FOUR EDGE-INDEPENDENT SPANNING TREES [1]

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ABSTRACT

We prove an ear-decomposition theorem for 4-edge-connected graphs and use it to prove that for every 4-edge-connected graph G and every $r \in V(G)$, there is a set of four spanning trees of G with the following property. For every vertex in G, the unique paths back to r in each tree are edge-disjoint. Our proof implies a polynomial-time algorithm for constructing the trees.

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1 Introduction

If r is a vertex of a graph G, two subtrees T_1, T_2 of G are *edge-independent with root* r if each tree contains r, and for each $v \in V(T_1) \cap V(T_2)$, the unique path in T_1 between r and v is edge-disjoint from the unique path in T_2 between r and v. Larger sets of trees are called *edge-independent with root* r if they are pairwise edge-independent with root r.

Itai and Rodeh [6] posed the Edge-Independent Tree Conjecture, that for every k-edge-connected graph G and every $r \in V(G)$, there is a set of k edge-independent spanning trees of G rooted at r. Here, we prove the case $k = 4$ of the Edge-Independent Tree Conjecture. That is, we prove the following:

Theorem 1. *If G is 4-edge-connected and $r \in V(G)$, then there exists a set of four edge-independent spanning trees of G rooted at r.*

There is a similar conjecture which has been studied in parallel, concerning vertices rather than edges. If r is a vertex of G, two subtrees T_1, T_2 of G are *independent with root* r if each tree contains r, and for each $v \in V(T_1) \cap V(T_2)$, the unique path in T_1 between r and v is internally vertex-disjoint from the unique path in T_2 between r and v. Larger sets of trees are called *independent with root* r if they are pairwise independent with root r.

Itai and Rodeh [6] also posed the Independent Tree Conjecture, that for every k-connected graph G and for every $r \in V(G)$, there is a set of k independent spanning trees of G rooted at r.

The case $k = 2$ of each conjecture was proven by Itai and Rodeh [6]. The case $k = 3$ of the Independent Tree Conjecture was proven by Cheriyan and Maheshwari [1], and then independently by Zehavi and Itai [11]. Huck [5] proved the Independent Tree Conjecture for planar graphs (with any k). Building on this work and that of Kawarabayashi, Lee, and Yu [7], the case $k = 4$ of the Independent Tree Conjecture was proven by Curran, Lee, and Yu across two papers [2, 3]. The Independent Tree Conjecture is open for nonplanar graphs with $k > 4$.

In 1992, Khuller and Schieber [8] published a later-disproven argument that the Independent Tree Conjecture implies the Edge-Independent Tree Conjecture. Gopalan and Ramasubramanian [4] demonstrated that Khuller and Schieber's proof fails, but salvaged the technique, and proved the case $k = 3$ of the Edge-Independent Tree Conjecture by reducing it to the case $k = 3$ of the Independent Tree Conjecture. Schlipf and Schmidt [10] provided an alternate proof of the case $k = 3$ of the Edge-Independent Tree Conjecture, which does not rely on the Independent Tree Conjecture. The case $k = 4$ of the Edge-Independent Tree Conjecture is proven here, while the case $k > 4$ remains open.

By adapting the technique of Schlipf and Schmidt [10], we prove an edge analog of the planar chain decomposition of Curran, Lee, and Yu [2]. We then use this decomposition to create two edge numberings which define the required trees.

The conjectures are related to network communication with redundancy. If G represents a communication network, one can wonder if information can be broadcast through the entire network with resistance to edge failures (i.e. it would require k simultaneous edge failures to disconnect a client from every broadcast). The Edge-Independent Tree Conjecture implies that the absence of edge bottlenecks of size less than k is necessary and sufficient for a redundant broadcast to be possible from any source r. The Independent Tree Conjecture answers the analogous problem where vertex failures are the concern, rather than edge failures.

2 The Chain Decomposition

In this paper, a *graph* will refer to what is commonly called a multigraph. That is, there may be multiple edges between the same pair of vertices ("parallel edges") and an edge may connect a vertex to itself (a "loop"). We consider a loop to induce a cycle of length one and a pair of parallel edges to induce a cycle of length two. Also, the presence of a loop increases the degree of a vertex by two.

Throughout this section, fix a non-null graph G and a vertex $r \in V(G)$. We begin by defining a decomposition analogous to the planar chain decomposition in [2].

Definition. An *up chain* of G with respect to the pair of subgraphs $(H, \overline{H})$ is a subgraph which is either:

i A path with at least one edge such that every vertex is either r or has degree at least two in $\overline{H}$, and the ends are either r or are in H, OR

ii A cycle such that every vertex is either r or has degree at least two in $\overline{H}$, and some vertex is either r or has degree at least two in H. We will consider this vertex to be both ends of the chain.

Chains which are paths will be called *open* and chains which are cycles will be called *closed*, analogous to the standard ear decomposition.

Definition. A *down chain* of G with respect to the pair of subgraphs $(H, \overline{H})$ is an up chain with respect to $(\overline{H}, H)$.

Definition. A *one-way chain* of G with respect to the pair of subgraphs $(H, \overline{H})$ is a subgraph induced by an edge with ends u and v, such that u is either r or has degree at least two in

H, and v is either r or has degree at least two in $\overline{H}$. We call u the *tail* of the chain and v the *head*.

Definition. Let $G_0, G_1, \ldots, G_m$ be a sequence of subgraphs of G. Denote $H_i = G_0 \cup G_1 \cup \cdots \cup G_{i-1}$ and $\overline{H_i} = G_{i+1} \cup G_{i+2} \cup \cdots \cup G_m$, so that H_0 and $\overline{H_m}$ are the null graph. We say that the sequence $G_0, G_1, \ldots, G_m$ is a *chain decomposition* of G rooted at r if:

1. The sets $E(G_0), E(G_1), \ldots, E(G_m)$ partition $E(G)$, AND
2. For $i = 0, \ldots, m$, the subgraph G_i is either an up chain, a down chain, or a one-way chain with respect to the subgraphs $(H_i, \overline{H_i})$.

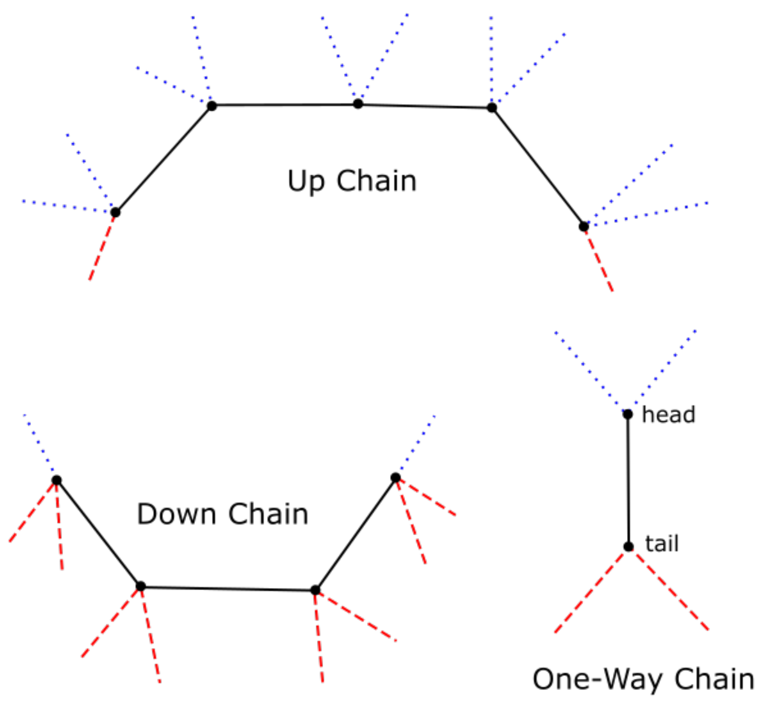

Figure 1: An illustration of an up chain of length 4, a down chain of length 3, and a one-way chain. The red/dashed edges are in earlier chains, while the blue/dotted edges are in later chains.

Remarks.

1. The chain decomposition is symmetric in the following sense. If $G_0, G_1, \ldots, G_m$ is a chain decomposition rooted at r, then $G_m, G_{m-1}, \ldots, G_0$ is a chain decomposition rooted at r, with the up and down chains switched and the heads and tails of one-way chains switched. Throughout this paper, we will refer to this fact as "symmetry".

2. G_0 is either a closed up chain containing r or a one-way chain with r as the tail, and G_m is either a closed down chain containing r or a one-way chain with r as the head.

3. In the planar chain decomposition in [2], up chains and down chains are analogous to the corresponding open chains. The trivial chain is analogous to a one-way chain.

Definition. The *chain index* of $e \in E(G)$, denoted $CI(e)$, is the index of the chain containing e.

Definition. An up chain G_i is **minimal** if no internal vertex of G_i is in $\{r\} \cup V(H_i)$.

Definition. A down chain G_i is **minimal** if no internal vertex of G_i is in $\{r\} \cup V\left(\overline{H_i}\right)$.

Definition. A chain decomposition is **minimal** if all of its up chains and down chains are minimal.

Remark 2. An up chain or down chain may be subdivided into several minimal chains by breaking at the offending internal vertices. These minimal chains may then be inserted consecutively to the decomposition at the index of the old chain. In this way, one can easily obtain a minimal chain decomposition from any chain decomposition.

We will prove Theorem 1 by combination of the following results:

Theorem 3. *If G is a 4-edge-connected graph and $r \in V(G)$, then G has a chain decomposition rooted at r.*

Theorem 4. *Suppose G is a graph with no isolated vertices. If G has a chain decomposition rooted at some $r \in V(G)$, then there exists a set of four edge-independent spanning trees of G rooted at r.*

3 Preliminary Results

While not needed for our main results, the following proposition demonstrates how the chain decomposition fits in with the various decompositions used in other cases of the Independent Tree Conjecture and Edge-Independent Tree Conjecture. A partial chain decomposition and its complement are "almost 2-edge-connected" in the following sense.

Proposition 5. *Suppose $G_0, G_1, \ldots, G_m$ is a chain decomposition of a graph G rooted at r. Then for $i = 1, \ldots, m$, H_i and $\overline{H_{i-1}}$ are connected. Further, if e is a cut edge of H_i (resp. $\overline{H_{i-1}}$), then e induces a one-way chain and one component of $H_i - e$ (resp. $\overline{H_{i-1}} - e$) contains one vertex and no edges.*

Proof. By symmetry, we need only prove the result for the H_i's. The connectivity follows from the fact that every type of chain is connected and contains at least one vertex in an earlier chain.

Suppose e is a cut edge of some H_i. Then e cannot be part of an up chain, or it would be part of a cycle either formed by the chain itself or the chain and a path in H_i between its ends. Also, e cannot be part of a down chain, or else it would be part of a cycle formed by e and a path in H_i between its ends. Therefore, e induces a one-way chain.

Let C be the component of $H_i - e$ not containing r, and suppose for the sake of contradiction that C contains an edge. Let e' be an edge of C with minimal chain index. Consider $G_{CI(e')}$, the chain containing e'. Regardless of the chain type, some vertices in $V(G_{CI(e')})$ are incident to at least two edges in $H_{CI(e')} \subset H_i$ since $r \notin C$, so one of these edges is not e. This contradicts the minimality of $CI(e')$. □

The next lemma and its corollary will allow us to ignore the possibility of loops in the graph when convenient.

Lemma 6. *Suppose $G_0, G_1, \ldots, G_m$ is a chain decomposition of G rooted at r. If $v \neq r$ is in H_i (resp $\overline{H_i}$), then v is incident to a non-loop edge in H_i (resp $\overline{H_i}$). If v has degree at least two in H_i (resp. $\overline{H_i}$), then v is incident to two distinct non-loop edges in H_i (resp. $\overline{H_i}$).*

Proof. Suppose v is incident to a loop, which by symmetry we may assume is in H_i. Of all loops incident to v, choose the one with minimal chain index $j < i$. Consider the chain classification of G_j. The chain definitions all coincide for a loop, and require that v has degree at least two in H_j. By the minimality of j, v is not incident to any loops in H_j. It follows that v is incident to two distinct non-loop edges in $H_j \subset H_i$. □

Corollary 7. *Suppose $G_0, G_1, \ldots, G_m$ is a chain decomposition of G rooted at r, and $e \in E(G_i)$ is a loop. Then $G_0, G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_m$ is a chain decomposition of $G - e$ rooted at r.*

Next, we prove the following useful fact about minimal chain decompositions.

Lemma 8. *Suppose G is a graph with no isolated vertices, $G_0, G_1, \ldots, G_m$ is a minimal chain decomposition of G rooted at r, and $v \in V(G)$ with $v \neq r$. Then there are indices i, j so that v has degree exactly two in H_i and $\overline{H_j}$.*

Proof. By symmetry, we need only find i. Since G has no isolated vertices, v is in some chain. Consider the chain G_{i_0} containing v so that i_0 is minimal. Note that $v \notin V(H_{i_0})$.

If G_{i_0} is an up chain, then v is in the interior of G_{i_0} since $v \notin V(H_{i_0})$, so v has degree two in G_{i_0} and degree at least two in $\overline{H_{i_0}}$. Therefore $\overline{H_{i_0}}$ is not null, so $i_0 < m$. Then $i = i_0 + 1$ completes the proof.

The chain G_{i_0} is not a down chain since $v \notin V(H_{i_0})$.

So we may assume that G_{i_0} is a one-way chain, and v must be the head since $v \notin V(H_{i_0})$. Therefore v has degree at least two in $\overline{H_{i_0}}$, so we may consider the next chain to contain v, say G_{i_1}. Note that v has degree one in H_{i_1} by the definition of i_1.

If G_{i_1} is an up chain, then v is one of two distinct ends of G_{i_1} since the chain decomposition is minimal and v has degree one in H_{i_1}. The chain G_{i_1} is not a down chain since v has degree one in H_{i_1}. If G_{i_1} is a one-way chain, then v is the head since v does not have degree at least two in H_{i_1}. In all cases, v has degree one in G_{i_1} and degree at least two in $\overline{H_{i_1}}$. Therefore $\overline{H_{i_1}}$ is not null, so $i_1 < m$. Then $i = i_1 + 1$ completes the proof. □

Finally, we show that the chain decomposition implies a minimum degree result.

Lemma 9. *Suppose G is a graph with no isolated vertices, $G_0, G_1, \dots, G_m$ is a chain decomposition of G rooted at r, and $v \in V(G)$ with $v \neq r$. Then v has degree at least 4.*

Proof. By Corollary 7, we may assume that there are no loops in G. If v is in an up chain G_i, then v has degree at least 2 in $\overline{H_i}$, and either degree 2 in G_i (if v is interior) or degree at least 1 in G_i and degree at least 1 in H_i (if v is an end). Either way, v has degree at least 4 in G. By symmetry, the same is true if v is in a down chain.

So we may assume that the only chains containing v are one-way chains. Since G has no isolated vertices, there is at least one such chain G_j. Then v has degree 1 in G_j and degree at least 2 in H_j (if v is the tail) or $\overline{H_j}$ (if v is the head). We conclude that v has degree at least 3 in G.

Assume for the sake of contradiction that v does not have degree at least 4. Then v has degree 3 and is in exactly three one-way chains, say G_{ℓ_1}, G_{ℓ_2}, G_{ℓ_3} with $\ell_1 < \ell_2 < \ell_3$. Consider G_{ℓ_2}. Since we know all of the chains containing v, we can say that v has degree 1 in H_{ℓ_2} and degree 1 in $\overline{H_{\ell_2}}$. This contradicts the definition of a one-way chain, as v can be neither the head nor the tail of the chain G_{ℓ_2}. We conclude that v has degree at least 4 as desired. □

Remark. If $|V(G)| \geq 2$, then r has degree at least 4 as well. However, we will not need this result, and it will follow from Theorem 4.

4 The Mader Construction

We will adapt the strategy of Schlipf and Schmidt [10] in order to construct a chain decomposition. In particular, we will use a construction method for k-edge-connected graphs due to Mader [9]. We limit our description of the construction to the needed case $k = 4$, since the method is more complicated for odd k.

Definition. A *Mader operation* is one of the following operations:

1. Add an edge between two (not necessarily distinct) vertices.
2. Consider two distinct edges, say e_1 with ends x, y and e_2 with ends z, w, and "pinch" them as follows. Delete the edges e_1 and e_2, add a new vertex v, then add the new edges e_x, e_y, e_z, e_w with one end v and the other end x, y, z, w respectively. While e_1 and e_2 must be distinct, the ends x, y, z, w need not be. In this case, v will have parallel edges to any repeated vertex.

Theorem 10 (Mader [9])**.** *A graph G is 4-edge-connected if and only if, for any $r \in V(G)$, one can construct G in the following way. Begin with a graph G^0 consisting of r and one other vertex of G, connected by four parallel edges. Then, repeatedly perform Mader operations to obtain G.*

Remark. Mader does not explicitly state that one can include a fixed vertex r in G^0, but it follows from his work. His proof starts with G, and then reverses one of the Mader operations while maintaining 4-edge-connectivity. An edge can be deleted unless G is minimally 4-edge-connected, in which case he finds two vertices of degree 4. He then shows that either of these vertices can be "split off" (the reverse of a pinch), so we can always remove a vertex not equal to r.

5 Proof of Theorem 3

Due to Theorem 10, it suffices to prove that a chain decomposition can be maintained through a Mader operation. The decomposition in the starting graph G^0 is as follows. Two of the edges form a closed up chain. The remaining two edges form a closed down chain.

Suppose the graph G' is obtained from the graph G by a Mader operation, with both graphs 4-edge-connected. Assume that we have a chain decomposition $G_0, G_1, \ldots, G_m$ of G. By Remark 2, we may assume that we have a minimal chain decomposition. We wish to create a new chain decomposition of G'.

5.1 Adding an Edge

Suppose G' is obtained from G by adding an edge with ends u, v. If one of the ends is the root r, we can classify the new edge as a one-way chain with tail r at, say, the very beginning of the chain decomposition. The head must have at least two incident edges in later chains, since all chains are later.

If neither end is r, choose the minimal index i such that either u or v has degree exactly two in H_i, guaranteed to exist by Lemma 8. Note that $i \geq 1$ since H_0 is null. Without loss of generality, u has degree exactly two in H_i. By the definition of i, v has degree at most two in H_i, and therefore degree at least two in $\overline{H_{i-1}}$. We classify the new edge as a one-way chain with tail u and head v, between the chains G_{i-1} and G_i.

We consider the impact of these changes on other chains in the graph. A new chain was added, but none of the other chains changed index relative to each other. Vertices may have increased degree in the H_i's or the $\overline{H_i}$'s due to the new edge, but increasing degree does not invalidate any chain types.

5.2 Pinching Edges

Suppose G' is obtained from G by pinching the edges e_1 with ends x, y and e_2 with ends z and w, replacing them with edges e_x, e_y, e_z, e_w. We will use the notation $J_1 = G_{CI(e_1)} = P_x e_1 P_y$ for the chain containing e_1, where P_x is the subpath between x and an end of J_1 so that $e_1 \notin E(P_x)$, and P_y is defined similarly. Note that P_x (resp. P_y) could have no edges if x (resp. y) is an end of J_1. In the same way, we will use the notation $J_2 = G_{CI(e_2)} = P_z e_2 P_w$ for the chain containing e_2.

We now prove several claims to deal with all possible chain classification and chain index combinations for J_1 and J_2.

Claim 1. *If $CI(e_1) = CI(e_2)$, then G' has a chain decomposition rooted at r.*

Proof. If $CI(e_1) = CI(e_2)$, then $J_1 = J_2$. Without loss of generality, $e_1 \in P_y$ and $e_2 \in P_z$, so that the chain can be written as $J_1 = J_2 = P_x e_1 (P_y \cap P_z) e_2 P_w$ (where $P_y \cap P_z$ may be have no edges if $y = z$). Recall that e_1 and e_2 are distinct, so $J_1 = J_2$ is not a one-way chain.

By symmetry, we may assume $J_1 = J_2$ is an up chain. In G', we replace the chain $J_1 = J_2$ with the following chains (in the listed order):

1. $P_x e_x e_w P_w$. This is an up chain. Since the edges e_y and e_z have not yet been used, the new vertex v is incident to two edges in later chains.

2. e_y. This is a one-way chain with tail v and head y. The tail v is incident to two edges in earlier chains, namely e_x and e_w. The head y is incident to two edges in later chains since it was an internal vertex in the old up chain $J_1 = J_2$.

3. e_z. This is a one-way chain with tail v and head z. The tail v is incident to two edges in earlier chains, namely e_x and e_w. The head z is incident to two edges in later chains since it was an internal vertex in the old up chain $J_1 = J_2$.

4. $(P_y \cap P_z)$. Only add this chain if $P_y \cap P_z$ contains an edge. This is an up chain. The new ends y, z are incident to edges in earlier chains, namely e_y and e_z.

We consider the impact of these replacements on other chains in the graph. We inserted most of the edges of the old chain $J_1 = J_2$ at the same chain index, relative to the unchanged chains, preventing any changes. The exception is the pinched edges e_1 and e_2 which were deleted, but the ends each received new incident edges e_x, e_y, e_z, e_w whose indices are the same as $CI(e_1) = CI(e_2)$, relative to the unchanged chains. Thus, we have maintained the chain decomposition. This proves Claim 1. $\square$

Without loss of generality, we assume the following for the remainder of the proof:

- $CI(e_1) < CI(e_2)$.
- If J_1 is a one-way chain, then x is the tail and y the head.
- If J_2 is a one-way chain, then z is the tail and w the head.

Claim 2. *Suppose that either J_1 is a one-way chain whose head y has degree one in $H_{CI(e_2)}$, or J_2 is a one-way chain whose tail z has degree one in $\overline{H_{CI(e_1)}}$. Then G' has a chain decomposition rooted at r.*

Proof. By symmetry, we may assume J_1 is a one-way chain whose head y has degree one in $H_{CI(e_2)}$.

First, we replace J_1 with e_x. This is a one-way chain with tail x and head v. The tail x was the tail of the old one-way chain J_1. The head v has two (in fact three) incident edges in later chains, namely e_y, e_z, e_w.

- Case 1: J_2 is an up chain. Since y has degree one in $H_{CI(e_2)}$, if J_2 is closed then y is not the end of J_2. By swapping z and w if necessary, we may assume that y is not the end of J_2 in P_z. Thus, the end of J_2 in P_z still has an incident edge in an earlier chain, despite having not placed e_y yet. We use the edges of J_2 and e_y, e_z, e_w to construct chains at the index $CI(e_2)$ as follows:

  1. $P_z e_z$. This is an up chain. The new end, v, has one incident edge in an earlier chain (e_x) and two incident edges in later chains (e_y, e_w). By assumption, the old end in P_z still has an incident edge in an earlier chain.
  2. e_y. This is a one-way chain with tail v and head y. The tail v is incident two edges in earlier chains (e_x, e_z). The head y has two incident edges in later chains since y has degree one in $H_{CI(e_2)}$ by assumption.

3. e_w. This is a one-way chain with tail v and head w. The tail v has two (in fact three) incident edges in earlier chains (e_x, e_y, e_z). The head w is incident to two edges in later chains since it was part of the old up chain J_2.
4. P_w. Only add this if P_w contains an edge. This is an up chain. The new end, w, has one incident edge in an earlier chain (e_w) and two incident edges in later chains since it was part of the old up chain J_1. Since we placed e_y above, the end of J_2 in P_w has an incident neighbor in an earlier chain even if that end is y.

- Case 2: J_2 is a down chain. Since y has degree one in $H_{CI(e_2)}$, $y \notin V(J_2)$, so each vertex of J_2 still has two incident edges in earlier chains, despite having not placed e_y yet. We use the edges of J_2 and e_y, e_z, e_w to construct chains at the index $CI(e_2)$ as follows:

1. P_w. Only add this if P_w contains an edge. This is a down chain. The new end, w, has one incident edge in a later chain (e_w) and two incident edges in earlier chains since it was part of the old down chain J_1.
2. e_w. This is a one-way chain with tail w and head v. The tail w is incident to two edges in later chains since it was part of the old down chain J_2. The head v is incident two edges in later chains (e_y, e_z).
3. $P_z e_z$. This is a down chain. The new end, v, has one incident edge in a later chain (e_y) and two incident edges in earlier chains (e_x, e_w).
4. e_y. This is a one-way chain with tail v and head y. The tail v has two (in fact three) incident edges in earlier chains (e_x, e_z, e_w). The head y has two incident edges in later chains since y has degree one in $H_{CI(e_2)}$ and $y \notin V(J_2)$ by assumption, so y has degree at least three in $\overline{H_{CI(e_2)}}$.

- Case 3: J_2 is a one-way chain. Since y has degree one in $H_{CI(e_2)}$, $y \neq z$ so the tail z still has two incident edges in earlier chains, despite having not placed e_y yet. We use the edges e_y, e_z, e_w to construct chains at the index $CI(e_2)$ as follows:

1. e_z. This is a one-way chain with tail z and head v. The tail z is incident to two edges in earlier chains since it was the tail of J_2. The head v is incident to two edges in later chains (e_y, e_w).
2. e_w. This is a one-way chain with tail v and head w. The tail v is incident to two edges in earlier chains (e_x, e_z). The head w has two incident edges in later chains since it was the head of J_2.

3. e_y. This is a one-way chain with tail v and head y. The tail v has two (in fact three) incident edges in earlier chains (e_x, e_z, e_w). The head y has two incident edges in later chains since y has degree one in $H_{CI(e_2)}$ and $y \notin V(J_2)$ by assumption, so y has degree at least three in $\overline{H_{CI(e_2)}}$.

We consider the impact of these replacements on other chains in the graph. As before, we inserted most of the edges of the old chains J_1 and J_2 at the same chain indices relative to the unchanged chains, preventing any changes. The pinched edges e_1 and e_2 were deleted, but the ends x, z, w each received new incident edges e_x, e_z, e_w whose indices are the same as $CI(e_1)$ and $CI(e_2)$ relative to the unchanged chains. Although e_y moved to a different chain index, since y has degree one in $H_{CI(e_2)}$, there are no chains containing y between $CI(e_1)$ and $CI(e_2)$, so no chains were affected. Thus, we have maintained the chain decomposition. This proves Claim 2. □

We may now assume the following for the remaining cases:

- If J_1 is a one-way chain, then y has degree at least two in $H_{CI(e_2)}$.
- If J_2 is a one-way chain, then z has degree at least two in $\overline{H_{CI(e_1)}}$.

We also make the following conditional definitions, which will aid in distinguishing the remaining cases:

- If J_1 is a one-way chain and y is not in $H_{CI(e_1)}$, then define the minimal index i such that $y \in V(G_i)$ and $CI(e_1) < i < CI(e_2)$. Since i is minimal, y has degree one in H_i (incident only to the pinched edge e_1). From this and the fact that G_i is a minimal chain, it follows that either y is one of two distinct ends of the up chain G_i, or y is the head of the one-way chain G_i which is not a loop.
- If J_2 is a one-way chain and z is not in $\overline{H_{CI(e_2)}}$, then define the maximal index j such that $z \in V(G_j)$ and $CI(e_1) < j < CI(e_2)$. Since j is maximal, z has degree one in $\overline{H_j}$ (incident only to the pinched edge e_2). From this and the fact that G_j is a minimal chain, it follows that either z is one of two distinct ends of the down chain G_j, or z is the tail of the one-way chain G_j which is not a loop.

Claim 3. *Suppose that either one of i, j is not defined, or $i < j$. Then G' has a chain decomposition rooted at r.*

Proof. We can replace the chains J_1, J_2 independently. We may be using chain indices which differ from $CI(e_1)$ and $CI(e_2)$ relative to the existing chains. However, the chains used to replace J_1 will have index i if it is defined or else an index inserted adjacent to $CI(e_1)$, and the chains used to replace J_2 will have index j if it is defined or else an index inserted adjacent to $CI(e_2)$. Thus, by the assumptions of this claim, the edges used in the first replacement will always have lower chain indices than the edges used in the second replacement. We begin by replacing J_1 as follows:

- Case 1: J_1 is an up chain. We replace it with $P_x e_x e_y P_y$. This is an up chain. The new vertex v has two incident edges in later chains, namely e_z and e_w.

- Case 2: J_1 is a down chain. We replace it with the following chains (in the listed order):

  1. P_x. Only add this chain if P_x contains an edge. This is a down chain. The new end x has an incident edge in a later chain, namely e_x.
  2. P_y. Only add this chain if P_y contains an edge. This is a down chain. The new end y has an incident edge in a later chain, namely e_y.
  3. e_x. This is a one-way chain with tail x and head v. The tail x has two incident edges in earlier chains since it was in the old down chain J_1. The head v has two incident edges in later chains, namely e_z and e_w.
  4. e_y. This is a one-way chain with tail y and head v. The tail y has two incident edges in earlier chains since it was in the old down chain J_1. The head v has two incident edges in later chains, namely e_z and e_w.

- Case 3: J_1 is a one-way chain whose head y is in $H_{CI(e_1)}$. We replace it with the following chains (in the listed order):

  1. e_x. This is a one-way chain with tail x and head v. The tail x was the tail of the old one-way chain J_1. The head v has two (in fact three) incident edges in later chains, namely e_y, e_z, e_w.
  2. e_y. This is an up chain. The vertex y has two incident edges in later chains since it was the head of the old one-way chain J_1, and it has an incident edge in an earlier chain by assumption. The vertex v has two incident edges in later chains, namely e_z and e_w, and is incident to e_x from the previous chain.

- Case 4: J_1 is a one-way chain whose head y is not in $H_{CI(e_1)}$. Then i is defined as above.

First, we replace J_1 with e_x. This is a one-way chain with tail x and head v. The tail x was the tail of the old one-way chain J_1. The head v has two (in fact three) incident edges in later chains, namely e_y, e_z, e_w.

- Subcase 1: y is one of two distinct ends of the up chain G_i. Replace G_i with $G_i e_y$. This is an up chain. Since G_i was a path and v is a new vertex, this new chain is a path. The new end v is adjancent to one edge in an earlier chain (e_x) and two edges in later chains (e_z and e_w).
- Subcase 2: y is the head of the one-way chain G_i which is not a loop. Then y is not required to be in H_i, or any earlier H subgraph. Thus, we can leave G_i as is and insert the chain e_y immediately after G_i. This is an up chain. The vertex y is incident to an edge in the previous chain G_i and two edges in later chains since it is the head of G_i. The vertex v is adjancent to one edge in an earlier chain (e_x) and two edges in later chains (e_z and e_w).

The procedure for replacing J_2 is symmetric, by following the above steps in the reversed chain decomposition.

We consider the impact of these replacements on other chains in the graph. We inserted most of the edges of the old chains J_1 and J_2 at the same relative chain indices, preventing any changes. The pinched edges e_1 and e_2 were deleted, but the ends x and z each received new incident edges e_x and e_z at the same relative indices. In subcases where the ends y and w had their new incident edges e_y and e_w change relative chain index, the assumption that $i < j$ if both are defined ensures that the affected chain is still valid. Thus, we have maintained the chain decomposition. This proves Claim 3. □

Claim 4. *Suppose that both of i, j are defined and $i = j$. Then G' has a chain decomposition rooted at r.*

Proof. Since $i = j$, we know that $G_i = G_j$ is a one-way chain with tail z and head y, and $y \neq z$ since i and j are defined. We can replace J_1 and J_2 with the following chains, in the listed order. The first two will be placed immediately before index $i = j$, and the last two immediately after index $i = j$.

1. e_x. This is a one-way chain with tail x and head v. The tail x was the tail of the old one-way chain J_1 and we are placing this chain after index $CI(e_1)$. The head v has two (in fact three) incident edges in later chains, namely e_y, e_z, e_w.
2. e_z. This is a one-way chain with tail z and head v. By the definition of j, the tail z has two incident edges in earlier chains than G_j, and we are placing this chain immediately before index j. The head v has two incident edges in later chains, namely e_y and e_w.

3. e_y. This is a one-way chain with tail v and head y. The tail v has two incident edges in earlier chains, namely e_x and e_z. By the definition of i, the head y has two incident edges in later chains than G_i, and we are placing this chain immediately after index i.

4. e_w. This is a one-way chain with tail v and head w. The tail v has two (in fact three) incident edges in earlier chains, namely e_x, e_y, e_z. The head w was the head of the old one-way chain J_2, and we are placing this chain before $CI(e_2)$.

We consider the impact of these replacements on other chains in the graph. This time, we changed relative indices for all new edges so we must be careful. The edge e_x has a later chain index than the old chain J_1, but x had degree at least two in $H_{CI(e_1)}$, so losing a degree in later H subgraphs will not invalidate any chains. The edge e_y has a later chain index than the old chain J_1, but the index is immediately after i, so by the definition of i, the only chain affected is G_i. Since G_i has y as a head, losing a degree in H_i will not invalidate the chain. By a symmetric argument, the changes caused by e_z and e_w do not invalidate any chains. This proves Claim 4. ☐

Claim 5. *Suppose that both of i, j are defined, and $i > j$. Then G' has a chain decomposition rooted at r.*

Proof. We can replace J_1 and J_2 with the following chains, at the indicated chain indices.

1. e_x. Add this chain at index $CI(e_1)$. This is a one-way chain with tail x and head v. The tail x was the tail of the old one-way chain J_1 and we are placing this chain at index $CI(e_1)$. The head v has two (in fact three) incident edges in later chains, namely e_y, e_z, e_w.

2. e_z. Add this chain immediately after G_j. This is a one-way chain with tail z and head v. By the definition of j, the tail z has two incident edges in earlier chains than G_j, and we are placing this chain after index j. The head v has two incident edges in later chains, namely e_y and e_w.

3. e_y. Add this chain immediately before G_i. This is a one-way chain with tail v and head y. The tail v has two incident edges in earlier chains, namely e_x and e_z. By the definition of i, the head y has two incident edges in later chains than G_i, and we are placing this chain before index i.

4. e_w. Add this chain at index $CI(e_2)$. This is a one-way chain with tail v and head w. The tail v has two (in fact three) incident edges in earlier chains, namely e_x, e_y, e_z. The head w was the head of the old one-way chain J_2, and we are placing this chain at index $CI(e_2)$.

We consider the impact of these replacements on other chains in the graph. The edge e_1 was deleted, but x received a new incident edge e_x at the same chain index. The edge e_y has a later chain index than e_1, but the index is still earlier than i, so by the definition of i, no chains are affected. By a symmetric argument, the changes caused by e_z and e_w also do not invalidate any chains. This proves Claim 5. □

The claims cover all possibilities of pinching edges. The proof of Theorem 3 is complete. The proof also implies a polynomial-time algorithm to construct a chain decomposition. □

6 Proof of Theorem 4

Assume that we have a chain decomposition $G_0, G_1, \ldots, G_m$ of G. By Remark 2, we may assume that the chain decomposition is minimal. We will adapt the strategy of Curran, Lee, and Yu [3] to prove Theorem 4. In particular, we will construct two partial numberings of the edges of G using the chain decomposition. We will then construct four spanning trees in two pairs, with one pair associated with each numbering. Within each pair, paths back to the root r will be monotonic in the associated numbering to ensure independence. Between pairs, paths back to the root r will be monotonic in chain index to ensure independence.

Using Corollary 7, we may assume that there are no loops in G. By Lemma 8, for each vertex $v \neq r$, there are two distinct non-loop edges incident to v whose chain indices are strictly smaller than the chain index of any other edge incident to v. Likewise there are two distinct edges whose chain indices are strictly larger than the chain index of any other edge adjacent to v. We will name these edges as follows:

Definition. For each vertex $v \neq r$, the two *f-edges* of v are the two incident edges with the lowest chain index. Similarly, the two *g-edges* of v are the two incident edges with the highest chain index.

Remark 11. By the definition of a down chain, the edges of down chains are never f-edges. Likewise, by the definition of an up chain, the edges of up chains are never g-edges.

Next, we will recursively define a numbering f, which will assign distinct values in $\mathbb{R}$ to all edges in up chains and one-way chains. We begin by numbering the edges in $E(G_0)$, and then number the edges of each up chain and one-way chain in order of chain index. When we reach a chain G_i, we may assume that all edges in $E(H_i)$ belonging to up chains and one-way chains have been numbered, which includes all f-edges in $E(H_i)$ by Remark 11. We use the following procedure to number the edges in $E(G_i)$:

- If G_i is a closed up chain containing r, then number the edges in $E(G_i)$ so that the values change monotonically between consecutive edges in the chain. The particular numbers used are arbitrary.

- If G_i is a closed up chain not containing r, then both f-edges of the common end have already been numbered. Call these two f-edges *numbering edges* of G_i. Say the numbering edges of G_i have f-values a and b. Number the edges in $E(G_i)$ so that the values change monotonically between consecutive edges in the chain, and all values are between a and b.

- If G_i is an open up chain containing r, then r is an end and the other end is some $u \neq r$. At least one f-edge of u has already been numbered. Choose an f-edge which has already been numbered and call it a *numbering edge* of G_i. Say that a is the f-value of the numbering edge. Number the edges in $E(G_i)$ so that the values increase between consecutive edges in the chain when moving from u to r, and all values are larger than a.

- If G_i is an open up chain not containing r, then at least one f-edge of each end has been numbered. If the ends are u and v, we can choose two distinct edges $e_u, e_v \in E(H_i)$ so that e_u is an f-edge of u and e_v is an f-edge of v. Otherwise, the only f-edge of either end in $E(H_i)$ would be a single edge between u and v, and then H_i would not be connected. Call the edges e_u, e_v *numbering edges* of G_i. Without loss of generality, $f(e_u) = a < b = f(e_v)$. Number the edges in $E(G_i)$ so that the values increase between consecutive edges in the chain when moving from u to v, and all values are between a and b.

- If G_i is a one-way chain whose tail is r, then number the edge of G_i arbitrarily.

- If G_i is a one-way chain whose tail is not r, then both f-edges of the tail are already numbered, say with f-values a and b. Number the edge of G_i between a and b.

We symmetrically define a numbering g, which assigns distinct values in $\mathbb{R}$ to the edges of down chains and one-way chains, by using the above procedure in the reversed chain decomposition.

We are finally ready to construct the trees. Define the subgraphs T_1, T_2, T_3, T_4 as follows. For each $v \neq r$, consider the two f-edges of v. Assign the edge with the lower f-value to T_1 and the edge with the higher f-value to T_2. Similarly, consider the two g-edges of v. Assign the edge with the lower g-value to T_3 and the edge with the higher g-value to T_4.

Several properties of T_1, T_2, T_3, T_4 will follow from the following claim.

Claim. *For any $v \neq r$, consider the edge e_1 assigned to T_1 at v. Let v' be the other end of e_1. If $v' \neq r$, let e_1' be the edge assigned to T_1 at v'. Then $CI(e_1') \leq CI(e_1)$ and $f(e_1') < f(e_1)$.*

Proof. Let e_2 be the edge assigned to T_2 at v. The edge e_1 is not in a down chain by Remark 11. We break into two cases.

- Suppose e_1 is in an up chain G_i. Since the chain decomposition is minimal and $v' \in V(G_i)$, its f-edges are either in $E(G_i)$, or else have chain index less than i. In either case, $CI(e_1') \leq i = CI(e_1)$ as desired.

  Note that e_2 is either in $E(G_i)$, or else is the numbering edge of G_i at the end v. By the numbering procedure, we know that $f(e_1)$ is between $f(e_2)$ and the f-value of one of the f-edges of v', say e^*. By the definition of T_1, $f(e_1) < f(e_2)$, so it follows that $f(e^*) < f(e_1)$. Again by the definition of T_1, $f(e_1') \leq f(e^*)$, so $f(e_1') < f(e_1)$ as desired.

- Suppose e_1 induces a one-way chain G_i. Since e_1 is an f-edge, v has degree at most one in H_i, so v must be the head of G_i. Then v' is the tail of G_i, so the f-edges of v' have chain indices smaller than i, which means $e_1' \neq e_1$ and $CI(e_1') < CI(e_1)$ as desired.

  From the numbering procedure, we know that $f(e_1)$ is between the f-values of the two f-edges of v', with $f(e_1')$ being the smaller by the definition of T_1. So, $f(e_1') < f(e_1)$ as desired.

In both cases we have $CI(e_1') \leq CI(e_1)$ and $f(e_1') < f(e_1)$. This proves the claim. □

With the claim proven, it follows that the edges assigned to T_1 are all distinct, there are no cycles in T_1, and following consecutive edges assigned to T_1 produces a path which is decreasing in chain index, strictly decreasing in f-value, and can only end at r. Thus, T_1 is connected and is a spanning tree of G. A similar argument shows that T_2 is a spanning tree of G where paths to r are decreasing in chain index and strictly increasing in f-value. Due to the opposite trends in f-values, T_1 and T_2 are edge-independent with root r.

By symmetry, we obtain analogous results for T_3 and T_4. It remains to show that a tree from $\{T_1, T_2\}$ and a tree from $\{T_3, T_4\}$ are edge-independent. The paths back to r from a vertex $v \neq r$ are decreasing in chain index in one tree and increasing in chain index in the other tree, but not strictly. The first edges in these paths are an f-edge and a g-edge of v, respectively. By Lemmas 8 and 9, there is a positive difference in chain index between these initial edges, so the paths are in fact edge-disjoint. The proof of Theorem 4 is complete. The proof also implies a polynomial-time algorithm to construct the edge-independent spanning trees. □

7 Summary of Results

With Theorems 3 and 4 proven, we obtain Theorem 1. In fact, we can examine the argument more carefully to extract a stronger, summarizing result.

Corollary 12. *Suppose G is a graph with no isolated vertices and $V(G) \geq 2$. Then the following statements are equivalent.*

1. *G is 4-edge-connected.*
2. *There exists $r \in V(G)$ so that G has a chain decomposition rooted at r.*
3. *For all $r \in V(G)$, G has a chain decomposition rooted at r.*
4. *There exists $r \in V(G)$ so that G has four edge-independent spanning trees rooted at r.*
5. *For all $r \in V(G)$, G has four edge-independent spanning trees rooted at r.*

Proof. Theorem 3 gives us (1) $\Rightarrow$ (3). Theorem 4 gives us (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5). Trivially, we have (3) $\Rightarrow$ (2) and (5) $\Rightarrow$ (4). Therefore, we need only show (4) $\Rightarrow$ (1).

Assume for the sake of contradiction that G has four edge-independent spanning trees rooted at some $r \in V(G)$, but is not 4-edge-connected. Suppose $S \subseteq E(G)$ is an edge cut with $|S| < 4$. Consider a vertex v in the component of $G - S$ not containing r. Using the paths in each of the edge-independent spanning trees, we find that there exist four edge-disjoint paths between v and r. This contradicts the existence of S. □

References

[1] J. Cheriyan and S. N. Maheshwari, Finding Nonseparating Induced Cycles and Independent Spanning Trees in 3-Connected Graphs, J. Algorithms **9** (1988), 507–587.

[2] S. Curran, O. Lee, X. Yu, Chain Decompositions of 4-Connected Graphs, SIAM J. Discrete Math **19(4)** (2005), 848–880.

[3] S. Curran, O. Lee, X. Yu, Finding four independent trees, SIAM J. Comput., **35(5)** (2006), 1023–1058.

[4] A. Gopalan and S. Ramasubramanian, On constructing three edge independent spanning trees, unpublished. Available at citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.442.3598&rep=rep1&type=pdf.

[5] A. Huck, Independent trees in graphs, Graphs and Combinatorics **10** (1994), 29–45.

[6] A. Itai and M. Rodeh, The multi-tree approach to reliability in distributed networks, Proceedings 25th Annual IEEE Sympos. on Fund. Of Comput. Sci. (1984), 137–147.

[7] K. Kawarabayashi, O. Lee, X. Yu, Non-Separating Paths in 4-Connected Graphs, Ann. Comb. **9** (2005), 47–56.

[8] S. Khuller and B. Schieber, On Independent Spanning Trees, Information Processing Letters **42** (1992), 321–323.

[9] W. Mader, A reduction method for edge-connectivity in graphs, Advances in Graph Theory **3** (1978), 145-164.

[10] L. Schlipf and J. M. Schmidt, Edge-Orders, arXiv:1607.04431

[11] A. Zehavi and A. Itai, Three tree-paths, J. Graph Theory **13** (1989): 175–188.

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