Recent Excluded Minor Theorems for Graphs

Robin Thomas

Summary A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. An excluded minor theorem describes the structure of graphs with no minor isomorphic to a prescribed set of graphs. Splitter theorems are tools for proving excluded minor theorems. We discuss splitter theorems for internally 4-connected graphs and for cyclically 5-connected cubic graphs, the graph minor theorem of Robertson and Seymour, linkless embeddings of graphs in 3-space, Hadwiger's conjecture on t-colorability of graphs with no K_{t+1} minor, Tutte's edge 3-coloring conjecture on edge 3-colorability of 2-connected cubic graphs with no Petersen minor, and Pfaffian orientations of bipartite graphs. The latter are related to the even directed circuit problem, a problem of Pólya about permanents, the 2-colorability of hypergraphs, and sign-nonsingular matrices.

1 Introduction

All *graphs* in this paper are finite, and may have loops and parallel edges. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. An H *minor* is a minor isomorphic to H. The following is Wagner's reformulation [75] of Kuratowski's theorem [27].

Theorem 1.1 *A graph is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$.*

Kuratowski's theorem is important, because it gives a good characterization (in the sense of J. Edmonds) of planarity, but we can also think of it as a structural theorem characterizing graphs with no K_5 or $K_{3,3}$ minor. What about excluding only one of these graphs? Wagner [75] characterized those classes. To state his theorems we need one definition.

Let G_1 and G_2 be graphs with disjoint vertex-sets, let $k \geq 0$ be an integer, and for $i = 1, 2$ let $X_i \subseteq V(G_i)$ be a set of cardinality k of pairwise adjacent vertices. For $i = 1, 2$ let G'_i be obtained from G_i by deleting a (possibly empty) set of edges with both ends in X_i. Let $f : X_1 \to X_2$ be a bijection, and let G be the graph obtained from the union of G'_1 and G'_2 by identifying x with $f(x)$ for all $x \in X_1$. In those circumstances we say that G is a k-*sum* of G_1 and G_2.

Theorem 1.2 *A graph has no minor isomorphic to $K_{3,3}$ if and only if it can be obtained from planar graphs and K_5 by means of* 0-, 1-, *and* 2-*sums.*

Published in Surveys in combinatorics, 1999 (Canterbury), 201–222, London Math. Soc. Lecture Note Ser., 267, Cambridge Univ. Press, Cambridge, 1999.

By V_8 we mean the graph obtained from a circuit of length eight by joining each pair of diagonally opposite vertices by an edge.

Theorem 1.3 *A graph has no minor isomorphic to K_5 if and only if it can be obtained from planar graphs and V_8 by means of* 0-, 1-, 2- *and* 3-*sums.*

There are many similar results in Graph Theory, known as excluded minor theorems (see for example [5, 6, 16, 19, 27, 75, 76]). Such characterizations can be useful: we often need to exclude certain minors when they are obvious obstructions to some desired property, but knowledge of the structure which their exclusion forces may enable us to establish that property for the remaining graphs. Surveys of excluded minor theorems are given in [12] (for finite minors) and [45] (for infinite minors). We show that Theorem 1.1 is not an isolated result, but rather a beginning of a rich theory. We do not attempt to give a complete survey, but instead concentrate on the developments of this decade.

2 Seymour's splitter theorem

Seymour's splitter theorem is a tool for proving excluded minor theorems. We say that a simple graph G is obtained from a simple graph H by *splitting a vertex* if H is obtained from G by contracting an edge e, where both ends of e have degree at least three in G. Since H is simple, it follows that e belongs to no triangle of G. A graph is a *wheel* if it is obtained from a circuit on at least three vertices by adding a vertex joined to every vertex on the circuit. (*Paths* and *circuits* have no "repeated" vertices.) A graph G is *k-connected* if it has at least $k+1$ vertices, and $G\backslash X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$. (We use $\backslash$ for deletion.) The following is a classical result of Tutte [71].

Theorem 2.1 *Every simple* 3-*connected graph can be obtained from some wheel by repeatedly applying the operations of adding an edge between two nonadjacent vertices and splitting a vertex.*

The converse also holds: if a graph can be obtained from a wheel (in fact, any simple 3-connected graph) by means of the above two operations, then it is simple and 3-connected. Seymour [61] proved the following strengthening.

Theorem 2.2 *Let H be a simple* 3-*connected minor of a simple* 3-*connected graph G such that if H is a wheel, then H is the largest wheel minor of G. Then a graph isomorphic to G can be obtained from H by repeatedly applying the operations of adding an edge between two nonadjacent vertices and splitting a vertex.*

Thus a simple 3-connected graph can be built starting from almost any simple 3-connected minor of itself, not necessarily a wheel. To illustrate the use of Seymour's theorem, let us deduce Theorem 1.2 from it. A *separation* in a graph G is a pair of subgraphs (G_1, G_2) such that $G_1 \cup G_2 = G$ and $E(G_1) \cap E(G_2) = \emptyset$. The order of (G_1, G_2) is $|V(G_1) \cap V(G_2)|$.

Proof of Theorem 1.2 The "if" part is easy. For the "only if" part let G be a graph with no minor isomorphic to $K_{3,3}$, and assume that the theorem holds for all graphs H with $|V(H)| + |E(H)| < |V(G)| + E(G)|$. Assume first that G is not a simple 3-connected graph. If v is an isolated vertex of G, then the theorem follows by considering $G\backslash v$. Otherwise, G has a separation (G_1, G_2) of order k, where $k \leq 2$, such that $|E(G_1)|, |E(G_2)| \leq |E(G)| - 1$, and the inequality is strict if $k = 2$. Moreover, we may assume that (G_1, G_2) is chosen with k minimum. For $i = 1, 2$ let G'_i be G_i if the order of (G_1, G_2) is less than two, and otherwise let G'_i be obtained from G_i by adding an edge joining the two vertices of $V(G_1) \cap V(G_2)$. It follows from the minimality of k that G'_1 and G'_2 are minors of G. Thus, both G'_1 and G'_2 can be obtained from planar graphs and K_5 by means of 0-, 1-, and 2-sums. However, G is a k-sum of G'_1 and G'_2, as desired.

Thus we may assume that G is simple and 3-connected. If G is planar, then the theorem holds, and so we may assume that G is not planar. By Theorem 1.1, G has a minor isomorphic to K_5. We claim that G is isomorphic to K_5. Indeed, if it is not, then, by Theorem 2.2 applied to $H = K_5$ and to G, a graph isomorphic to G can be obtained from K_5 as stated in Theorem 2.2. Since K_5 is a complete graph, the next graph in the sequence is obtained from it by splitting a vertex. There is, up to isomorphism, only one way to split a vertex of K_5. It is easy to check that the resulting graph has a minor isomorphic to $K_{3,3}$, and hence so does G, a contradiction. Thus G is isomorphic to K_5, as desired. ■

To prove Theorem 1.3 we need the following lemma. We say that a graph G is *internally* 4*-connected* if G is simple, 3-connected and for every separation (G_1, G_2) of G of order three, either $|E(G_1)| \leq 3$ or $|E(G_2)| \leq 3$.

Lemma 2.3 *Let G be an internally 4-connected nonplanar graph. Then either G is isomorphic to $K_{3,3}$, or it has a minor isomorphic to K_5 or V_8.*

Proof Let G be an internally 4-connected nonplanar graph. By Theorem 1.1 the graph G has a minor isomorphic to K_5 or $K_{3,3}$. In the former case we are done, and so we may assume that G has a $K_{3,3}$ minor, and that it is not isomorphic to $K_{3,3}$. Thus G has six distinct vertices $v_1, v_2, \ldots, v_6$ and nine paths P_{ij} $(i = 1, 2, 3; j = 4, 5, 6)$ such that P_{ij} has ends v_i and v_j, and the paths are disjoint, except possibly for their ends. Let H denote the union of the nine paths. We claim that we may assume the following.

(∗) *For distinct integers* $i, k = 1, 2, 3$ *and* $j, l = 4, 5, 6$ *the graph* G *has no path with one end in* $V(P_{ij}) - \{v_i, v_j\}$*, the other end in* $V(P_{kl}) - \{v_k, v_l\}$*, and otherwise disjoint from* H*.*

Indeed, otherwise the union of H and the path give a V_8 minor, as desired. Thus we may assume that (∗) holds.

Since G is internally 4-connected and is not isomorphic to $K_{3,3}$, we deduce that at least two of the graphs $(P_{14} \cup P_{15} \cup P_{16})\backslash\{v_4, v_5, v_6\}$, $(P_{24} \cup P_{25} \cup P_{26})\backslash\{v_4, v_5, v_6\}$, $(P_{34} \cup P_{35} \cup P_{36})\backslash\{v_4, v_5, v_6\}$ belong to the same component of $G\backslash\{v_4, v_5, v_6\}$. By symmetry and (∗) we may assume that G has a path P with one end in $V(P_{14}) - \{v_4\}$ and the other end in $V(P_{24}) - \{v_4\}$.

Similarly, there exist an integer $i \in \{1, 2, 3\}$, distinct integers $k, l \in \{4, 5, 6\}$ and a path Q in $G\backslash\{v_1, v_2, v_3\}$ with one end in $V(P_{ik}) - \{v_i\}$ and the other end in $V(P_{il}) - \{v_i\}$. By considering the graph $H \cup P \cup Q$ we deduce that G has a K_5 minor, as desired. ■

Proof of Theorem 1.3 Again, the "if" part is easy. For the "only if" part let G be a graph with no minor isomorphic to K_5, and assume that the theorem holds for all graphs with fewer edges. If G is not internally 4-connected, then we conclude the proof in a similar way as in the proof of Theorem 1.2. Thus we may assume that G is internally 4-connected. If G is planar, then it satisfies the conclusion of the theorem, and so we may assume that G is not planar. By Theorem 1.1, G has a minor isomorphic to $K_{3,3}$. By Lemma 2.3 either G is isomorphic to $K_{3,3}$, or it has a V_8 minor. In the former case the theorem holds, because $K_{3,3}$ is a 3-sum of two planar graphs. Thus we may assume that G has a V_8 minor. Now it follows from Theorem 2.2 as in the proof of Theorem 1.2 that G is isomorphic to V_8, as desired. ■

3 A splitter theorem for internally 4-connected graphs

Many excluded minor theorems (e.g. the results of [17, 18, 75, 77]) can be deduced using Theorem 2.2 as in the above proofs of Theorems 1.2 and 1.3. For others, however, it is desirable to have versions of Theorem 2.2 for different kinds of connectivity. Robertson [38] and Kelmans [25] obtained one such version. This section discusses a splitter theorem for internally 4-connected graphs, and its applications. We consider yet another splitter theorem in the next section.

The straightforward analogue of Theorem 2.2 does not hold for internally 4-connected graphs for various reasons. Let us consider the following example. Let H be a graph, and let C be a circuit in H with vertices $v_1, v_2, \ldots, v_t$ (in order). Assume that each v_i has degree three, and let u_i be the neighbor of v_i other than its two neighbors on C. Let G be obtained from H by adding, for $i = 1, 2, \ldots, t$, an edge e_i joining v_i and u_{i+1} (where u_{t+1} means u_1). Then, in general, there is no sequence $J_0, J_1, \ldots, J_k$ of internally 4-connected graphs such that $J_0 = H$, $J_k = G$, and for $i = 1, 2, \ldots, k$, J_{i-1} is isomorphic to a minor of J_i and differs from J_i only "a little". (Notice that if H' is obtained from H by adding a nonempty proper subset of $\{e_1, e_2, \ldots, e_t\}$, then H' is not internally 4-connected, because it has a vertex of degree three that belongs to a circuit of length three.) Thus in the theorem to follow we allow the intermediate graphs to fail the requirement of internal 4-connectivity, but only in one area, and we insist that the next operation to be performed repairs this connectivity violation, possibly at the expense of creating another violation elsewhere.

Let us make this precise now. Let e be an edge of a graph G, and let v be a vertex of degree three adjacent to both ends of e. We say that e is a *violating edge*, and that (v, e) is a *violating pair*. We say that a graph G is *almost* 4-*connected* if G is simple, 3-connected and, for every separation (G_1, G_2) of order three, either $|E(G_1)| \leq 4$ or $|E(G_2)| \leq 4$. Thus if a graph G is obtained from an internally 4-connected graph H by applying one of the two operations of Theorem 2.2, then G is almost 4-connected, and has at most two violating edges. It turns out that we need two additional operations, which we now introduce.

Let H be a graph, let e be a violating edge in H, let v be a vertex of H such that v is not incident with or adjacent to either end of e, and let H have no violating pair (w, e) such that v is adjacent to w in H. Let G be a graph obtained from H by deleting e, and adding a new vertex and three edges joining the new vertex to v and the two ends of e. We say that G was obtained from H by a *special addition*.

Let H be a simple graph, let (v, e) be a violating pair in H, let u be the neighbor of v that is not incident with e, let u have degree at least five, and let G be obtained from H by splitting u, and then adding an edge between v and the new vertex not adjacent to v in such a way that both new vertices have degree at least four in G. We say that G was obtained from H by a *special split*.

Finally, we need several exceptional families that will play the same roles that the wheels played in Theorem 2.2. We say that an internally 4-connected graph G is a *biwheel* if G has two vertices u, v such that $G \backslash \{u, v\}$ is a circuit, and we say that it is a *ladder* if it belongs to one of the four infinite families indicated in Figure 1. The following is a result of [21].

Theorem 3.1 *Let H be an internally 4-connected minor of an internally 4-connected graph G such that H has at least seven vertices and, if H is a ladder or a biwheel, then it has at least nine vertices and it is the largest ladder or biwheel minor of G. Then a graph isomorphic to G can be obtained from H by repeatedly applying the operations of adding an edge between two nonadjacent vertices, splitting a vertex, special addition and special split in such a way that each intermediate graph is almost 4-connected, with at most one violating edge, and no edge is a violating edge of two consecutive graphs*

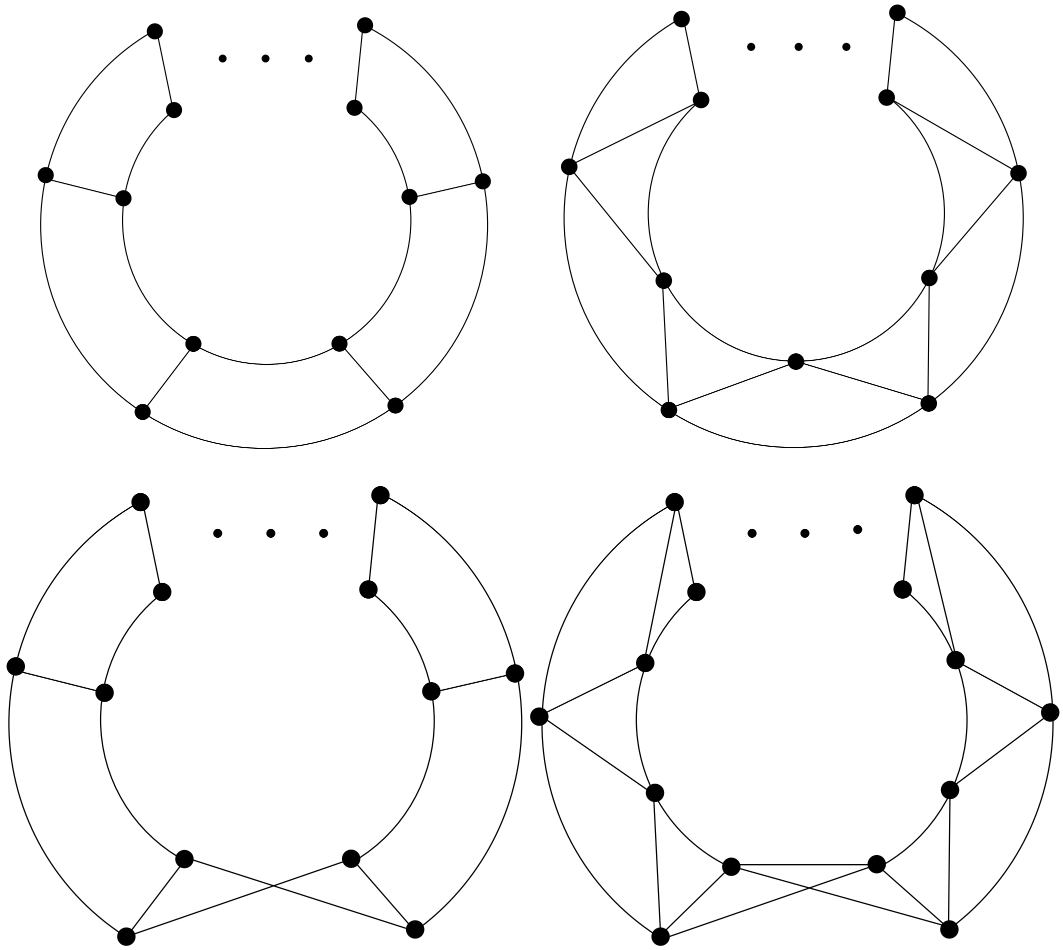

Figure 1: Ladders

in the sequence.

For an application, let us consider the following unpublished theorem of N. Robertson.

Theorem 3.2 *An internally 4-connected graph G has no V_8 minor if and only if one of the following holds.*

(a) *G is planar, or*

(b) *G has two vertices u, v such that $G\backslash\{u, v\}$ is a circuit, or*

(c) *there is a set $X \subseteq V(G)$ of cardinality four such that every edge of G has at least one end in X, or*

(d) *G is isomorphic to the line graph of $K_{3,3}$, or*

(e) *G has at most seven vertices.*

It is easy to see that in order to characterize graphs with no V_8 minor it suffices to restrict oneself to internally 4-connected graphs. Thus Theorem 3.2 can be turned into a characterization of all graphs with no V_8 minor. One step in the proof of the theorem is to show the following.

Lemma 3.3 *If an internally* 4*-connected graph has a minor isomomorphic to the line graph of* $K_{3,3}$*, and has no* V_8*-minor, then it is isomorphic to the line graph of* $K_{3,3}$.

Lemma 3.3 can be proved using Theorem 3.1 just as we used Theorem 2.2 to prove Theorems 1.2 and 1.3. In fact, all we need to do is to verify that two graphs have V_8 minors. This time we do need the stronger Theorem 3.1, because the operations used in Theorem 2.2 produce graphs which are not internally 4-connected and have no V_8 minors.

4 A splitter theorem for cyclically 5-connected cubic graphs

A graph is *cubic* if every vertex has degree three. To motivate the next splitter theorem let us mention a special case of a theorem of Tutte [70] (the proof is easy).

Theorem 4.1 *Let* G, H *be* 3*-connected cubic graphs, and let* H *be a minor of* G*. Then a graph isomorphic to* G *can be obtained from* H *by repeatedly subdividing two distinct edges and joining the new vertices by an edge.*

A cubic graph G is *cyclically* 5*-connected* if it is simple, 3-connected, and for every set $F \subseteq E(G)$ of cardinality at most four, at most one component of $G \backslash F$ has circuits. For the results discussed in Section 10 below we need a similar theorem for cyclically 5-connected cubic graphs. An ideal analogue of Theorem 4.1 for cyclically 5-connected cubic graphs would assert that G can be obtained as in Theorem 4.1 in such a way that all the intermediate graphs are cyclically 5-connected. That is, unfortunately, not true, but the exceptions can be conveniently described. We will do so now.

Let G be a cyclically 5-connected cubic graph. Let e, f be distinct edges of G with no common end and such that no edge of G is adjacent to both e and f, and let G' be obtained from G by subdividing e and f and joining the new vertices by an edge. We say that G' is a *handle expansion* of G. It can be shown that G' is cyclically 5-connected. Let e_1, e_2, e_3, e_4, e_5 (in order) be the edges of a circuit of G of length five. Let us subdivide e_i by a new vertex v_i, add a circuit (disjoint from G) with vertices u_1, u_2, u_3, u_4, u_5 (in order), and for $i = 1, 2, \ldots, 5$ let us add an edge joining u_i and v_i to form a graph G''. In these circumstances we say that G'' is a *circuit expansion* of G.

Let p be an integer such that $p \geq 5$ if p is odd and $p \geq 10$ if p is even. Let G be a cubic graph with vertex-set $\{u_0, u_1, \ldots, u_{p-1}, v_0, v_1, \ldots, v_{p-1}\}$ such that for $i = 0, 1, \ldots, p-1$, u_i has neighbors u_{i-1}, u_{i+1} and v_i, and v_i has neighbors

u_i, v_{i-2} and v_{i+2}, where the index arithmetic is taken modulo p (see Figure 2). We say that G is a *biladder* on $2p$ vertices. We remark that the Petersen graph is a biladder on 10 vertices, and that the dodecahedron is a biladder on 20 vertices. The following theorem [51] generalizes [1, 7, 10, 33, 34].

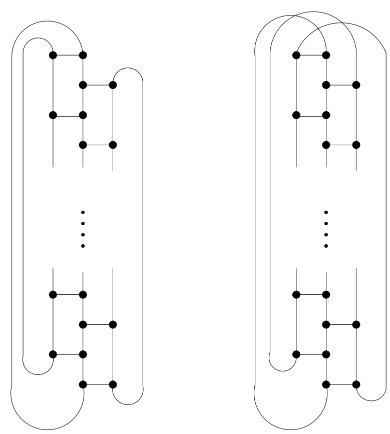

Figure 2: Biladders

Theorem 4.2 *Let G, H be cyclically 5-connected cubic graphs, let H be a minor of G, and assume that if H is a biladder, then it is the largest biladder minor of G. Then a graph isomorphic to G can be obtained from H by repeatedly applying the operations of handle expansion and circuit expansion.*

5 Excluding a general graph

We have mentioned several excluded minor theorems for specific excluded minors. Robertson and Seymour [43] found a general structure theorem for graphs with no minor isomorphic to an arbitrary fixed graph H. The theorem gives only a necessary condition for excluding H, but the condition is necessary and sufficient in the sense that no graph that possesses the structure has a minor isomorphic to some other graph H', where H' is much larger than H.

We say that a graph G is a *clique-sum* of two graphs G_1 and G_2 if G is an i-sum of G_1 and G_2 for some integer $i \geq 0$. Roughly speaking, the theorem of Robertson and Seymour says that for every graph H there exists an integer k such that that every graph with no H minor can be obtained by means of clique-sums from the class of graphs that are obtained by adding at most k vertices (and any number of edges incident with these vertices) to graphs that can be "almost" drawn on a surface the graph H cannot be drawn on. The word almost means that the graph can be drawn in the surface, except for k disks, where crossings are permitted, but that the number of edges crossing is controlled in a certain way depending on k. Let us be more precise now.

Let G be a graph, and let U be a cyclic ordering of a subset of its vertices. We say that $(X_u)_{u\in U}$ is a *vortex decomposition* of the pair (G, U) if

(V1) $u \in X_u$ for every $u \in U$,

(V2) $\bigcup_{u\in U} X_u = V(G)$, and every edge of G has both ends in some X_u, and

(V3) if u_1, u_2, u_3, u_4 occur in U in the order listed, then $X_{u_1} \cap X_{u_3} \subseteq X_{u_2} \cup X_{u_4}$.

Let us remark that axiom (V3) is equivalent to saying that, for every vertex $v \in V(G)$, the set of all $u \in U$ with $v \in X_u$ is empty, or a contiguous interval, or the whole of U. We say that $(X_u)_{u\in U}$ has width less than k if $|X_u| \leq k$ for every $u \in U$.

A *surface* is a compact connected 2-manifold with (possibly empty) boundary; the surface is *closed* if its boundary is empty. The unique surface obtained from a closed surface Σ by removing the interiors of k disjoint closed discs will be denoted by $\Sigma - k$. The components of the boundary of a surface Σ are the *cuffs* of Σ. Thus, each cuff of a surface is homeomorphic to the unit circle.

Let G be a graph, and Σ a surface with cuffs $C_1, \ldots, C_k$. We say that G *can be nearly drawn* in Σ if G has a set X of at most k vertices (where k is the number of cuffs of Σ) such that $G \backslash X$ can be written as $G_0 \cup G_1 \cup \ldots \cup G_k$, where

(N1) G_0 is embedded in Σ;

(N2) the graphs G_i $(i = 1, \ldots, k)$ are pairwise disjoint, and $U_i := V(G_0) \cap V(G_i) = V(G_0) \cap C_i$ for each $i = 1, 2, \ldots, k$;

(N3) for each $i = 1, \ldots, k$, the pair (G_i, U_i) has a vortex decomposition $(X_u)_{u\in U_i}$ of width less than k, where the ordering of U_i is determined by the cyclic ordering of points on C_i.

We can now state the excluded minor theorem of Robertson and Seymour [43].

Theorem 5.1 *For every graph H there exists an integer $k \geq 0$ such that every finite graph with no H minor can be obtained by means of clique-sums from graphs that can be nearly drawn in $\Sigma - k$ for some closed surface Σ such that H cannot be drawn in Σ.*

6 The graph minor theorem

Is there an analogue of Theorem 1.1 for other surfaces? The following is a result of Archdeacon [5], and Glover, Huneke and Wang [14].

Theorem 6.1 *A graph G admits an embedding in the projective plane if and only if G has no minor isomorphic to a member of an explicit list of 35 graphs.*

For other surfaces no such theorem is known, and there is some evidence that the list of graphs is too large to be useful. On the other hand, the following landmark result of Robertson and Seymour [44] guarantees that the lists are finite.

Theorem 6.2 *Every infinite set of graphs includes two distinct elements such that one is isomorphic to a minor of the other.*

The proof is based on Theorem 5.1. Let $\mathcal{F}$ be an infinite set of graphs, and let $F \in \mathcal{F}$. We may assume that no other member of $\mathcal{F}$ has an F minor, and hence every member of $\mathcal{F}$ has a structure as described in Theorem 5.1. That structure can be expoited to conclude the proof, but the argument is lengthy and depends on the results of several other papers.

The following is another deep result of Robertson and Seymour [42].

Theorem 6.3 *For every graph H there exists an $O(n^3)$ algorithm to decide whether an input graph on n vertices has a minor isomorphic to H.*

Theorems 6.2 and 6.3 have some surprising consequences.

Corollary 6.4 *For every class of graphs closed under isomorphisms and taking minors there exists an $O(n^3)$ algorithm to decide if an input graph on n vertices belongs to the class.*

Proof Let $\mathcal{L}'$ be the class of all graphs G such that $G \notin \mathcal{F}$, but every proper minor of G belongs to $\mathcal{F}$, and let $\mathcal{L}$ contain one graph from each isomorphism class of graphs in $\mathcal{L}'$. Then no member of $\mathcal{L}$ is isomorphic to a minor of another, and hence $\mathcal{L}$ is finite by Theorem 6.2. Thus membership to $\mathcal{F}$ can be tested using Theorem 6.3 by testing the absence of minors isomorphic to a member of $\mathcal{L}$. ■

The above proof guarantees the existence of an algorithm, but gives no clue as to how to construct one. Let us look at a special case. We say that a piecewise-linear embedding of a graph G in 3-space is *knotless* if every circuit of G forms a trivial knot. It is easy to see that contracting an edge in a knotless embedding results in a knotless embedding. Thus, by Corollary 6.4 there *exists* a polynomial-time algorithm to test whether an input graph has a knotless embedding. Curiously, at the moment we know of *no* explicit algorithm (let alone a polynomial-time one) to decide whether a given graph has a knotless embedding.

7 Linklessly embeddable graphs

Related to knotless embeddings are the following two concepts, introduced by Sachs [56, 57] and Böhme [8], respectively. We say that a (piecewise-linear) embedding of a graph in 3-space is *linkless* if every two disjoint circuits of the graph have zero linking number. We say that an embedding is *flat* if every circuit of the graph bounds a (topological) disk disjoint from the rest of the graph. By the *Petersen family* we mean the set of seven graphs depicted in Figure 3. Those are precisely the graphs that can be obtained from K_6 by

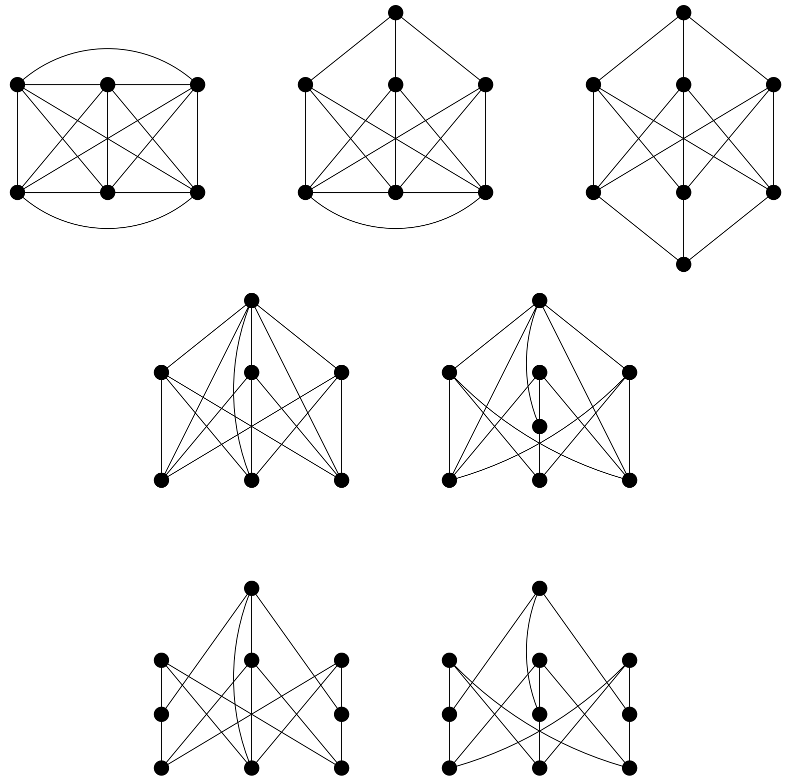

Figure 3: The Petersen family

means of Y–Δ and Δ–Y exchanges. The Petersen graph belongs to this set, and hence the name.

Sachs showed that no member of the Petersen family has a linkless embedding, and conjectured that conversely every graph has a linkless embedding unless it has a minor isomorphic to a member of the Petersen family. It turns out that the related notion of flat embeddings has an interesting theory. The following three results are proved in [50].

Theorem 7.1 *A piecewise-linear embedding of a graph G in 3-space is flat if and only if the fundamental group of the complement in 3-space of every subgraph of G is free.*

Theorem 7.2 *Every two flat embeddings of a 4-connected graph in 3-space are related by a homeomorphism of the 3-space.*

Theorem 7.2 can be regarded as an analogue of Whitney's classical result [78] which states that every 3-connected planar graph has a unique planar embedding. The following implies Sach's conjecture.

Theorem 7.3 *For a graph G, the following conditions are equivalent:*

(i) *G has a flat embedding,*

(ii) *G has a linkless embedding,*

(iii) *G has no minor isomorphic to a member of the Petersen family.*

As a structure theorem, Theorem 7.3 is not completely satisfactory, because we do not know how to test in nondeterministic polynomial time whether a given embedding is flat. It would be nice to have a graph-theoretical description of all linklessly embeddable graphs, perhaps along the lines of Theorem 10.3.

There is a related result of Lovász and Schrijver [31], concerning the parameter μ introduced by Colin de Verdière [11]. Let G be a connected graph with vertex-set $\{v_1, v_2, \ldots, v_n\}$. Then $\mu(G)$ is defined as the maximum dimension of a kernel of a matrix $M = (m_{ij})_{i,j=1}^n$ satisfying

(i) M is symmetric,

(ii) for distinct $i, j \in \{1, 2, \ldots, n\}$, $m_{ij} = 0$ if v_i and v_j are not adjacent in G, and $m_{ij} < 0$ otherwise,

(iii) M has exactly one negative eigenvalue of multiplicity one,

(iv) there is no nonzero symmetric matrix $X = (x_{ij})_{i,j=1}^n$ such that $MX = 0$ and such that $x_{ij} = 0$ whenever $i = j$ or $m_{ij} \neq 0$.

If G is not connected we define $\mu(G)$ to be the maximum of $\mu(H)$ over all components H of G.

Colin de Verdière [11] showed that a graph is planar if and only if $\mu(G) \leq 3$. This is a surprising result, given the way in which μ is defined. Lovász and Schrijver [31] proved the following generalization, conjectured in [46].

Theorem 7.4 *A graph G has a linkless embedding if and only if $\mu(G) \leq 4$.*

It follows from [11] that this is indeed a generalization of Colin de Verdiére's result. It is tempting to ask whether there is any relationship between knotlessly embeddable graphs and $\mu(G) \leq 5$. As far as I am aware, it is not even known whether $K_{1,1,3,3}$, the complete 4-partite graph with parts of sizes $1, 1, 3, 3$, respectively, has a knotless embedding.

8 The four color theorem

Our work on linkless embeddings was partly motivated by the fact that the conjectured answer involved the Petersen family, which was of interest to us because it includes both K_6 and the Petersen graph—two graphs whose exclusion is important for the $p = 5$ case of Hadwiger's conjecture and Tutte's conjectures (see Sections 9 and 10 below). The latter problems generalize the

Four Color Theorem (4CT), whose history dates back to 1852 when Francis Guthrie, while trying to color the map of the counties of England, noticed that four colors sufficed, and asked whether the same could be true for any map. Since then the conjecture has attracted a lot of attention and motivated many new developments. A proof was finally found by Appel and Haken [2, 3], reprinted in [4], formally as follows.

Theorem 8.1 *Every loopless planar graph is 4-colorable.*

However, the history seems not to end here. The proof by Appel and Haken is not completely satisfactory, because it relies on the use of computers, and even the computer-free part is so complicated that no one has been able to check it. This was partly remedied in a new proof recently found by Robertson, Sanders, Seymour and the author [41], but their proof is still computer-assisted. See [39, 40, 65] for recent surveys.

Another aspect of the 4CT is that there are several conjectures that, if true, would generalize the 4CT. It might be possible to reduce some of them to the 4CT, while others may require a strengthening of the proof of the Four Color Theorem. We will discuss two such generalizations in the next two sections.

9 Hadwiger's conjecture

Hadwiger [15] made the following conjecture.

Conjecture 9.1 *For every integer $p \geq 1$, every loopless graph with no K_{p+1} minor is p-colorable.*

Conjecture 9.1 is trivial for $p \leq 2$, for $p = 3$ it was shown by Hadwiger [15] and Dirac [13] (the proof is not very difficult), but for $p \geq 4$ it seems very difficult, because it implies the Four Color Theorem. To see this let $p \geq 4$, and let G be a planar graph. Let H be obtained from G by adding $p - 4$ vertices adjacent to each other and to every vertex of G. Then H has no K_{p+1} minor (because no planar graph has a K_5 minor by the "easy" half of Theorem 1.1), and hence H has a p-coloring by the assumed truth of Conjecture 9.1. In this p-coloring vertices of G receive at most four colors, and so G is 4-colorable, as desired.

Theorem 1.3 implies that Hadwiger's conjecture for $p = 4$ is, in fact, equivalent to the 4CT. Robertson, Seymour and the author managed to prove that the next case (that is, $p = 5$) is also equivalent to the 4CT. More specifically, in [47] they proved the following (without using the 4CT), which immediately implies (assuming the 4CT) Hadwiger's conjecture for $p = 5$. We say that a graph G is *apex* if $G \backslash v$ is planar for some $v \in V(G)$.

Theorem 9.2 *Let G be a loopless graph with no K_6 minor such that G is not 5-colorable, and, subject to that, $|V(G)|$ is minimum. Then G is apex.*

While Theorem 1.3 gives a structural description of graphs with no K_5 minor, Theorem 9.2 does not do the same for graphs with no K_6 minor. Jorgensen [22] made the following beautiful conjecture, which implies Theorem 9.2 by a result of Mader [32].

Conjecture 9.3 *Every 6-connected graph with no minor isomorphic to K_6 is apex.*

At present, Hadwiger's conjecture is open for all $p \geq 6$.

10 Tutte's edge 3-coloring conjecture

Tait [64] showed that the Four Color Theorem is equivalent to the following statement.

Theorem 10.1 *Every 2-connected cubic planar graph is edge 3-colorable.*

The smallest 2-connected cubic graph that is not edge 3-colorable is the Petersen graph. Tutte [72] conjectured that Theorem 10.1 holds with "planar" replaced by "no Petersen minor". Robertson, Sanders, Seymour and the author were recently able to settle Tutte's conjecture, as follows.

Theorem 10.2 *Every 2-connected cubic graph with no minor isomorphic to the Petersen graph is edge 3-colorable.*

The proof proceeds in two steps. First we showed in [53] that Theorem 10.2 holds in general as long as it holds for two classes of graphs: apex (defined above) and *doublecross* graphs (graphs that can be drawn in the plane with two crossings on the same region). Then we adapted our proof of the Four Color Theorem [41] to show the edge 3-colorability of 2-connected apex [59] and doublecross graphs [58]. For the first part we used Theorem 4.2 to prove the following in [52]. (*Starfish* is the graph depicted in Figure 4.)

Theorem 10.3 *Let G be a cyclically 5-connected cubic graph with no Petersen minor, and assume that for every set $A \subseteq V(G)$ with $|A|, |V(G)-A| \geq 6$ there are at least six edges of G incident with both A and $V(G)-A$. Then G is apex, or it is doublecross, or it is isomorphic to Starfish.*

Another consequence of Theorem 10.3 is the result [55] that every cubic graph of girth at least six has a subgraph isomorphic to a minor of the Petersen graph. Huck [20] used this to show that the 5-cycle double cover conjecture holds for cubic graphs with no Petersen minor.

We say that a graph G has a *nowhere-zero 4-flow* if there exists a function f mapping $E(G)$ into the nonzero elements of the Abelian group $\mathbf{Z}_2 \times \mathbf{Z}_2$ in such a way that, for every vertex v of G, the sum of $f(e)$, over all edges e incident with v, is zero. It follows that a cubic graph has a nowhere-zero 4-flow if and only if it is edge 3-colorable. Tutte [72] also made the following more general conjecture, known as the 4-flow conjecture.

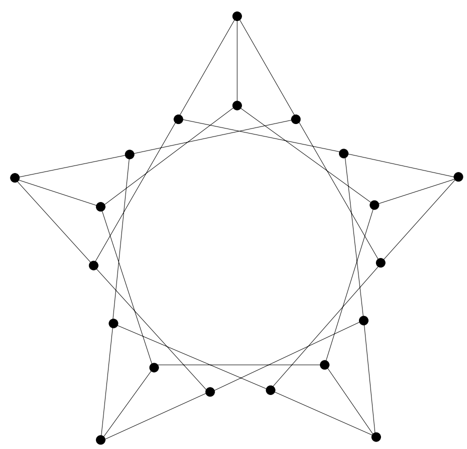

Figure 4: Starfish

Conjecture 10.4 *Every 2-connected graph with no Petersen minor has a nowhere-zero 4-flow.*

It may be possible to extend the proof of Theorem 10.2 to prove Conjecture 10.4, but no work in that direction has yet been done. Tutte made two other conjectures about nowhere-zero flows, known as the 3-flow [72] and 5-flow conjectures [69]. Both of them are still open. We refer to [62] for a survey on nowhere-zero flows.

11 Pfaffian orientations

Finally, I discuss a structural result pertaining to matching theory. An orientation D of a graph G is *Pfaffian* [23, 24, 30] if every even circuit C of G such that $G \backslash V(C)$ has a perfect matching has an odd number of edges directed in D in the direction of each orientation of C. The significance of Pfaffian orientations is that if a graph G has one, then the number of perfect matchings of G can be computed in polynomial time. Furthermore, the problem of deciding whether a bipartite graph has a Pfaffian orientation is equivalent to several other problems of interest—we mention these later. Little [28] obtained the following "excluded minor" characterization. We say that a graph H is a *matching minor* of a graph G if G has a subgraph K such that $G \backslash V(K)$ has a perfect matching, and H is obtained from K by repeatedly contracting pairs of edges incident with a common vertex of degree two.

Theorem 11.1 *A bipartite graph has a Pfaffian orientation if and only if it has no matching minor isomorphic to $K_{3,3}$.*

Theorem 11.1 is a beautiful result, but unfortunately it seems not to imply a polynomial-time algorithm to test if a given bipartite graph has a Pfaffian orientation. The next theorem, proven independently by McCuaig [35, 36] and by Robertson, Seymour and Thomas [54], can be used to design such an algorithm. We say that a bipartite graph is a *brace* if every matching of size at most two can be extended to a perfect matching. An argument similar to the one in the proof of Theorem 1.2 shows that it suffices to characterize braces that have a Pfaffian orientation. The *Heawood* graph is depicted in Figure 5.

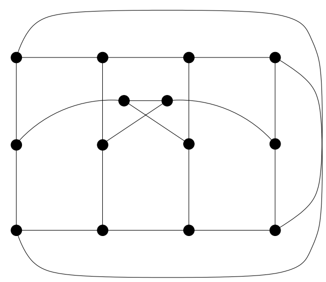

Figure 5: The Heawood graph.

Let G_0 be a graph, and let C be a circuit of G_0 of length four such that $G_0 \backslash V(C)$ has a perfect matching. Let G_1, G_2 be two subgraphs of G_0 such that $G_1 \cup G_2 = G_0$, $G_1 \cap G_2 = C$, $V(G_1) - V(G_2) \neq \emptyset$ and $V(G_2) - V(G_1) \neq \emptyset$, and let G be obtained from G_0 by deleting a (possibly empty) subset of $E(C)$. In these circumstances we say that G is a C_4*-sum* of G_1 and G_2. The following result gives the desired characterization.

Theorem 11.2 *A brace has a Pfaffian orientation if and only if either it is isomorphic to the Heawood graph, or it can be obtained from planar braces by repeated applications of the C_4-sum operation.*

Using Theorem 11.2 we were able to design a polynomial-time algorithm [54] to decide if an input graph has a Pfaffian orientation:

Theorem 11.3 *There exists an $O(n^3)$ algorithm that, given an input graph G on n vertices, either outputs a Pfaffian orientation of G, or a valid statement that G has no Pfaffian orientation.*

I now describe some consequences of Theorem 11.2. Pólya [37] asked whether given a square $0,1$-matrix A there is a matrix B obtained from A by changing some of the 1's into -1's in such a way that the determinant of B equals the permanent of A. This cannot be done for all matrices. However, given that the computing of permanents is #P-complete [73] it would seem desirable to have a characterization of matrices for which this is possible. Theorem 11.2 gives such a characterization by a result of Vazirani and Yannakakis [74].

Another consequence of Theorem 11.2 is a solution of the even directed circuit problem [66, 68, 63, 74]. The question is whether there exists a polynomial-time algorithm to decide if a digraph has a circuit of even length. Again, Theorem 11.3 provides such an algorithm by [74]. There are other equivalent formulations of the result in terms of 2-coloring of hypergraphs [29, 60], and several others in terms of sign-nonsingular matrices [9, 26, 67].

Acknowledgements

The author acknowledges partial support by NSF under Grant No. DMS-9623031, and by NSA under Contract No. MDA904-98-1-0517.

References

[1] E. R. L. Aldred, D. A. Holton & B. Jackson, Uniform cyclic edge connectivity in cubic graphs, *Combinatorica*, **11** (1991), 81–96.

[2] K. Appel & W. Haken, Every planar map is four colorable, Part I: discharging, *Illinois Journal of Mathematics*, **21** (1977), 429–490.

[3] K. Appel, W. Haken & J. Koch, Every planar map is four colorable, Part II: reducibility, *Illinois Journal of Mathematics*, **21** (1977), 491–567.

[4] K. Appel & W. Haken, Every planar map is four colorable, *Contemporary Mathematics*, **98** (1989).

[5] D. Archdeacon, A Kuratowski theorem for the projective plane, PhD thesis, Ohio State University 1980.

[6] S. Arnborg & A. Proskurowski, Characterization and recognition of partial 3-trees, *SIAM Journal of Algebraic and Discrete Methods*, **7** (1986), 305–314.

[7] D. Barnette, On Generating Planar Graphs, *Discrete Mathematics*, **7** (1974), 199–208.

[8] T. Böhme, On spatial representations of graphs, in Contemporary Methods in Graph Theory, (ed. R. Bodendieck), Bibliographisches Institut, Mannheim (1990), 151–167.

[9] R. A. Brualdi & B. L. Shader, Matrices of sign-solvable linear systems, *Cambridge Tracts in Mathematics*, **116** (1995).

[10] J. W. Butler, A Generation Procedure for the Simple 3–polytopes with Cyclically 5–Connected Graphs, *Canadian Journal of Mathematics*, **26** (1974), 686–708.

[11] Y. Colin de Verdière, Sur un nouvel invariant des graphes et un critère de planarité, *Journal of Combinatorial Theory Series B*, **50**, (1990), 11–21.

[12] R. Diestel, Graph Decompositions. A Study in Infinite Graph Theory, Oxford University Press, Oxford (1990).

[13] G. A. Dirac, Homomorphism theorems for graphs, *Mathematische Annalen*, **153** (1964), 69–80.

[14] H. H. Glover, J. P. Huneke & C. S. Wang, 103 graphs that are irreducible for the projective plane, *Journal of Combinatorial Theory Series B*, **27** (1979), 332–370.

[15] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, *Viertel Jahres Schrift der Naturforschenden Gesellschaft Zürich*, **88** (1943), 133–142.

[16] R. Halin, Über einen graphentheoretischen Basisbegriff und seine Anwendung auf Färbungsprobleme, Dissertation, University of Köln, 1962.

[17] R. Halin, Über einen Satz von K. Wagner zum Vierfarbenproblem, *Mathematische Annalen*, **153** (1964), 47–62.

[18] R. Halin, Zur Klassifikation der endlichen Graphen nach H. Hadwiger und K. Wagner, *Mathematische Annalen*, **172** (1967), 46–78.

[19] R. Halin & K. Wagner, Homomorphiebasen von Graphenmengen, *Mathematische Annalen*, **147** (1962), 126–142.

[20] A. Huck, Reducible configurations for the cycle double cover conjecture, manuscript 1997.

[21] T. Johnson & R. Thomas, A splitter theorem for internally 4-connected graphs, manuscript.

[22] L. Jorgensen, Contraction to K_8, *Journal of Graph Theory*, **18** (1994), 431–448.

[23] P. W. Kasteleyn, Dimer statistics and phase transitions, *Journal of Mathematical Physics*, **4** (1963), 287–293.

[24] P. W. Kasteleyn, Graph theory and crystal physics, in Graph Theory and Theoretical Physics (ed. F. Harary), Academic Press, New York, 1967, 43–110.

[25] A. K. Kelmans, On 3-connected graphs without essential 3-cuts or triangles, *Soviet Mathematics. Doklady*, **33** (1986), 698–703.

[26] V. Klee, R. Ladner & R. Manber, Sign-solvability revisited, *Linear Algebra and Applications*, **59** (1984), 131–158.

[27] K. Kuratowski, Sur le problème des courbes gauches en topologie, *Fundamenta Mathematicae*, **15** (1930), 271–283.

[28] C. H. C. Little, A characterization of convertible $(0, 1)$-matrices, *Journal of Combinatorial Theory Series B*, **18** (1975), 187–208.

[29] L. Lovász, On chromatic number of finite set-systems, *Acta Mathematica Academiae Scientiarum Hungaricae*, **19** (1968), 59–67.

[30] L. Lovász & M. Plummer, Matching theory, *Annals of Discrete Mathematics*, **29**, North-Holland, Amsterdam (1986).

[31] L. Lovász & A. Schrijver, A Borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs, *Proceedings of the American Mathematical Society*, **126** (1998), 1275–1285.

[32] W. Mader, Über trennende Eckenmengen in homomorphiekritische Graphen, *Mathematische Annalen*, **175** (1968), 245–252.

[33] W. McCuaig, Edge-reductions in cyclically k–connected cubic graphs, PhD thesis, University of Waterloo, Waterloo, Ontario, October 1987.

[34] W. McCuaig, Edge-reductions in cyclically k–connected cubic graphs, *Journal of Combinatorial Theory Series B*, **56** (1992), 16–44.

[35] W. McCuaig, Pólya's permanent problem, manuscript (81 pages), June 1997.

[36] W. McCuaig, Generating braces, manuscript (25 pages).

[37] G. Pólya, Aufgabe 424, *Arch. Math. Phys. Ser.*, **20** (1913), 271.

[38] N. Robertson, Minimal cyclic-4-connected graphs, *Transactions of the American Mathematical Society*, **284** (1984), 665–687.

[39] N. Robertson, D. P. Sanders, P. D. Seymour & R. Thomas, Efficiently four-coloring planar graphs, *Proceedings of the ACM Symposium on the Theory of Computing*, **28** (1996), 571–575.

[40] N. Robertson, D. P. Sanders, P. D. Seymour & R. Thomas, A new proof of the four–color theorem, *Electronic Research Announcements of the AMS*, **2** (1996), 17–25.

[41] N. Robertson, D. P. Sanders, P. D. Seymour & R. Thomas, The four-colour theorem, *Journal of Combinatorial Theory Series B*, **70** (1997), 2-44.

[42] N. Robertson & P. D. Seymour, Graph Minors XIII. The disjoint paths problem, *Journal of Combinatorial Theory Series B*, **63** (1995), 65–110.

[43] N. Robertson & P. D. Seymour, Graph Minors XVII. Excluding a non-planar graph, submitted.

[44] N. Robertson & P. D. Seymour, Graph Minors XX. Wagner's conjecture, manuscript.

[45] N. Robertson, P. D. Seymour & R. Thomas, Excluding infinite minors, *Discrete Mathematics*, **95** (1991), 303–319.

[46] N. Robertson, P. D. Seymour & R. Thomas, Linkless embeddings of graphs in 3-space, *Bulletin of the American Mathematical Society*, **28** (1993), 84–89.

[47] N. Robertson, P. D. Seymour & R. Thomas, Hadwiger's conjecture for K_6-free graphs, *Combinatorica*, **13** (1993), 279–361.

[48] N. Robertson, P. D. Seymour & R. Thomas, Kuratowski chains, *Journal of Combinatorial Theory Series B*, **64** (1995), 127–154.

[49] N. Robertson, P. D. Seymour & R. Thomas, Petersen family minors, *Journal of Combinatorial Theory Series B*, **64** (1995), 155–184.

[50] N. Robertson, P. D. Seymour & R. Thomas, Sach's linkless embedding conjecture, *Journal of Combinatorial Theory Series B*, **64** (1995), 185–227.

[51] N. Robertson, P. D. Seymour & R. Thomas, Cyclically 5-connected cubic graphs, manuscript.

[52] N. Robertson, P. D. Seymour & R. Thomas, Excluded minors in cubic graphs, manuscript.

[53] N. Robertson, P. D. Seymour & R. Thomas, Tutte's edge-coloring conjecture, *Journal of Combinatorial Theory Series B*, **70** (1997), 166–183.

[54] N. Robertson, P. D. Seymour & R. Thomas, Permanents, Pfaffian orientations, and even directed circuits, to appear in *Annals of Mathematics.*

[55] N. Robertson, P. D. Seymour & R. Thomas, Girth six graphs have Petersen minors, manuscript.

[56] H. Sachs, On a spatial analogue of Kuratowski's theorem on planar graphs—an open problem, Proceedings of a Conference held in Łagów, Poland, February 10–13, 1981, Lecture Notes in Mathematics, Vol. 1018, Springer-Verlag, Berlin, 1983.

[57] H. Sachs, On spatial representation of finite graphs, Colloquia Mathematica Societatis János Bolyai, 37. Finite and infinite sets, (eds. A. Hajnal, L. Lovász & V. T. Sós), North-Holland, Budapest 1984, pp. 649–662.

[58] D. P. Sanders, P. D. Seymour & R. Thomas, Edge three-coloring cubic doublecross graphs, in preparation.

[59] D. P. Sanders & R. Thomas, Edge three-coloring cubic apex graphs, manuscript.

[60] P. D. Seymour, On the two-colouring of hypergraphs, *The Quarterly Journal of Mathematics. Oxford Second Series*, **25** (1974), 303–312.

[61] P. D. Seymour, Decomposition of regular matroids, *Journal of Combinatorial Theory Series B*, **28** (1980), 305–359

[62] P. D. Seymour, Nowhere-zero flows, in: Handbook of combinatorics, (eds. R. L. Graham, M. Grötschel, L. Lovász), North-Holland, 1995.

[63] P. D. Seymour & C. Thomassen, Characterization of even directed graphs, *Journal of Combinatorial Theory Series B*, **42** (1987), 36-45.

[64] P. G. Tait, Note on a theorem in geometry of position, *Transactions of the Royal Society of Edinburgh*, **29** (1880), 657–660.

[65] R. Thomas, An update on the Four-Color Theorem, *Notices of the American Mathematical Society*, **45** (1998), 848–859.

[66] C. Thomassen, Even cycles in directed graphs, *European Journal of Combinatorics*, **6** (1985), 85–89.

[67] C. Thomassen, Sign-nonsingular matrices and even cycles in directed graphs, *Linear Algebra and its Applications*, **75** (1986), 27–41.

[68] C. Thomassen, The even cycle problem for directed graphs, *Journal of the American Mathematical Society*, **5** (1992), 217–229.

[69] W. T. Tutte, A contribution to the theory of chromatic polynomials, *Canadian Mathematical Journal*, **6** (1954), 80-91.

[70] W. T. Tutte, Convex Representations of Graphs, *Proceedings of the London Mathematical Society*, **10** (1960), 304–320.

[71] W. T. Tutte, A theory of 3-connected graphs, *Indagationes Mathematicae*, **23** (1961), 441-455.

[72] W. T. Tutte, On the algebraic theory of graph colorings, *Journal of Combinatorial Theory Series B*, **1** (1966), 15–50.

[73] L. G. Valiant, The complexity of computing the permanent, *Theoretical Computer Science*, **8** (1979), 189–201.

[74] V. V. Vazirani & M. Yannakakis, Pfaffian orientations, 0-1 permanents, and even cycles in directed graphs, *Discrete Applied Mathematics*, **25** (1989), 179–190.

[75] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Mathematische Annalen*, **114** (1937), 570-590.

[76] K. Wagner, Über eine Erweiterung des Satzes von Kuratowski, *Deutsche Mathematik*, **2** (1937), 280–285.

[77] K. Wagner, Bemerkungen zu Hadwigers Vermutung, *Mathematische Annalen*, **141** (1960), 433-451.

[78] H. Whitney, 2-isomorphic graphs, *American Journal of Mathematics*, **55** (1933), 245–254.

School of Mathematics
Georgia Institute of Technology
Altanta, GA 30332-0160, USA
`thomas@math.gatech.edu`