ARRANGEABILITY AND CLIQUE SUBDIVISIONS

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ABSTRACT

Let k be an integer. A graph G is *k-arrangeable* (concept introduced by Chen and Schelp) if the vertices of G can be numbered $v_1, v_2, \ldots, v_n$ in such a way that for every integer i with $1 \le i \le n$, at most k vertices among $\{v_1, v_2, \ldots, v_i\}$ have a neighbor $v \in \{v_{i+1}, v_{i+2}, \ldots, v_n\}$ that is adjacent to v_i. We prove that for every integer $p \ge 1$, if a graph G is not p^8-arrangeable, then it contains a K_p-subdivision. By a result of Chen and Schelp this implies that graphs with no K_p-subdivision have "linearly bounded Ramsey numbers," and by a result of Kierstead and Trotter it implies that such graphs have bounded "game chromatic number."

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In this paper graphs are finite, may have parallel edges, but may not have loops. We begin by defining the concept of admissibility, introduced by Kierstead and Trotter [6].

Let G be a graph, let $M \subseteq V(G)$, and let $v \in M$. A set $A \subseteq V(G)$ is called an *M-blade* with center v if either

(i) $A = \{a\}$ and $a \in M$ is adjacent to v, or

(ii) $A = \{a, b\}$, $a \in M - \{v\}$, $b \in V(G) - M$, and b is adjacent to both v and a.

An *M-fan with center v* is a set of pairwise disjoint *M-blades* with center v. Let k be an integer. A graph G is *k-admissible* if the vertices of G can be numbered $v_1, v_2, \ldots, v_n$ in such a way that for every $i = 1, 2, \ldots, n$, G has no $\{v_1, v_2, \ldots, v_i\}$-fan with center v_i of size $k + 1$.

As pointed out in [6] the concepts of arrangeability and admissibility are asymptotically equivalent in the sense that if a graph is k-arrangeable, then it is $2k$-admissible, and if it is k-admissible, then it is $(k^2 - k + 1)$-arrangeable.

Let p be an integer. A graph G has a K_p-subdivision if G contains p distinct vertices $v_1, v_2, \ldots, v_p$ and $\binom{p}{2}$ paths P_{ij} $(i, j = 1, 2, \ldots, p, i < j)$ such that P_{ij} has ends v_i and v_j, and if a vertex of G belongs to both P_{ij} and $P_{i'j'}$ for $(i, j) \neq (i', j')$, then it is an end of both. The following is our main result.

(1) Theorem. *Let $p \geq 1$ be an integer. If a graph G is not $\frac{1}{2}p^2(p^2 + 1)$-admissible, then it has a K_p-subdivision.*

We first prove Theorem (1), and then discuss its applications. For the proof, we need the following result of Komlós and Szemerédi [7].

(2) Theorem. *Let $p \geq 1$ be an integer. If a simple graph on n vertices has at least $\frac{1}{2}p^2 n$ edges, then it has a K_p-subdivision.*

We first prove a lemma.

(3) Lemma. *Let $p \geq 1$ be an integer, let G be a graph, and let M be a non-empty subset of $V(G)$. If for every $v \in M$ there is an M-fan in G with center v of size $\frac{1}{2}p^2(p^2 + 1)$, then G has a K_p-subdivision.*

Proof. Let p, G, and M be as stated in the lemma, and for $v \in M$ let F_v be a fan in G

with center v of size $\frac{1}{2}p^2(p^2+1)$. We may assume that G is minimal subject to $M \subseteq V(G)$ and the existence of all F_v ($v \in M$). Let $|M| = m$, let e_1 be the number of edges of G with both ends in M, and let e_2 be the number of edges of G with one end in M and the other in $V(G) - M$. Then from the existence of the fans F_v for $v \in M$ we deduce that $2e_1 + e_2 \geq \frac{1}{2}p^2(p^2+1)m$.

We claim that if $|V(G) - M| \geq p^2 m - e_1$, then G has a K_p-subdivision. Indeed, by our minimality assumption for every $w \in V(G) - M$ there exist vertices $u, v \in M$ such that $\{u, w\} \in F_v$. For $w \in V(G) - M$ let us denote by $e(w)$ some such pair of vertices. Let J be the graph obtained from G by deleting $V(G) - M$ and for every $w \in V(G) - M$ adding an edge between the vertices in $e(w)$. Then $|E(J)| \geq p^2 m$, and since every pair of vertices is joined by at most two (parallel) edges, J has a simple subgraph J' on the same vertex-set with at least $\frac{1}{2}p^2 m$ edges. By (2) J' has a K_p-subdivision L. Every edge of L that does not belong to G joins two vertices u, v with $\{u, v\} = e(w)$ for some $w \in V(G) - M$. By replacing each such edge by the edges uw, vw we obtain a K_p-subdivision in G. This proves our claim, and so we may assume that $|V(G) - M| \leq p^2 m - e_1$.

Now $|V(G)| \leq (p^2+1)m - e_1$, and

$$
\begin{aligned}
|E(G)| &\geq e_1 + e_2 = 2e_1 + e_2 - e_1 \geq \frac{1}{2}p^2(p^2+1)m - e_1 \\
&\geq \frac{1}{2}p^2((p^2+1)m - e_1) \geq \frac{1}{2}p^2|V(G)|,
\end{aligned}
$$

and hence G has a K_p-subdivision by (2), as required. □

Proof of Theorem (1). Let p be an integer, and let G be a graph on n vertices with no K_p-subdivision. We are going to show that G is $\frac{1}{2}p^2(p^2+1)$-admissible by exhibiting a suitable ordering of $V(G)$. Let $i \in \{0, 1, \ldots, n\}$ be the least integer such that there exist vertices $v_{i+1}, v_{i+2}, \ldots, v_n$ with the property that for all $j = i, i+1, \ldots, n$, G has no $(V(G) - \{v_{j+1}, v_{j+2}, \ldots, v_n\})$-fan with center v_j of size $p^2(\frac{1}{2}p^2+1)+1$. We claim that $i = 0$. Indeed, otherwise by Lemma (3) applied to $M = V(G) - \{v_{i+1}, v_{i+2}, \ldots, v_n\}$ there exists a vertex v_i with no M-fan with center v_i of size $\frac{1}{2}p^2(p^2+1)$, and so the sequence $v_i, v_{i+1}, \ldots, v_n$ contradicts the choice of i. Hence $i = 0$, and $v_1, v_2, \ldots, v_n$ is the desired enumeration of the vertices of G. □

We now mention two applications of Theorem (1). Let $\mathcal{G}$ be a class of graphs. We say that $\mathcal{G}$ has *linearly bounded Ramsey numbers* if there exists a constant c such that if $G \in \mathcal{G}$ has n vertices, then for every graph H on at least cn vertices, either H or its complement contain a subgraph isomorphic to G. The class of all graphs does not have linearly bounded Ramsey numbers, but some classes do. Burr and Erdös [3] conjectured the following.

(4) Conjecture. *Let t be an integer, and let $\mathcal{G}$ be the class of all graphs whose edge-sets can be partitioned into t forests. Then $\mathcal{G}$ has linearly bounded Ramsey numbers.*

Chvátal, Rödl, Szemerédi and Trotter [5] proved that for every integer d, the class of graphs of maximum degree at most d has linearly bounded Ramsey numbers, and Chen and Schelp [4] extended that to the class of k-arrangeable graphs. Chen and Schelp also showed that every planar graph has arrangeability at most 761, a bound that has been subsequently lowered to 10 by Kierstead and Trotter [6]. From Chen and Schelp's result and Theorem (1) we deduce

(5) Corollary. *For every integer $p \geq 1$, the class of graphs with no K_p-subdivision has linearly bounded Ramsey numbers.*

For the second application we need to introduce the following two-person game, first considered by Bodlaender [2]. Let G be a graph, and let t be an integer, both fixed in advance. The game is played by two players Alice and Bob. Alice is trying to color the graph, and Bob is trying to prevent that from happening. They alternate turns with Alice having the first move. A move consists of selecting a previously uncolored vertex v and assigning it a color from $\{1, 2, \dots, t\}$ distinct from the colors assigned previously (by either player) to neighbors of v. If after $|V(G)|$ moves the graph is (properly) colored, Alice wins, otherwise Bob wins. More precisely, Bob wins if after less than $|V(G)|$ steps either player cannot make his or her next move. The *game chromatic number* of a graph G is the least integer t such that Alice has a winning strategy in the above game. Kierstead and Trotter [6] have shown the following.

(6) Theorem. *Let k and t be positive integers. If a k-admissible graph has chromatic number t, then its game chromatic number is at most $kt + 1$.*

They have also shown that planar graphs have admissibility at most 8, and hence planar graphs have game chromatic number at most 33 by (6) and the Four Color Theorem [1]. From (1), (2) and (6) we deduce

(7) Corollary. *Let p be a positive integer. Then every graph with no K_p-subdivision has game chromatic number at most $\frac{1}{2}p^4(p^2+1)$.*

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