by one of the corresponding edges of G, then we say that C is a *lift* of C'.

Lemma 6.2. *Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying* (I0)*,* (I3)*,* (I8) *and* (I9)*, let γ be a good configuration that strongly appears in G, and let G' be the γ-reduction of G with respect to a 3-coloring ϕ of $\mathcal{R}$. If G is $\mathcal{R}$-critical and C' is a cycle in G' of length at most four, then either a lift of C' is a cycle in G, or C' is noncontractible and there exists a noncontractible cycle C in G such that C touches γ and $|C| - |C'| \leq 3$. Furthermore, all ring vertices of C' belong to C; and if C' is a triangle disjoint from the rings and its vertices have distinct pairwise non-adjacent neighbors in a ring R of length 6, then two vertices of C have distinct non-adjacent neighbors in R.*

Proof. Suppose that $C' \subseteq G'$ is a cycle of length 3 or 4 such that no lift of C' is a cycle in G. Let us discuss the possible configurations γ:

- γ is isomorphic to one of R1, R2, R6, R6$'$, R7$'$, R7$''$, R7$'''$ or R7$''''$, or to R4 and both x_4 and x_5 belong to $\mathcal{R}$ and $\phi(x_4) = \phi(x_5)$. We are adding an edge e between vertices $x, x' \in \mathcal{A}_\gamma$ along the replacement path $P \subset G$ of length at most 4. Note that $e \in E(C')$. Let $C \subseteq G$ be the cycle obtained from C' by replacing e with P. Clearly, $|C| \leq |C'| + 3 \leq 7$. Let us remark that C is indeed a cycle (i.e., if γ is R4, then $v_2 \notin V(C')$), since no non-ring cycle of length at most four touches γ by the definition of weak appearance. Note that P is not a part of a boundary of a face in any of the configurations; thus C does not bound a face in G. By (I9), C is not contractible; hence C' is not contractible, either.

- γ is R3: Let w be the vertex of G' obtained by identifying v_1 with v_3 and v_5. Note that $w \in V(C')$ and consider the edges $e_1, e_2 \in E(C')$ incident with w. Unless C' corresponds to a cycle of length $|V(C')|$ in G, e_1 and e_2 are incident with distinct vertices $a, b \in \mathcal{I}_\gamma$, and the cycle C obtained from C' by adding the replacement path avb between a and b has length at most $|C'|+2 \leq 6$. Note that C' and C have the same homotopy. Suppose that they are contractible. By (I9) that implies that C' bounds a face h and v has degree two. By (I0), v belongs to $\mathcal{R}$, and since at least one of a and b is internal, v is a vertex ring. However, then h is the cuff face of v and C' is not contractible. This is a contradiction.

- γ is R4 and at least one of x_4 and x_5 is internal: Let w be the vertex obtained by identifying x_4 and x_5. If x_1x_3 is not an edge of C', then (since C' is not a cycle of G) the cycle C obtained from C' by replacing w by the path $x_4v_4v_5x_5$ satisfies $6 \leq |C| \leq 7$ and does not bound a face; thus neither C nor C' is contractible. Let us assume that $x_1x_3 \in E(C')$. Similarly, we deal with the case that $w \notin V(C')$ or that both edges incident with w in C' correspond to edges incident to one of x_4 and x_5.

  Suppose now that the neighbors of w in C' are adjacent to x_4 and x_5. Since no non-ring cycle of length at most four touches γ by the definition of weak appearance, we have $x_1x_5, x_3x_4 \notin E(G)$; thus by symmetry we may assume that $x_1x_4 \in E(C')$ and x_3 and x_5 are joined by a path P of length at most two in C'. By (I8), the 5-cycle $K = x_1v_1v_5v_4x_4$ separates x_3 from x_5; thus P is not disjoint from K. However, then a cycle of length at most four touches γ.

- γ is R4, neither x_4 nor x_5 is internal and $\phi(x_4) \neq \phi(x_5)$: Let w be the vertex created by identifying v_2 and x_5. The claim of the lemma follows by considering the non-facial cycle C obtained from C' by replacing w with $v_2v_1v_5x_5$.

- γ is R5: Let w be the vertex obtained by identifying v_4 and x_6. Let C be the cycle obtained from C' by replacing v_2x_8 by $v_2v_1v_8x_8$ or w by $v_4v_5v_6x_6$ or both. If we performed at most one replacement, then $|C| \leq |C'| + 3$ and the claim follows from (I9).

  Otherwise, $v_2x_8 \in E(C')$ and $w \in V(C')$, and since no non-ring cycle of length at most four touches γ, there exist paths P_1 between v_2 and x_6 and P_2 between v_4 and x_8 of total length at most three. Let K_1 be the cycle consisting of $v_2v_3v_7v_6x_6$ and P_1 and K_2 the cycle consisting of $v_4v_3v_7v_8x_8$ and P_2, and by symmetry assume that $|K_1| = 5$ and $|K_2| \leq 6$. By (I8) the cycle K_1 separates v_4 from v_8; thus P_2 intersects K_1. However, that contradicts the fact that no non-ring cycle of length at most four touches γ.

- γ is R7: Let w be the vertex obtained by identifying x_6 and x_7. Let C_1 be the cycle obtained from C' by replacing x_1x_3 by $x_1v_1v_2v_3x_3$ or w by $x_6v_6v_7x_7$ or both. If we performed only one replacement, then $|C_1| = |C'| + 3$ and the claim of the lemma follows from (I9), with $C = C_1$.

  Otherwise, let C_2 be the closed walk obtained from C_1 by replacing $x_6v_6v_7x_7$ by x_6zx_7; we have $|C_2| = |C'| + 5 \leq 9$. Since γ appears, observe that all vertices of C' are internal and at most one of them has a neighbor in a ring. Note that C_2 is a cycle, since otherwise a non-ring cycle of length at most four touching γ is a subgraph of C_2. Suppose now that C_2 consists of $x_1v_1v_2v_3x_3$, a path P_1 from x_3 to x_7, the path x_7zx_6 and a path P_2 from x_6 to x_1, where the total length of P_1 and P_2 is at most three. Let K_1 be the cycle consisting of P_1 and $x_3v_3v_{12}v_6v_7x_7$ and K_2 the cycle consisting of P_2 and $x_1v_1v_{11}v_{12}v_6x_6$. Note that $\min(|K_1|, |K_2|) \leq 6$, and by (I8), the shorter of the two cycles is separating. It follows that K_1 and K_2 intersect in a vertex distinct from v_{12} and v_6, This is a contradiction, since the vertices of C_2 are mutually distinct and none of them is equal to $v_7, v_{11} \notin V(G')$.

  Therefore, C_2 consists of $x_1v_1v_2v_3x_3$, a path Q_1 of length $l_1 \geq 1$ from x_3 to x_6, the path x_6zx_7 and a path Q_2 of length l_2 from x_7 to x_1, where $l_1 + l_2 \leq 3$. Let L_1 be the cycle consisting of Q_1 and $x_3v_3v_{12}v_6x_6$ and L_2 the cycle consisting of Q_2 and $x_1v_1v_{11}v_8v_7x_7$. Note that neither L_1 nor L_2 bounds a face, $|L_1| = 4 + l_1 \leq 7$ and $|L_2| = 5 + l_2 \leq 7$, thus by (I9) neither

L_1 nor L_2 is contractible. Furthermore, $|L_1|+|L_2| \le 9+l_1+l_2 \le 12$, thus there exists a cycle $C \in \{L_1, L_2\}$ of length at most $6 \le |C'|+3$ touching γ.

Let us now show that the cycle C' is not contractible. Assume for a contradiction that C', and hence also C_2, is contractible. Let $\Delta \subseteq \Sigma$ be an open disk bounded by C_2. Note that Δ does not consist of a single face, since at least one edge incident with v_1 or v_2 lies inside Δ. By (I9), Δ consists of two or three faces, and in the latter case, $|C_2| = 9$ and three vertices of C_2 have a common neighbor.

It follows that $v_{11}, v_{12} \notin \Delta$, and thus the edge joining v_2 with its neighbor $x_2 \notin \{v_1, v_3\}$ lies in Δ. Since γ appears strongly in G, we have that $x_2 \neq z$ and that z is an internal vertex. We conclude that $\deg(z) = 3$ and z has a neighbor inside Δ distinct from x_6 and x_7. By (I3) and (I9), this neighbor is equal to x_2. However, this contradicts the assumption that γ appears strongly in G.

If γ is R7 and C is one of the cycles L_1 and L_2, then since γ appears in G, the vertices x_1, x_3, x_6, x_7 and all their neighbors in G are internal. Consequently, x_1, x_3 and all their neighbors are internal in G'. It follows that C' contains no ring vertex, and that at most two distinct ring vertices have a neighbor in C', hence the last claim of the lemma holds trivially.

Otherwise, C is obtained from C' by replacing a new edge by a path in G, or by adding a replacement path between vertices of $\mathcal{I}_\gamma$, or both. Therefore, every ring vertex of C' also belongs to C. Suppose that C' is a triangle with vertices c_1, c_2 and c_3, that $R = r_1r_2r_3r_4r_5r_6$ is a ring and that c_1r_1, c_2r_3 and c_3r_5 are edges of G'. If, say, r_1 has no neighbor in C, then either r_1c_1 is a new edge, or one of r_1 and c_1 is the new vertex created by the identification of the vertices of $\mathcal{I}_\gamma$. Since C is not a lift of C', in the former case C' contains a new vertex that is replaced by a path in C, and in the latter case C' contains a new edge. Therefore, c_2r_3 and c_3r_5 are edges of G.

□

7 Contributions of faces

Let G be a graph in a surface Σ with rings $\mathcal{R}$ satisfying (I3). Let γ be a good configuration that strongly appears in G, let G' be the γ-reduction of G, and let G'' be a subgraph of G' that includes all the rings and satisfies (I0).

Let f'' be an internal face of G'', and let H be the subgraph of G'' that forms the boundary of f''. We wish to define a subgraph $J_{f''}$ of G that will correspond to H, and a union of faces of $J_{f''}$ that will correspond to f''.

Let us recall that during the construction of the graph G', parallel edges may have been removed (e.g., if γ is R5 and v_4 and x_6 have a common neighbor), but we have retained the correspondence of each non-squashed edge e of G' to a unique edge of G (which also determined the placement of e in the embedding of G'). We now define the edge-set of $J_{f''}$, by replacing pieces of the boundary of f'' by appropriate replacement paths. More precisely, we apply the following construction to each boundary walk C of f''. Let C be $v_1, e_1, v_2, e_2, \ldots, v_m, e_m$ and let $e_{m+1} = e_1$, $v_{m+1} = v_1$, $e_0 = e_m$ and $v_0 = v_m$. Replace each edge e_i of C by a path P_i defined as follows:

- If e_i is a new edge, then P_i is the corresponding replacement path.
- If e_i is a squashed edge, then a vertex v_j with $j \in \{i, i+1\}$ is a new vertex. Let $e = e_{2j-i-1}$ be the other edge of C incident with v_j. Note that $e \neq e_i$, since otherwise v_j would have degree one; however, by the assumption (I0), v_j would be a vertex ring, and the corresponding vertex of G would belong to $\mathcal{I}_\gamma$, contrary to the definition of appearance. Consequently, e is not a squashed edge, as the inspection of the good configurations shows that a new vertex is only incident with at most one squashed edge. The edge of G corresponding to e is incident with a vertex $v \in \mathcal{I}_\gamma$. Let P_i be the edge vv_{2i-j+1}.
- Suppose that e_i is neither new nor squashed. If v_{i+1} is a new vertex, e_{i+1} is not a squashed edge, e_i is incident in G with a vertex $u \in \mathcal{I}_\gamma$, e_{i+1} is incident in G with $v \in \mathcal{I}_\gamma$ and $u \neq v$, then let P_i consist of e_i and the replacement path between u and v.
- Otherwise, let P_i be the path with edge-set $\{e_i\}$.

The newly constructed walk has the same homotopy as C. The graph $J_{f''}$ is defined as the result of applying the above construction to every boundary walk of f''. It should be noted that even though f'' is a face of G'', it may correspond to several faces of $J_{f''}$. Let the set of these faces of $J_{f''}$ be denoted by $S_{f''}$. For example, suppose that G' was created by reducing the configuration R3, G'' does not contain any of the squashed edges and f'' is bounded by a cycle that contains the new vertex and edges that were incident with v_3 and v_5, and suppose that v_4 is incident with f'' as well. Then $J_{f''}$ contains the replacement path $v_3v_4v_5$, which can split f'' to two faces sharing the vertex v_4.

Let us also remark on another somewhat subtle issue. In the definition of "appear", we require that vertex rings do not belong to $\mathcal{I}_\gamma$. The reason for this restriction is the following. Later, we use the fact that the faces in $S_{f''}$ do not contain any rings. If say $u \in \mathcal{I}_\gamma$ were a vertex ring identified with another vertex $v \in \mathcal{I}_\gamma$ to a new vertex w and all edges in f'' that are incident with w corresponded to edges of G incident with v, then this condition could fail.

The *elasticity* of f'' is $\mathrm{el}(f'') = \left(\sum_{f\in S_{f''}} |f|\right) - |f''|$. Note that f'' can have non-zero elasticity only if $J_{f''}$ contains at least one replacement path. Using this fact and the inspection of the configurations, we observe the following.

Lemma 7.1. *Let G, γ, G', G'' be as above. Then G'' has at most three faces with non-zero elasticity, and the sum of the elasticities of the faces of G'' is at most 10. Furthermore, if an internal face f'' of G'' is closed 2-cell or omnipresent, then el$(f'') \le 5$, and if the inequality is strict, then el$(f'') \le 3$.*

Let G be a graph in a surface Σ with rings $\mathcal{R}$, let J be a subgraph of G, and let S be a subset of the set of faces of J such that

(1) *J is equal to the union of the boundaries of the faces in S and whenever C is a cuff intersecting a face $f \in S$, then C is incident with a vertex ring belonging to J.*

We define $G[S]$ to be the subgraph of G consisting of J and all the vertices and edges drawn inside the faces of S. Let $C_1, C_2, \ldots, C_k$ be the boundary walks of the faces in S (in case that a vertex ring $R \in \mathcal{R}$ forms a component of a boundary of a face in S, we consider R itself to be such a walk). We would like to view $G[S]$ as a graph with rings $C_1, \ldots, C_k$. However, the C_i's do not necessarily have to be disjoint, and they do not have to be cycles or isolated vertices. To overcome this difficulty, we proceed as follows: Let Z be the set of cuffs incident with the vertex rings that form a component of J by themselves, and let $\hat{Z} = \bigcup_{R \in Z} \hat{R}$, where $\hat{R}$ is as in Section 2. Suppose that $S = \{f_1, \ldots, f_m\}$. For $1 \le i \le m$, let Σ'_i be a surface with boundary B_i such that $\Sigma'_i \setminus B_i$ is homeomorphic to f_i. Let $\theta_i : \Sigma'_i \setminus B_i \to f_i$ be a homeomorphism that extends to a continuous mapping $\theta_i : \Sigma'_i \to \overline{f_i}$, where $\overline{f_i}$ denotes the closure of f_i. Let $\Sigma_i = \Sigma'_i \setminus \theta_i^{-1}(\hat{Z} \cap f_i)$, and let G_i be the inverse image of $G \cap \overline{f_i}$ under θ_i. Then G_i is a graph normally embedded in Σ_i. We say that the set of embedded graphs $\{G_i : 1 \le i \le m\}$ is a *G-expansion of S*. Note that there is a one-to-one correspondence between the boundary walks of the faces of S and the rings of the graphs in the G-expansion of S; however, each vertex of J may be split to several copies.

We define the *G-expansion of f''* to be the G-expansion of $S_{f''}$. The following lemma is straightforward.

Lemma 7.2. *Let G, γ, G', G'' be as above, and let f be an internal face of G. Then either there exists a unique internal face f'' of G'' such that f corresponds to an internal face of a member of the G-expansion of f'' or γ is isomorphic to* R3 *and f is the 6-face of $\mathcal{F}_\gamma$.*

Let us now give an informal summary of what we are trying to achieve in this section. We assign weights to the faces of embedded graphs according to the function s as described in Section 5, and we aim to show that the sum of the weights of the faces of G is bounded by the sum of the weights of the faces of G''. To do so, we would like to claim that the sum w of the weights of the faces of members of the G-expansion of f'' is bounded by the weight w'' of f''. In Theorem 8.5, we will show that this claim holds, provided that the elasticity of f'' is small and the G-expansion of f'' is not a singleton set consisting of one of a few exceptional graphs. Here, we assign a *contribution* $c(f'')$ to each face f'' of G'' according to the criteria that we later prove to ensure that $w \le w'' - c(f'')$. Furthermore, we argue that the sum of the contributions of all faces is non-negative.

Let us now proceed more formally. Let G be a graph in a surface Σ with rings $\mathcal{R}$. If Σ is a disk and $\mathcal{R} = \{R\}$, then we say that G is a *plane graph with*

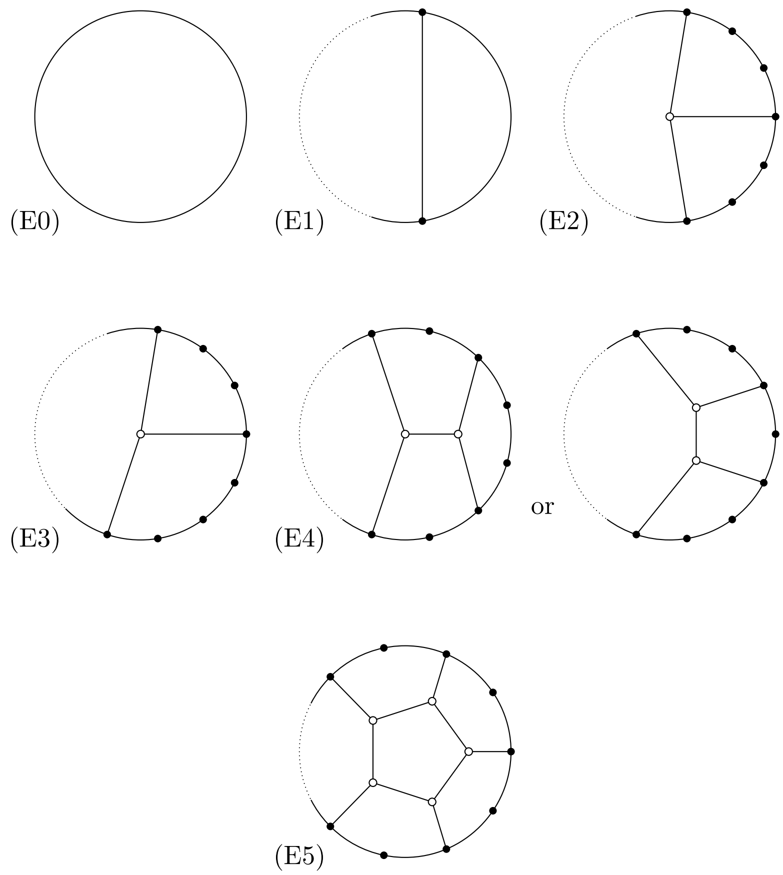

Figure 2: Exceptional graphs.

one ring R. We say that a plane graph G with one ring R of length $l \geq 5$ is *exceptional* if it satisfies one of the conditions below (see Figure 2):

(E0) $G = R$,

(E1) $l \geq 8$ and $|E(G)| - |E(R)| = 1$,

(E2) $l \geq 9$, $V(G) - V(R)$ has exactly one vertex of degree three, and the internal faces of G have lengths $5, 5, l-4$,

(E3) $l \geq 11$, $V(G) - V(R)$ has exactly one vertex of degree three, and the internal faces of G have lengths $5, 6, l-5$,

(E4) $l \geq 10$, $V(G) - V(R)$ consists of two adjacent degree three vertices, and the internal faces of G have lengths $5, 5, 5, l-5$,

(E5) $l \geq 10$, $V(G) - V(R)$ consists of five degree three vertices forming a facial cycle of length five, and the internal faces of G have lengths $5, 5, 5, 5, 5, l-5$.

We say that G is *very exceptional* if it satisfies (E0), (E1), (E2) or (E3).

Let us now show the following lemma, which we use to analyze omnipresent faces.

Lemma 7.3. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, let G be well-behaved, let it satisfy (I0), (I4) and (I8), let γ be a good configuration that strongly appears in G, let G' be the γ-reduction of G, let G'' be a subgraph of G' that includes all the rings and satisfies (I0), and let H be a component of G'' that contains a new edge or a new vertex. Assume that either Σ is a disk and $|\mathcal{R}| = 1$ and every face of G'' is closed 2-cell, in which case we let R be the unique member of $\mathcal{R}$, or that G'' has an omnipresent face, in which case we let R be the boundary walk of the omnipresent face that is a subgraph of H. In either case H can be regarded as a plane graph with one ring R. Then H is not very exceptional. Furthermore, if γ is isomorphic to one of* R6, R6′, R7, R7′, R7″, R7‴ *or* R7⁗, *then H is not exceptional.*

Proof. Since H contains a new vertex or a new edge, it does not satisfy (E0). If γ is isomorphic to one of R7, R7′, R7″, R7‴ or R7⁗, then all vertices in $\mathcal{A}_\gamma \cup \mathcal{I}_\gamma$ and all their neighbors are internal by the definition of appearance, and thus each new edge or new vertex is at distance at least two from R. It follows that H cannot be exceptional. Similarly, we exclude the case that γ is isomorphic to R6 or R6′. Thus, assume that γ is one of R1, …, R5.

Suppose that H contains a new edge xy. Note that since γ does not touch a non-ring cycle of length at most four by the definition of weak appearance, neither x nor y is a new vertex. Since γ appears strongly in G, we may assume that x is an internal vertex; thus H does not satisfy (E1). Suppose that H satisfies (E2) or (E3). Then, in H the vertex x has three neighbors in R. On the other hand, x has at most one neighbor in R in G, by (I4). We conclude that x is adjacent to a new vertex in G'' that belongs to R. It follows that γ is R4 or R5, and in the former case at least one of x_4 and x_5 is internal. Let $\mathcal{I} = \mathcal{I}_\gamma$ if γ is R5 and $\mathcal{I} = \{x_4, x_5\}$ if γ is R4. Note that there exists a vertex in $\mathcal{I}$ belonging to R, and another vertex of $\mathcal{I}$ is adjacent to x in G. If γ is isomorphic to R4, then by symmetry we may assume that x_1 is adjacent to x_4 and x_3 and x_5 belong to R. However, by (I8), the cycle $x_1v_1v_5v_4x_4$ consisting of internal vertices separates x_3 from x_5, which is a contradiction. If γ is isomorphic to R5, then by the conditions of appearance, v_4 is an internal vertex; hence x_6 belongs to R. Since v_2 and v_4 are not adjacent, we conclude that v_4 is adjacent to x_8 and that v_2 belongs to R. However, this again contradicts (I8).

Therefore, we may assume that H contains a new vertex, but not a new edge. Suppose first that γ is not isomorphic to R4. If H satisfied (E1), then by (I4) there would exist vertices $x \in \mathcal{I}_\gamma \cap V(R)$ and $y \in \mathcal{I}_\gamma \setminus V(R)$ and a neighbor z of y in R, where z is not adjacent to x. However, this contradicts the assumption that γ appears strongly in G. If H satisfies (E2) or (E3), then by (I4) we have $|\mathcal{I}_\gamma| = 3$ (thus γ is R3), all elements of $\mathcal{I}_\gamma$ are internal and each of them has exactly one neighbor in R. This is excluded, since γ appears in G.

Finally consider the case that γ is R4 and H does not contain a new edge. By (I4), H does not satisfy (E2) or (E3); thus suppose that H satisfies (E1). If x_4 is an internal vertex, this implies that $x_5 \in V(R)$ and x_4 has a neighbor w in R distinct from z. By (I4), z is an internal vertex. Since G is well-behaved, the path x_5zx_4w forms a part of a boundary of a 5-face; thus z has degree two, contrary to (I0). The case that x_5 is internal is symmetric; thus assume that both x_4 and x_5 belong to R. Then v_2 is an internal vertex of degree at least four by the definition of weak appearance and has a neighbor $w \in V(R)$. However, since G is well-behaved, the subpaths v_3v_2w and v_1v_2w of the paths $x_4v_4v_3v_2w$ and $x_5v_5v_1v_2w$ form parts of boundaries of faces, implying on the contrary that v_2 has degree three. □

Let $s : \{5, 6, \ldots\} \to \mathbb{R}^+$ be an increasing function, to be specified later, such that

(S4) $14s(5) \leq s(6)$, $135s(5) \leq s(7)$, $4s(6) \leq s(7)$, $3s(7) \leq s(8)$, $2s(8) \leq s(7) + s(9)$ *and* $s(l) = l - 8$ *for* $l \geq 9$.

It follows that the function s satisfies

(S5) $s(x + a) - s(x) \leq s(y + a) - s(y)$ *for all integers* $y \geq x \geq 5$ *and* $a \geq 0$.

We will refer to condition (S5) as *convexity*.

Let G be a graph in a surface Σ with rings $\mathcal{R}$, let γ be a good configuration that strongly appears in G, let G' be the γ-reduction of G, and let G'' be a subgraph of G' that includes all the rings and satsfies (I0). For every internal face f'' of G'' we define its *contribution* $c(f'')$ as follows.

Let f'' be a closed 2-cell internal face of G'', and let $G_{f''}$ be a member of the G-expansion of f''. Then the contribution of f'' is defined according to the following rules:

- If $|S_{f''}| = 1$ and $G_{f''}$ satisfies (E0), then $c(f'') = -\infty$ if f'' has non-zero elasticity and $c(f'') = 0$ otherwise.
- If $|S_{f''}| = 1$ and $G_{f''}$ satisfies (E1), then $c(f'') = -\infty$ if $\mathrm{el}(f'') = 5$ and $c(f'') = s(8 - \mathrm{el}(f'')) - 2s(5)$ otherwise.
- If $|S_{f''}| = 1$ and $G_{f''}$ satisfies (E2), then $c(f'') = -\infty$ if $\mathrm{el}(f'') = 5$ and $c(f'') = s(9 - \mathrm{el}(f'')) - 3s(5)$ otherwise.
- If $|S_{f''}| = 1$ and $G_{f''}$ satisfies (E3), then $c(f'') = s(11 - \mathrm{el}(f'')) - 2s(6) - s(5)$.
- If $|S_{f''}| = 1$ and $G_{f''}$ satisfies (E4) or (E5), or if $|S_{f''}| = 2$ and $G_{f''}$ consists of two cycles such that one of them has length 5, then $c(f'') = s(10 - \mathrm{el}(f'')) - 6s(5)$.
- If $|S''_f| = 1$ and $G_{f''}$ is not exceptional, and
  - $G_{f''}$ contains a path $P = v_1v_2v_3v_4$ such that $v_1, v_4 \in V(J_{f''})$, $v_2, v_3 \notin V(J_{f''})$ and both of the open disks bounded by P and paths in $J_{f''}$ contain at least two vertices of G, then $c(f'') = s(7)$.
  - Otherwise, $c(f'') = s(11 - \mathrm{el}(f'')) - s(6) + 5s(5)$.

- If $|S_{f''}| = 2$ and $G_{f''}$ does not consist of two cycles such that one of them has length 5, or if $|S_{f''}| \geq 3$, then $c(f'') = s(12 - \mathrm{el}(f'')) - 2s(6)$.

Suppose now that f'' is an omnipresent face of G''. Let $G''_1, G''_2, \ldots, G''_k$ be the components of G'' such that G''_i contains the ring $R_i \in \mathcal{R}$. If there exist $i \neq j$ such that $G''_i \neq R_i$ and $G''_j \neq R_j$, then $c(f'') = 1$. Otherwise, we may assume that $G''_i = R_i$ for $i \geq 2$. If G''_1 satisfies (E0), (E1), (E2) or (E3), then $c(f'') = -\infty$. If G''_1 satisfies (E4) or (E5), then $c(f'') = 5 - \mathrm{el}(f'') - 5s(5)$, otherwise $c(f'') = 5 - \mathrm{el}(f'') + 5s(5)$.

This completes the definition of contribution of faces of G''. We define the contribution of G'' as $c(G'') = -\delta + \sum_{f'' \in F(G'')} c(f'')$, where δ is $s(6)$ if γ is isomorphic to R3 and 0 otherwise.

Lemma 7.4. *Let G be a well-behaved graph in a surface Σ with rings $\mathcal{R}$ satisfying (I0)–(I4) and (I8), let γ be a good configuration strongly appearing in G, let G' be the γ-reduction of G. Suppose that G'' is a subgraph of G' that includes $\mathcal{R}$, G'' satisfies (I6), and G'' contains a new vertex or a new edge. If each face of G'' is closed 2-cell or omnipresent, then $c(G'') \geq 0$. Furthermore, if each face of G'' is closed 2-cell, then G'' has a face of length at least six, and if Σ is a disk and $|\mathcal{R}| = 1$, then $c(G'') \geq 10s(5)$.*

Proof. We first prove the first and third statement of the lemma. The proof will show that there is a face of positive contribution, and we will utilize that face in the proof of the second statement of the lemma. Let us note that G'' satisfies the assumptions of Lemma 7.3, and thus the contribution of the omnipresent face (if G'' has such a face) is not $-\infty$.

We may assume that there exists a face of non-zero elasticity, for otherwise all faces have non-negative contribution and the face f'' of G'' that includes all vertices of $\mathrm{dom}(d_\gamma)$ satisfies $c(f'') \geq s(7)$.

Let us argue that if a face f'' that is closed 2-cell has non-zero elasticity, then either $S_{f''}$ has at least two faces or the interior of the unique face of $S_{f''}$ contains an edge of G. Indeed, most replacement paths are incident with edges on both of its sides; thus if such a replacement path is used in $J_{f''}$, then at least one such edge lies in $S_{f''}$. The exceptions are the replacement paths in R3, R4 and the replacement path between the vertices of $\mathcal{I}_\gamma$ in R7. In these configurations, the middle vertex v of the replacement path could also lie on the boundary walk of f'', in which case all the edges incident with v could belong to $J_{f''}$ or lie outside of $S_{f''}$. However, then $S_{f''}$ has at least two faces. We conclude that if $c(f'') = -\infty$, then $\mathrm{el}(f'') = 5$ and two replacement paths are used in the construction of $J_{f''}$.

Let us first consider the case when every face of G'' has elasticity at most three. Then the contribution of each face is greater or equal to $-s(5)$. Furthermore, there are at most two faces of elasticity three, at most one of them has contribution $-s(5)$, and every other face has non-negative contribution. If G'' has an omnipresent face, then $c(G'') \geq 2 - 6s(5) - s(6)$ or $c(G'') \geq 1 - s(6)$, and hence we may assume that all faces of G'' are closed 2-cell. Observe that G has a face f'' such that at least one vertex of $\mathrm{dom}(d_\gamma)$ is contained inside a face of $S_{f''}$. For this face, we have $c(f'') \geq s(6) - 3s(5)$. Furthermore, if γ is R3, then the elasticity of f'' is at most two; thus $c(f'') \geq s(7) - 3s(5)$, and all other faces of G'' have non-negative contribution. Therefore, $c(G'') \geq \min(s(6) - 4s(5), s(7) - s(6) - 3s(5)) \geq 10s(5)$, because $14s(5) \leq s(6)$. This completes the case when every face of G'' has elasticity at most three.

Thus we may assume that G'' has a face f_2 of elasticity five. It follows that γ is R4, R5, or R7, and G'' contains a new edge incident with two faces of non-zero elasticity, say f_1 and f_2, where f_2 contains all vertices of $\mathrm{dom}(d_\gamma)$. Furthermore, G'' contains a new vertex w incident with f_2 and possibly another face f_3 of non-zero elasticity.

Then the elasticity of f_2 is 5, and by inspection of the configurations, we conclude that $c(f_2) \geq -5s(5)$. Furthermore, if γ is isomorphic to R7, then $c(f_2) = s(7)$ if f_2 is closed 2-cell, and by Lemma 7.3, we have $c(f_2) \geq 5s(5)$ if f_2 is omnipresent.

Assume now that either f_2 is the only face of G'' with non-zero elasticity that is incident with w, or that $f_1 \neq f_3$. Consider a face $f \in \{f_1, f_3\}$ with non-zero elasticity. Since $\mathrm{el}(f) \leq 3$, we have $c(f) \geq -s(5)$. Furthermore, if f is omnipresent, then by Lemma 7.3, we have $c(f) \geq 2 - 5s(5)$ or $c(f) \geq 1$, and hence $c(G'') > 10s(5)$; thus we may assume that each such face f is closed 2-cell. If γ is R5, then $\mathrm{el}(f_1) = 2$ and $c(f_1) \geq s(6) - 2s(5)$. Similarly if γ is R4, then by the definition of appearance v_2 has degree at least 4 in G_{f_1}; hence $c(f_1) \geq s(7) - 6s(5)$. In both cases we get $c(G'') \geq 10s(5)$. Thus we may assume that γ is R7. If Σ is a disk and $|\mathcal{R}| = 1$, then f_2 is not omnipresent, and hence $c(G'') \geq s(7) - 2s(5) \geq 10s(5)$, because (using the numbering of the vertices as in R7) the path $v_3v_{12}v_6x_6$ shows that the contribution of f_2 is $s(7)$. Otherwise, $c(G'') \geq 3s(5)$, because $c(f_2) \geq 5 - \mathrm{el}(f_2) + 5s(5) = 5s(5)$, using Lemma 7.3.

Therefore, we may assume that $f_1 = f_3$ and f_1 has elasticity 5. If Σ were a disk and $|\mathcal{R}| = 1$, or if f_1 or f_2 were omnipresent, then w together with a vertex of the new edge would form a 2-cut in G'', contradicting (I6). We conclude that both f_1 and f_2 are closed 2-cell and that either Σ is not a disk or $|\mathcal{R}| \neq 1$; hence, it suffices to show that $c(G'')$ is non-negative.

Suppose that γ is R4. Since γ weakly appears in G, we have that no cycle of length at most 4 touches γ, and thus $z \neq v_2$. The fact that v_2 has degree at least four in G_{f_1} implies that $c(f_1) \geq 5s(5)$, unless G_{f_1} consists of a 5-cycle and a $|f_1|$-cycle. In that case z would be a vertex of degree two, and by (I0) it would form a vertex ring. However, then f_1 could not be closed 2-cell, since z would be either an isolated vertex or a vertex of degree one forming a part of the boundary of f_1. This is a contradiction; thus $c(G'') \geq 0$.

Assume next that γ is R5. By (I1) and (I2) we have that G_{f_1} is not an exceptional graph (considering the cycle formed by the path $v_1v_8v_7v_6v_5$ together with a path in G_{f_1}), thus again $c(f_1) \geq 5s(5)$ and $c(G'') \geq 0$.

Finally, let γ be R7. If $|S_{f_1}| \geq 2$, then $c(f_1) \geq -5s(5)$. Otherwise, note that z is not a vertex ring, thus by (I0), it is incident with an edge lying inside the face of S_{f_1}. Since γ appears strongly in G, we have that v_2 is not adjacent to z, and v_2 and z have no common neighbor distinct from v_1, v_3, x_6 and x_7. It follows that G_{f_1} does not satisfy (E1), (E2) or (E3), and thus $c(f_1) \geq -5s(5)$.

Therefore, $c(G'') \geq s(7) - 5s(5) > 0$.

Therefore, both inequalities from the statement of the lemma hold. Furthermore, note that in all the cases, at least one face f'' of G'' has positive contribution; and if f'' is closed 2-cell, then $|f''| \geq 6$. □

8 Plane graphs with one ring

Before we turn our attention to plane graphs with one ring, let us show several properties of critical graphs. Let us recall that $\mathcal{R}$-critical graphs were defined at the end of Section 2.

Lemma 8.1. *Let G be a graph in a surface Σ with rings $\mathcal{R}$. If G is $\mathcal{R}$-critical, then it satisfies* (I0), (I1) and (I2).

Proof. If G contains an internal vertex v of degree at most two, then let $G' = G - v$. If G contains a cycle C consisting of internal vertices of degree three that has even length or two vertices of C have adjacent neighbors, then let $G' = G - V(C)$. For any precoloring ψ of $\mathcal{R}$ that extends to a 3-coloring ϕ of G', observe that ϕ can be extended to a 3-coloring of G. This contradicts the assumption that G is $\mathcal{R}$-critical. □

By the theorem of Grötzsch, no component of a critical graph is a triangle-free planar graph. This observation can be strengthened as follows.

Lemma 8.2. *Let G be a graph in a surface Σ with rings $\mathcal{R}$. Suppose that each component of G is a planar graph containing exactly one of the rings. If G is $\mathcal{R}$-critical and contains no non-ring triangle, then each component of G is 2-connected and G satisfies (I6).*

Proof. We can consider each component of G separately; thus assume that Σ is the sphere and G has only one ring R. Firstly, observe that G is 2-connected; otherwise, it contains proper subgraphs G_1 and G_2 such that $G = G_1 \cup G_2$ and $|V(G_1) \cap V(G_2)| \leq 1$. Since R is 2-connected, we can assume that $R \subseteq G_1$. However, G_2 is 3-colorable by Theorem 1.1, and since we can permute the colors arbitrarily, any precoloring of the common vertex of G_1 and G_2 extends to a 3-coloring of G_2. It follows that any 3-coloring of G_1 extends to a 3-coloring of G, contrary to the criticality of G.

Suppose now that G has an internal 2-cut, i.e., there exist proper induced subgraphs G_1 and G_2 of G such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{u, v\}$ for some vertices $u, v \in V(G)$, and $R \subseteq G_1$. Since G is 2-connected and planar, both u and v are incident with the same face of an embedding of G_2 in the plane. If u and v are adjacent, then we argue as in the previous paragraph that every precoloring of u and v by distinct colors extends to a 3-coloring of G_2, contrary to the criticality of G. If u and v are not adjacent, then let G_2' be the graph obtained from G_2 by adding vertices z_1 and z_2 and edges of paths uz_1v and uz_2v. The resulting graph is triangle-free, and by [6], every precoloring of the cycle uz_1vz_2 using three colors extends to a 3-coloring of G_2'; hence, every precoloring of u and v extends to a 3-coloring of G_2. Again, this contradicts the criticality of G. $\square$

If G is a plane graph with one ring R, then we abbreviate $\{R\}$-critical to *R-critical*. Such graphs are very important for the study of critical graphs on surfaces, for the following reason:

Lemma 8.3. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, and assume that G is $\mathcal{R}$-critical. Let C be a cycle in G bounding an open disk $\Delta \subseteq \hat{\Sigma}$ such that Δ is disjoint from the rings, and let G' be the graph consisting of the vertices and edges of G drawn in the closure of Δ. Then G' may be regarded as a plane graph with one ring C, and as such it is C-critical.*

Proof. If G' is not C'-critical, then let $e \in E(G') \setminus E(C')$ be an edge such that every precoloring of C' that extends to $G' - e$ also extends to G. Observe that every precoloring of $\mathcal{R}$ that extends to $G - e$ also extends to G, contrary to the assumption that G is $\mathcal{R}$-critical. $\square$

Critical plane graphs with one ring of length at most twelve were described by Thomassen [12] and independently by Walls [14] (actually, both papers describe ϕ-critical graphs for some fixed precoloring ϕ of R, but Theorem 8.4 follows straightforwardly from the characterizations):

Theorem 8.4. *Let G be a plane graph of girth 5 with one ring R such that $|V((R)| \leq 12$. If G is R-critical and R is an induced cycle, then*

(a) $|V(R)| \geq 9$ and $G - V(R)$ is a tree with at most $|V(R)| - 8$ vertices, or

(b) $|V(R)| \geq 10$ and $G - V(R)$ is a connected graph with at most $|V(R)| - 5$ vertices containing exactly one cycle, and the length of this cycle is 5, or

(c) $|V(R)| = 12$ and every second vertex of R has degree two and is contained in a facial 5-cycle.

In this section, we generalize this result by giving bounds on the weight of planar critical graphs with one ring.

Theorem 8.5. *Let $\epsilon \leq 1/1278$ be a fixed positive real number and let $s : \{5, 6, \ldots\} \to \mathbb{R}$ be an increasing function satisfying conditions (S1)–(S5) formulated in Sections 5 and 7. Let G be a plane graph with one ring R of length $l \geq 5$ such that G is R-critical and has no cycle of length at most four, and let w be the weight function arising from s as described in Section 5. Then*

- *$w(G, \{R\}) \leq s(l-3) + s(5)$, and furthermore,*
- *if R does not satisfy (E1), then $w(G, \{R\}) \leq s(l-4) + 2s(5)$,*
- *if (G, R) is not very exceptional, then $w(G, \{R\}) \leq s(l-5) + 5s(5)$, and*
- *if (G, R) is not exceptional, then $w(G, \{R\}) \leq s(l-5) - 5s(5)$.*

Proof. Let is note that $s(l-4)+2s(5) \leq s(l-3)+s(5)$ for $l \geq 9$ by (S4), and hence whenever G satisfies the second conclusion, it satisfies the first. If G satisfies ($E1$), then $l \geq 8$ and G has a face of length a such that $a \leq 7$. We can assume that the other face of G is at least as long as a, that is, $l+2-a \geq a$. Then, $w(G,\{R\}) = s(a)+s(l+2-a) \leq s(l-3)+s(5)$, where the inequality holds by convexity. If G satisfies (E2), then it is very exceptional, $l \geq 9$ and $w(G,\{R\}) = s(l-4)+2s(5)$. If G satisfies (E3), then it is very exceptional, $l \geq 11$ and $w(G,\{R\}) = s(l-5)+s(5)+s(6) \leq s(l-4)+2s(5)$, where the inequality follows from convexity. If G satisfies (E4) or (E5), then $l \geq 10$ and $w(G,\{R\}) \leq s(l-5)+5s(5) \leq s(l-4)+2s(5)$, where the second inequality follows from convexity and (S4). Finally, suppose that G is not exceptional. By Theorem 8.4, we have $l \geq 11$; thus $s(l-5)-5s(5) \leq s(l-5)+5s(5) \leq s(l-4)+2s(5)$ by convexity and (S4). Therefore, it suffices to prove that $w(G,\{R\}) \leq s(l-5)-5s(5)$ whenever G is not exceptional.

Suppose for a contradiction that (G,R) is not exceptional, and yet $w(G,\{R\}) > s(l-5)-5s(5)$. We may assume that the theorem holds for all graphs with fewer edges than G.

(1) $\quad l \geq 12$

To prove (1) let $l \leq 11$. Since G is not exceptional, it follows from Theorem 8.4 that $l = 11$, every internal face of G has length five, and there are at most seven internal faces. Thus $w(G,\{R\}) \leq 7w(5) \leq s(6)-5w(5) = s(l-5)-5w(5)$ by (S4), a contradiction. This proves (1).

(2) $\quad$ *There is no path of length at most two with both ends in R that is otherwise disjoint from R (i.e., G satisfies* (I4)*).*

To prove (2) let P be a path in G of length one or two with ends $u,v \in V(R)$, and otherwise disjoint from R. Let C_1, C_2 be the two cycles of $R \cup P$ other than R, and for $i = 1,2$ let G_i be the subgraph of G drawn in the closed disk bounded by C_i and $l_i = |C_i|$. Note that $l_1+l_2 = l+2|E(P)|$.

Since G does not satisfy (E1) and satisfies (I0), we can assume that $G_1 \neq C_1$. Hence G_1 is C_1-critical by Lemma 8.3. Assume for a moment that $G_2 = C_2$. If G_1 is not very exceptional, then using the minimality of G, we have $w(G,\{R\}) = w(G_1,\{C_1\})+s(l_2) \leq s(l_1-5)+5s(5)+s(l_2) \leq s(l_1+l_2-10)+6s(5) \leq s(l-5)-5s(5)$ by the convexity and (S4), a contradiction. Similarly, we exclude the case that P has length one and G_1 is very exceptional. Finally, if G_1 is very exceptional and $|E(P)| = 2$, then $G-V(R)$ consists of one or two adjacent vertices of degree three in G. Let $a_1 \leq a_2 \leq \ldots$ be the lengths of the internal faces of G. If $G-V(R)$ has one vertex, then since G does not satisfy (E2) or (E3), we have either $a_1 \geq 6$ (and $l \geq 12$), in which case $w(G,\{R\}) = s(a_1)+s(a_2)+s(a_3) \leq 2s(6)+s(l-6) \leq s(l-5)-5s(5)$, by convexity and (S4), or $a_1 = 5$ and $a_2 \geq 7$ (and $l \geq 13$), in which case $w(G,\{R\}) = s(a_1)+s(a_2)+s(a_3) \leq s(5)+s(7)+s(l-6) \leq s(l-5)-5s(5)$, again by convexity and (S4). If $G-V(R)$ consists of two adjacent vertices of degree three, then, since G does not satisfy (E4), we have $a_3 \geq 6$ and $l \geq 12$; thus $w(G,\{R\}) = s(a_1)+s(a_2)+s(a_3)+s(a_4) \leq 2s(5)+s(6)+s(l-6) \leq s(l-5)-5s(5)$. This is a contradiction.

Thus we may assume that $G_1 \neq C_1$ and $G_2 \neq C_2$. Therefore, G_1 is C_1-critical and G_2 is C_2-critical by Lemma 8.3. Furthermore, we may assume that P cannot be chosen so that $G_2 = C_2$. That implies that G_1 and G_2 are not very exceptional, and hence $w(G,\{R\}) \leq s(l_1-5)+5s(5)+s(l_2-5)+5s(5) \leq s(l-5)-5s(5)$. a contradiction. This proves (2).

Let ϕ be a precoloring of R that does not extend to a 3-coloring of G.

(3) *G is ϕ-critical.*

To prove (3) suppose to the contrary that G is not ϕ-critical. Then G contains a proper ϕ-critical subgraph G'. By Lemma 8.2, G' is 2-connected; thus all its faces are bounded by cycles. Note that G' is not very exceptional by (2). Since G' has fewer edges than G, we have $w(G',\{R\}) \leq s(l-5)+5s(5)$ by induction. For $f \in \mathcal{F}(G')$ let G_f be the subgraph of G drawn inside the closure of f, and let C be the cycle bounding f. By Lemma 8.3, G_f is either equal to C, or it is C-critical. Thus by induction, the convexity of s and (S4), we have $w(G_f,\{C\}) \leq s(|f|)$. Furthermore, if G_f is not equal to C, then $w(G_f,\{C\}) \leq s(|f|-3)+s(5)$. Let f_0 be a face of G' such that G_{f_0} is not equal to f_0. Note that $|f_0| \geq 8$ by Theorem 8.4. We have

$$\begin{aligned} w(G,\{R\}) &= \sum_{f\in\mathcal{F}(G)} s(|f|) = \sum_{f'\in\mathcal{F}(G')} w(G_{f'},\{f'\}) \\ &\leq s(|f_0|-3)+s(5)-s(|f_0|)+\sum_{f'\in\mathcal{F}(G')} s(|f'|) \\ &= s(|f_0|-3)+s(5)-s(|f_0|)+w(G',\{R\}) \\ &\leq s(|f_0|-3)-s(|f_0|)+s(l-5)+6s(5) \leq s(l-5)-5s(5), \end{aligned}$$

where the last inequality holds by convexity and (S4). This proves (3).

(4) *The graph G does not have two adjacent vertices of degree two (i.e., G satisfies* (I5)*). Furthermore, every vertex of degree two is incident with a face of length at most six.*

To prove (4) let u and v be two adjacent vertices of degree two in R. Let G' and R' be the graphs obtained from G and R, respectively, by identifying u and v into a single vertex w. Let ϕ' be a 3-coloring of R' matching ϕ on $R'-w$. Note that G' is ϕ'-critical, and by (2), G' has no cycle of length at most four. Let d be the length of the internal face f of G incident with the edge uv. By (2), if G' is exceptional, then it satisfies (E5); hence G has four faces of length five, a 6-face and a face of length $l-6$ and $w(G,\{R\}) = s(l-6)+s(6)+4s(5) \leq s(l-5)-5s(5)$ by (1) and (S4). Therefore, assume that G' is not exceptional. By the minimality of G we have $w(G', \{R'\}) \leq s(l-6) - 5s(5)$, and since the face corresponding to f contributes $s(d-1)$ to $w(G', \{R'\})$, we conclude that $d-1 < l-6$. Thus $w(G, \{R\}) = w(G', \{R'\}) - s(d-1) + s(d) \leq s(l-6) - 5s(5) - s(d-1) + s(d) \leq s(l-5) - 5s(5)$ by convexity. The case that a vertex v of degree two is incident with a face of length at least 7 is handled similarly, with G' obtained either by suppressing v or by identifying its neighbors, depending on whether the colors of these neighbors according to ϕ differ or not. This proves (4).

(5) *Some good configuration appears in G.*

To prove (5) suppose for a contradiction that no good configuration appears in G. By Lemma 8.1 the graph satisfies (I0), (I1) and (I2). By Lemma 8.2, the graph G satisfies (I3) and (I6). By (2) and (4) it satisfies (I4) and (I5). The assumptions (I7) and (I8) are trivially satisfied by planar graphs with only one ring. Let M be the null graph. We deduce from Lemma 5.10 that $w(G, \mathcal{R}) \leq 4n_2/3 + 52\epsilon n_3 - 8$, where n_2 and n_3 are as in Lemma 5.4. By (I5) we have $n_2 \leq l/2$, thus $4n_2/3 + 52\epsilon n_3 \leq (2/3 + 26\epsilon)l$. If $l \geq 16$, then

$$w(G, \mathcal{R}) \leq (2/3 + 26\epsilon)l - 8 \leq l - 13 - 10\epsilon = s(l-5) - 5s(5)$$

because $\epsilon \leq 1/1278$, a contradiction. Thus we may assume that $l \leq 15$, and hence $n_2 \leq 7$. If $l = 15$, then $w(G, \mathcal{R}) \leq 28/3 + 8 \cdot 52\epsilon - 8 \leq l - 13 - 10\epsilon = s(l-5) - 5s(5)$, again a contradiction. If $l = 13$, then we $n_2 \leq 6$ and $w(G, \mathcal{R}) \leq 7 \cdot 52\epsilon \leq s(8) - 5s(5)$.

Suppose that $l = 12$. If $n_2 \leq 5$, then $w(G, \mathcal{R}) \leq 20/3 + 52 \cdot 12\epsilon - 8 \leq 0 \leq s(7) - 5s(5)$, because $270\epsilon \leq s(7) \leq s(8)/3 \leq (s(7) + s(9))/6$ by (S4), implying that $\epsilon \leq 1/1350$. Thus we may assume that $n_2 = 6$ and $n_3 = 6$. By Theorem 8.4, all internal faces sharing an edge with R have length 5, thus the internal vertices that have a neighbor in R form a 6-cycle K. By Lemma 8.3 and Theorem 8.4, we have that K bounds a face, thus all its vertices have degree three. This contradicts (I1). It follows that if $l = 12$ and $n_2 = 6$, then $n_3 \leq 5$; thus $w(G, \mathcal{R}) \leq 260\epsilon \leq s(7) - 5s(5)$ by (S4).

Thus by (1) we may assume that $l = 14$. If $n_2 \leq 6$, then we have $w(G, \mathcal{R}) \leq 8 \cdot 52\epsilon \leq s(9) - 5s(5)$; hence $n_2 = 7$. Furthermore, using Lemma 5.11, we conclude that $b = 0$, where b is as in that lemma. Then vertices of degree two and three alternate on R, and every internal face that shares an edge with R has length five. The neighbors of the vertices of R of degree three form a 7-cycle, which bounds a face by Theorem 8.4. Then, $w(G, \{R\}) = s(7) + 7s(5) \leq s(9) - 5s(5)$. This proves (5).

(6) *The graph G is well-behaved.*

To prove (6), assume to the contrary that G is not well-behaved. Thus there exists a path P of length at most four, with ends $u, v \in V(R)$ and otherwise disjoint from R, that is not allowable. We may assume that P is such a path of the shortest possible length. By (2), the path P has length at least three.

Let C_1, C_2, R be the three cycles of $R \cup P$, and for $i = 1,2$ let G_i be the subgraph of G consisting of all vertices and edges drawn in the closed disk bounded by C_i. We claim that C_1 and C_2 are induced cycles. To prove this claim suppose to the contrary that some edge has ends $x, y \in V(C_i)$ for some $i \in \{1,2\}$, but that the edge itself does not belong to C_i. Then one of x, y, say x, belongs to the interior of P, and y does not belong to P. By (2), the vertex x is not a neighbor of u or v, and hence P has length four, and x is the middle vertex of P. Let the vertices of P be u, u', x, v', v, in order. Since P was chosen minimal, the two paths $uu'xy$ and $vv'xy$ are allowable; hence G_i consists of two 5-faces and the path P is allowable, a contradiction. This proves that C_1 and C_2 are induced cycles.

It follows from (2) and (4) that G_1 and G_2 are not very exceptional and that $G_i \neq C_i$. By Lemma 8.3 the graph G_i is C_i-critical for $i = 1,2$. Let $l_i = |C_i|$. By induction we have

$$\begin{aligned} w(G,\{R\}) &= w(G_1,\{C_1\}) + w(G_2,\{C_2\}) \\ &\leq s(l_1-5) + 5s(5) + s(l_2-5) + 5s(5) \\ &\leq s(l_1+l_2-15) + 11s(5) \leq s(l-5) - 5s(5), \end{aligned}$$

by convexity and (S4). This proves (6).

It follows from (5), (6) and Lemma 6.1 that some good configuration strongly appears in G, for if the second outcome of Lemma 6.1 holds, then (G,R) either is exceptional or satisfies the conclusion of the theorem. Let γ be a good configuration that strongly appears in G, and let G' be the γ-reduction of G. By Lemma 4.1 the 3-coloring ϕ does not extend to a 3-coloring of G'. Thus G' has a ϕ-critical subgraph G''. By Lemma 6.2 the graph G'' has no cycles of length at most four (G satisfies (I9) by Lemma 8.3 and Theorem 8.4). By Lemma 8.2, the graph G'' satisfies (I3) and (I6). Since G is ϕ-critical by (3), G'' is not a subgraph of G; hence G'' contains a new vertex or edge.

For an internal face f'' of G'' let $G^1_{f''}, G^2_{f''}, \ldots, G^{k_{f''}}_{f''}$ be the members of the G-expansion of $S_{f''}$, defined in Section 7, and let $C^1_{f''}, C^2_{f''}, \ldots, C^{k_{f''}}_{f''}$ be the corresponding rings so that $C^i_{f''}$ is a subgraph of $G^i_{f''}$.

(7) *Let f'' be a face of G''. Then*

$$\sum_{i=1}^{k_{f''}} w(G^i_{f''}, \{C^i_{f''}\}) \leq s(|f''|) - c(f'').$$

Note that by Lemma 8.3, we have that either $G^i_{f''} = C^i_{f''}$ or $G^i_{f''}$ is $C^i_{f''}$-critical for each i. To prove (7), let us discuss the possible cases in the definition of the contribution of a face:

- If $|S_{f''}| = 1$ and $G^1_{f''}$ satisfies (E0), then by Lemma 7.4 we have $c(f'') \neq -\infty$, hence f'' has zero elasticity, $c(f'') = 0$ and $w(G^1_{f''}, \{C^1_{f''}\}) = s(|f''|)$.

- If $|S_{f''}| = 1$ and $G^1_{f''}$ satisfies (E1), then similarly we have $\mathrm{el}(f'') < 5$ and $c(f'') = s(8 - \mathrm{el}(f'')) - 2s(5)$. Note that by Lemma 7.1 we have $\mathrm{el}(f'') \le 3$. By induction, $w(G^1_{f''}, \{C^1_{f''}\}) \le s(|C^1_{f''}| - 3) + s(5) = s(|f''| + \mathrm{el}(f'') - 3) + s(5)$, and $s(|f''| + \mathrm{el}(f'') - 3) + s(5) \le s(|f''|) - s(8 - \mathrm{el}(f'')) + 2s(5)$ by convexity.

- If $|S_{f''}| = 1$ and $G^1_{f''}$ satisfies (E2), then $\mathrm{el}(f) \le 3$, $c(f'') = s(9 - \mathrm{el}(f'')) - 3s(5)$, and $w(G^1_{f''}, \{C^1_{f''}\}) = s(|C^1_{f''}| - 4) + 2s(5) = s(|f''| + \mathrm{el}(f'') - 4) + 2s(5) \le s(|f''|) - c(f'')$ by convexity.

- If $|S_{f''}| = 1$ and $G^1_{f''}$ satisfies (E3), then $c(f'') = s(11 - \mathrm{el}(f'')) - 2s(6) - s(5)$ and $w(G^1_{f''}, \{C^1_{f''}\}) = s(|C^1_{f''}| - 5) + s(5) + s(6) = s(|f''| + \mathrm{el}(f'') - 5) + s(5) + s(6) \le s(|f''|) - c(f'')$.

- If $|S_{f''}| = 1$ and $G^1_{f''}$ satisfies (E4) or (E5), then $c(f'') = s(10 - \mathrm{el}(f'')) - 6s(5)$ and $w(G^1_{f''}, \{C^1_{f''}\}) \le s(|C^1_{f''}| - 5) + 5s(5) = s(|f''| + \mathrm{el}(f'') - 5) + 5s(5) \le s(|f''|) - c(f'')$.

- Suppose that $k_{f''} = 2$, $G^1_{f''} = C^1_{f''}$ and $G^2_{f''} = C^2_{f''}$, where $|C^1_{f''}| \le |C^2_{f''}|$. If $|C^1_{f''}| = 5$, then $c(f'') = s(10 - \mathrm{el}(f'')) - 6s(5)$ and $w(G^1_{f''}, \{C^1_{f''}\}) + w(G^2_{f''}, \{C^2_{f''}\}) = s(|C^2_{f''}|) + s(5) = s(|f''| + \mathrm{el}(f'') - 5) + s(5) < s(|f''|) - c(f'')$. Otherwise, $c(f'') = s(12 - \mathrm{el}(f'')) - 2s(6)$ and $w(G^1_{f''}, \{C^1_{f''}\}) + w(G^2_{f''}, \{C^2_{f''}\}) = s(|C^1_{f''}|) + s(|C^2_{f''}|) \le s(6) + s(|f''| + \mathrm{el}(f'') - 6) \le s(|f''|) - c(f'')$.

- Suppose that $k_{f''} = 1$ and $G^1_{f''}$ is not exceptional.
  - Let us consider the case that $G^1_{f''}$ contains a path $P = v_1v_2v_3v_4$ such that $v_1, v_4 \in V(C^1_{f''})$, $v_2, v_3 \notin V(C^1_{f''})$ and both of the open disks Δ_1 and Δ_2 bounded by P and paths in $C^1_{f''}$ contain at least two vertices of G. In this case, $c(f'') = s(7)$. Let H_i be the subgraph of $G^1_{f''}$ drawn in Δ_i and K_i the cycle bounding Δ_i, for $i \in \{1, 2\}$. Neither H_1 nor H_2 is very exceptional, thus we have $w(G^1_{f''}, \{C^1_{f''}\}) = w(H_1, K_1) + w(H_2, K_2) \le s(|K_1| - 5) + s(|K_2| - 5) + 10s(5) \le s(|K_1| + |K_2| - 15) + 11s(5) = s(|f''| + \mathrm{el}(f'') - 9) + 11s(5) < s(|f''|) - s(7)$, since $\mathrm{el}(f'') \le 5$ and $|f''| + \mathrm{el}(f'') \ge |K_1| + |K_2| - 6 \ge 14$, because G'' satisfies (I9) by Lemma 8.3 and Theorem 8.4.
  - Otherwise, $c(f'') = s(11 - \mathrm{el}(f'')) - s(6) + 5s(5)$. In this case, we have $w(G^1_{f''}, \{C^1_{f''}\}) \le s(|C^1_{f''}| - 5) - 5s(5) = s(|f''| + \mathrm{el}(f'') - 5) - 5s(5) \le s(|f''|) - c(f'')$.

- If $k_{f''} = 2$ and $G^1_{f''} \ne C^1_{f''}$, then $c(f'') = s(12 - \mathrm{el}(f'')) - 2s(6)$ and $w(G^1_{f''}, \{C^1_{f''}\}) + w(G^2_{f''}, \{C^2_{f''}\}) \le s(|C^1_{f''}| - 3) + s(5) + s(|C^2_{f''}|) \le s(|f''| + \mathrm{el}(f'') - 8) + 2s(5) < s(|f''|) - c(f'')$

- If $k_{f''} \ge 3$, then $c(f'') = s(12 - \mathrm{el}(f'')) - 2s(6)$ and $\sum_{i=1}^{k_{f''}} w(G^i_{f''}, \{C^i_{f''}\}) \le s(|f''| + \mathrm{el}(f'') - (k_{f''} - 1)5) + (k_{f''} - 1)s(5) < s(|f''|) - c(f'')$.

Therefore, in all the cases, (7) holds.

By Lemma 7.2, we have $w(G,\{R\}) \le \delta + \sum_{f''\in\mathcal{F}(G'')} \sum_{i=1}^{k_{f''}} w(G^i_{f''}, \{C^i_{f''}\})$, where $\delta = s(6)$ if γ is isomorphic to R3 and $\delta = 0$ otherwise, and by (7) this implies that

$$\begin{aligned} w(G,\{R\}) &\le \delta + \sum_{f''\in\mathcal{F}(G'')} (s(|f''|) - c(f'')) \\ &= w(G'',\{R\}) + \delta - \sum_{f''\in\mathcal{F}(G'')} c(f'') \\ &= w(G'',\{R\}) - c(G''). \end{aligned}$$

By Lemma 7.3, G'' is not very exceptional; hence $w(G'',\{R\}) \le s(l-5)+5s(5)$ by induction. Note that $c(G'') \ge 10s(5)$ by Lemma 7.4; thus

$$w(G,\{R\}) \le w(G'',\{R\}) - c(G'') \le s(l-5) - 5s(5),$$

which is a contradiction finishing the proof of the theorem. □

Proof of Theorem 1.6. Let $\epsilon = 2/4113$, $s(5) = 4/4113$, $s(6) = 72/4113$, $s(7) = 540/4113$, $s(8) = 2184/4113$, and $s(l) = l-8$ for $l \ge 9$. Then conditions (S1)–(S4) hold. Let G be a plane graph of girth at least five with a cycle C and let ϕ be a precoloring of C that does not extend to a 3-coloring of G. We may assume that G is ϕ-critical, and hence C is a face of G. By Theorem 8.5, we have $w(G,\{C\}) < w(|C|) < |V(C)|$. Note that $3|V(G)| - 2|V(C)| = \sum_f |f| \le \sum_f 5s(|f|)/s(5) = 5w(G,\{C\})/s(5)$, where the sum is over all faces of G, except the one bounded by C. Therefore, $|V(G)| \le (5/s(5)+2)|V(C)|/3 \le 1715|V(C)|$, as desired. □

9 Summary

In this section, we provide a summary result that will be used as a basis for the proofs in the next paper of the series, to avoid the need to repeat many of the definitions used here. Let Π be a surface with boundary and c a simple curve intersecting the boundary of Π exactly in its ends. The compact topological space obtained from Π by cutting along c (i.e., removing c and adding two new pieces of boundary corresponding to c) is a union of at most two surfaces. If $\Pi_1, \ldots, \Pi_k$ are obtained from Π by repeating this construction, we say that they are *fragments* of Π.

Consider a graph H embedded in Π with rings $\mathcal{Q}$, and let f be an internal face of H. Let us recall that f is an open subset of $\hat{\Pi}$. Let f' be obtained from f by removing $\hat{C}$ for each cuff C such that some facial walk of f consists only of a vertex ring incident with C. Let Π_f be a surface whose interior is homeomorphic to f'. Note that Π_f is unique up to homeomorphism and that the cuffs of Π_f correspond to the facial walks of f.

Theorem 9.1. *Let G be a well-behaved graph embedded in a surface Σ with rings $\mathcal{R}$ satisfying* (I0)–(I9) *and let M be a subgraph of G that captures (≤ 4)-cycles. Let $\ell(\mathcal{R})$ be the sum of the lengths of the rings in $\mathcal{R}$ and g the Euler genus of Σ, and assume that either $g > 0$ or $|\mathcal{R}| > 1$. Let ϵ be a real number satisfying $0 < \epsilon \leq 1/1278$, let $s : \{5, 6, \ldots\} \to \mathbb{R}$ be a function satisfying* (S1)–(S4)*, and suppose that $w(G, \mathcal{R}) > 8g + 8|\mathcal{R}| + (2/3 + 26\epsilon)\ell(\mathcal{R}) + 20|E(M)|/3 - 16$. If G is $\mathcal{R}$-critical, then there exists an $\mathcal{R}$-critical graph G' embedded in Σ with rings $\mathcal{R}$ such that $|E(G')| < |E(G)|$ and the following conditions hold.*

1. *If G has girth at least five, then there exists a set $Y \subseteq V(G')$ of size at most two such that $G' - Y$ has girth at least five.*

2. *If C' is a (≤ 4)-cycle in G', then C' is non-contractible and G contains a non-contractible cycle C of length at most $|C'| + 3$ such that $C \not\subseteq M$. Furthermore, all ring vertices of C' belong to C, and if C' is a triangle disjoint from the rings and its vertices have distinct pairwise non-adjacent neighbors in a ring R of length* 6*, then two vertices of C have distinct non-adjacent neighbors in R.*

3. *G' has an internal face that either is not closed* 2*-cell or has length at least* 6.

4. *There exists a collection $\{(J_f, S_f) : f \in F(G')\}$ of pairs of subgraphs $J_f \subseteq G$ and sets S_f of faces of J_f satisfying condition (1) stated after Lemma 7.1, such that*

   (a) *no subgraph J_f is equal to the union of the rings $\mathcal{R}$, for $f \in F(G')$,*

   (b) *for every $f \in F(G')$, the surfaces of the G-expansion of S_f are fragments of the surface Σ_f.*

   (c) *For every face $h \in F(G)$ but at most one, there exists unique $f \in F(G')$ such that h is a face of a member of the G-expansion of S_f. If there exists a face $h \in F(G)$ without a corresponding face in the G-expansions of S_f for $f \in F(G')$, then h is a* 6*-face, and set $\delta = s(6)$. Otherwise, set $\delta = 0$.*

   (d) *For $f \in F(G')$, let $el(f) = \left(\sum_{h \in S_f} |h|\right) - |f|$ and if f is closed* 2*-cell or omnipresent, let its contribution $c(f)$ be defined as in Section 7. Then $\sum_{f \in F(G')} el(f) \leq 10$ and if f is an omnipresent face, then $el(f) \leq 5$. Furthermore, if every internal face of G' is closed* 2*-cell or omnipresent and G' satisfies (I6), then $\sum_{f \in F(G')} c(f) \geq \delta$.*

   (e) *if $f \in F(G')$ is closed* 2*-cell and $G_1, \ldots, G_k$ are the components of the G-expansion of S_f, where for $1 \leq i \leq k$, G_i is embedded in a disk with one ring R_i, then $\sum_{i=1}^{k} w(G_i, \{R_i\}) \leq s(|f|) - c(f)$.*

Proof. Let n_2 and n_3 be the number of vertices of degree two and three, respectively, incident with facial rings of G. By (I5) we have $n_2 \leq \ell(R)/2$, and since $n_2 + n_3 \leq \ell(R)$, we have $4n_2/3 + 52\epsilon n_3 \leq (2/3 + 26\epsilon)\ell(R)$. Consequently,

$w(G,\mathcal{R}) > 8g + 8|\mathcal{R}| + 4n_2/3 + 52\epsilon n_3 + 20|E(M)|/3 - 16$, and by Lemma 5.10, a good configuration γ appears in G and does not touch M. By Lemma 6.1, we can assume that γ appears strongly in G. Let ϕ be a precoloring of $\mathcal{R}$ that does not extend to a 3-coloring of G, and let G_1 be a γ-reduction of G with respect to ϕ. By Lemma 4.1, ϕ does not extend to a 3-coloring of G_1, and thus G_1 contains an $\mathcal{R}$-critical subgraph G'. Clearly, $|E(G')| < |E(G)$. Let us now show that G' has the required properties:

1. If G has girth at least five, then every (≤ 4)-cycle in G' contains a new vertex or a new edge, and thus they can all be intersected by at most two vertices.

2. Follows from Lemma 6.2.

3. Suppose that all internal faces of G' are closed 2-cell. In particular, G' does not have an omnipresent face, and thus it satisfies (I6). If G' contains a new edge or a new vertex, then the claim holds by Lemma 7.4. Otherwise, G' is a proper subgraph of G, and thus there exists a cycle C bounding a face in G', but not in G. Let H be the subgraph of G drawn in the closed disk bounded by C. By Lemma 8.3, H is C-critical, and by Theorem 8.4, we conclude that C has length at least 8.

4. For each $f \in F(G')$, we define S_f and J_f as in Section 7.

   (a) This follows by the construction of S_f, since G' is not equal to the union of $\mathcal{R}$.

   (b) The construction of J_f and S_f ensures that the surfaces corresponding to the faces of S_f are constructed from Σ_f by cutting along simple curves with ends in cuffs, as described in the definition of fragments.

   (c) The claim follows from Lemma 7.2.

   (d) The first part follows from Lemma 7.1. If G' contains a new vertex or a new edge, then the second part follows from Lemma 7.4. Otherwise, G' is a proper subgraph of G and all its faces have elasticity 0. If f is a closed 2-cell of G', then $c(f) \geq 0$, and if additionally f is not a face of G, then $c(f) \geq s(8) - 2s(5) > s(6)$. If f is an omnipresent face, then note that no component of G' satisfies (E1), (E2) or (E3), since G satisfies (I4). Since G' is $\mathcal{R}$-critical, at least one component of G' does not satisfy (E0), and thus $c(f) \geq 5 - 5s(5) > s(6)$. Since $G' \neq G$, we conclude that $\sum_{f\in F(G')} c(f) > s(6) \geq \delta$.

   (e) This was proved as (7) in Section 8.

□

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