MINORS OF TWO-CONNECTED GRAPHS
OF LARGE PATH-WIDTH

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Abstract

Let $P$ be a graph with a vertex $v$ such that $P \setminus v$ is a forest, and let $Q$ be an outerplanar graph. We prove that there exists a number $p = p(P, Q)$ such that every 2-connected graph of path-width at least $p$ has a minor isomorphic to $P$ or $Q$. This result answers a question of Seymour and implies a conjecture of Marshall and Wood.

1 Introduction

All graphs in this paper are finite and simple; that is, they have no loops or parallel edges. Paths and cycles have no “repeated” vertices or edges. A graph $H$ is a minor of a graph $G$ if we can obtain $H$ by contracting edges of a subgraph of $G$. An $H$ minor is a minor isomorphic to $H$. A tree-decomposition of a graph $G$ is a pair $(T, X)$, where $T$ is a tree and $X$ is a family $(X_t : t \in V(T))$ such that:

(W1) $\bigcup_{t \in V(T)} X_t = V(G)$, and for every edge of $G$ with ends $u$ and $v$ there exists $t \in V(T)$ such that $u, v \in X_t$, and

(W2) if $t_1, t_2, t_3 \in V(T)$ and $t_2$ lies on the path in $T$ between $t_1$ and $t_3$, then $X_{t_1} \cap X_{t_3} \subseteq X_{t_2}$.

The width of a tree-decomposition $(T, X)$ is $\max\{|X_t| - 1 : t \in V(T)\}$. The tree-width of a graph $G$ is the smallest width among all tree-decompositions of $G$. A path-decomposition of $G$ is a tree-decomposition $(T, X)$ of $G$, where $T$ is a path. We will often denote a path-decomposition as $(X_1, X_2, \ldots, X_n)$, rather than having the constituent sets indexed by the vertices of a path. The path-width of $G$ is the smallest width among all path-decompositions of $G$. Robertson and Seymour [11] proved the following:

Theorem 1.1. For every planar graph $H$ there exists an integer $n = n(H)$ such that every graph of tree-width at least $n$ has an $H$ minor.

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Robertson and Seymour [10] also proved an analogous result for path-width:

**Theorem 1.2.** For every forest $F$, there exists an integer $p = p(F)$ such that every graph of path-width at least $p$ has an $F$ minor.

Bienstock, Robertson, Seymour and the second author [11] gave a simpler proof of Theorem 1.2 and improved the value of $p$ to $|V(F)| - 1$, which is best possible, because $K_k$ has path-width $k - 1$ and does not have any forest minor on $k + 1$ vertices. A yet simpler proof of Theorem 1.2 was found by Diestel [5].

While Geelen, Gerards and Whittle [7] generalized Theorem 1.1 to representable matroids, it is not a priori clear what a version of Theorem 1.2 for matroids should be, because excluding a forest in matroid setting is equivalent to imposing a bound on the number of elements and has no relevance to path-width. To overcome this, Seymour [4, Open Problem 2.1] asked if there was a generalization of Theorem 1.2 for 2-connected graphs with forests replaced by the two families of graphs mentioned in the abstract. Our main result answers Seymour’s question in the affirmative:

**Theorem 1.3.** Let $P$ be a graph with a vertex $v$ such that $P \setminus v$ is a forest, and let $Q$ be an outerplanar graph. Then there exists a number $p = p(P, Q)$ such that every 2-connected graph of path-width at least $p$ has a $P$ or $Q$ minor.

Theorem 1.3 is a generalization of Theorem 1.2. To deduce Theorem 1.2 from Theorem 1.3 given a graph $G$, we may assume that $G$ is connected, because the path-width of a graph is equal to the maximum path-width of its components. We add one vertex and make it adjacent to every vertex of $G$. Then the new graph is 2-connected, and by Theorem 1.3 it has a $P$ or $Q$ minor. By choosing suitable $P$ and $Q$, we can get an $F$ minor in $G$.

Marshall and Wood [8] define $g(H)$ as the minimum number for which there exists a positive integer $p(H)$ such that every $g(H)$-connected graph with no $H$ minor has path-width at most $p(H)$. Then Theorem 1.2 implies that $g(H) = 0$ iff $H$ is a forest. There is no graph $H$ with $g(H) = 1$, because path-width of a graph $G$ is the maximum of the path-widths of its connected components. Let $A$ be the graph that consists of a cycle $a_1a_2a_3a_4a_5a_6a_1$ and extra edges $a_1a_3,a_3a_5,a_5a_1$. Let $C_{3,2}$ be the graph consisting of two disjoint triangles. In Section 2 we prove a conjecture of Marshall and Wood [8]:

**Theorem 1.4.** A graph $H$ has no $K_4, K_{2,3}, C_{3,2}$ or $A$ minor if and only if $g(H) \leq 2$.

In Section 3 we describe a special tree-decomposition, whose existence we establish in [3]. In Section 4 we introduce “cascades”, our main tool, and prove that in any tree-decomposition with no duplicate bags of bounded width of a graph of big path-width there is an “injective” cascade of large height. In Section 5 we prove that every 2-connected graph of big path-width and bounded tree-width admits a tree-decomposition of bounded width and a cascade with linkages that are minimal. In Section 6 we analyze those minimal linkages and prove that there are essentially only two types of linkage. This is where we use the properties of tree-decompositions from Section 3. Finally, in Section 7 we convert the two types of linkage into the two families of graphs from Theorem 1.3.
2 Proof of Theorem 1.4

In this section we prove that Theorem 1.4 is implied by Theorem 1.3.

**Definition** Let $h \geq 0$ be an integer. By a *binary tree of height $h$* we mean a tree with a unique vertex $r$ of degree two and all other vertices of degree one or three such that every vertex of degree one is at distance exactly $h$ from $r$. Such a tree is unique up to isomorphism and so we will speak of the binary tree of height $h$. We denote the binary tree of height $h$ by $CT_h$ and we call $r$ the root of $CT_h$. Each vertex in $CT_h$ with distance $k$ from $r$ has height $k$. We call the vertices at distance $h$ from $r$ the *leaves of $CT_h$*. If $t$ belongs to the unique path in $CT_h$ from $r$ to a vertex $t' \in V(T_h)$, then we say that $t'$ is a *descendant of $t$* and that $t$ is an *ancestor of $t'$*. If, moreover, $t$ and $t'$ are adjacent, then we say that $t$ is the *parent of $t'$* and that $t'$ is a *child of $t$*.

Let $P_k$ be the graph consisting of $CT_k$ and a separate vertex that is adjacent to every leaf of $CT_k$.

**Lemma 2.1.** If a graph $H$ has no $K_4, C_{3,2}$, or $A$ minor, then $H$ has a vertex $v$ such that $H \setminus v$ is a forest.

**Proof.** We proceed by induction on $|V(H)|$. The lemma clearly holds when $|V(H)| = 0$, and so we may assume that $H$ has at least one vertex and that the lemma holds for graphs on fewer than $|V(H)|$ vertices. If $H$ has a vertex of degree at most one, then the lemma follows by induction by deleting such vertex. We may therefore assume that $H$ has minimum degree at least two.

If $H$ has a cutvertex, say $v$, then $v$ is as desired, for if $C$ is a cycle in $H \setminus v$, then $H \setminus V(C)$ also contains a cycle (because $H$ has minimum degree at least two), and hence $H$ has a $C_{3,2}$ minor, a contradiction. We may therefore assume that $H$ is 2-connected.

We may assume that $H$ is not a cycle, and hence it has an ear-decomposition $H = H_0 \cup H_1 \cup \cdots \cup H_k$, where $k \geq 1$, $H_0$ is a cycle and for $i = 1, 2, \ldots, k$ the graph $H_i$ is a path with ends $u_i, v_i \in V(H_0 \cup H_1 \cup \cdots \cup H_{i-1})$ and otherwise disjoint from $H_0 \cup H_1 \cup \cdots \cup H_{i-1}$. If $u_1 \in \{u_i, v_i\}$ for all $i \in \{2, 3, \ldots, k\}$, then $u_1$ satisfies the conclusion of the lemma, and similarly for $v_1$. We may therefore assume that there exist $i, j \in \{2, 3, \ldots, k\}$ such that $u_1 \notin \{u_i, v_i\}$ and $v_1 \notin \{u_j, v_j\}$. It follows that $H$ has a $K_4, C_{3,2}$, or $A$ minor, a contradiction.

**Lemma 2.2.** If a graph $H$ has a vertex $v$ such that $H \setminus v$ is a forest. then there exists an integer $k$ such that $H$ is isomorphic to a minor of $P_k$.

**Proof.** Let $v$ be such that $T := H \setminus v$ is a forest. We may assume, by replacing $H$ by a graph with an $H$ minor, that $T$ is isomorphic to $CT_i$ for some $t$, and that $v$ is adjacent to every vertex of $T$. It follows that $H$ is isomorphic to a minor of $P_{2t}$, as desired.

**Definition** Let $Q_1$ be $K_3$. An arbitrary edge of $Q_1$ will be designated as *base edge*. For $i \geq 2$ the graph $Q_i$ is constructed as follows: Now assume that $Q_{i-1}$ has already been defined, and let $Q_1$ and $Q_2$ be two disjoint copies of $Q_{i-1}$ with base edges $u_1v_1$ and $u_2v_2$,
respectively. Let \( T \) be a copy of \( K_3 \) with vertex-set \( \{w_1, w_2, w\} \) disjoint from \( Q_1 \) and \( Q_2 \). The graph \( Q_i \) is obtained from \( Q_1 \cup Q_2 \cup T \) by identifying \( u_1 \) with \( w_1 \), \( u_2 \) with \( w_2 \), and \( v_1 \) and \( v_2 \) with \( w \). The edge \( w_1w_2 \) will be the base edge of \( Q_i \).

A graph is outerplanar if it has a drawing in the plane (without crossings) such that every vertex is incident with the unbounded face. A graph is a near-triangulation if it is drawn in the plane in such a way that every face except possibly the unbounded one is bounded by a triangle.

Let \( H \) and \( G \) be graphs. If \( G \) has an \( H \) minor, then to every vertex \( u \) of \( H \) there corresponds a connected subgraph of \( G \), called the node of \( u \).

**Lemma 2.3.** Let \( H \) be a 2-connected outerplanar near-triangulation with \( k \) triangles. Then \( H \) is isomorphic to a minor of \( Q_k \). Furthermore, the minor inclusion can be chosen in such a way that for every edge \( a_1a_2 \in E(H) \) incident with the unbounded face and for every \( i \in \{1, 2\} \), the vertex \( w_i \) belongs to the node of \( a_i \), where \( w_1w_2 \) is the base edge of \( Q_k \).

**Proof.** We proceed by induction on \( k \). The lemma clearly holds when \( k = 1 \), and so we may assume that \( H \) has at least two triangles and that the lemma holds for graphs with fewer than \( k \) triangles. The edge \( a_1a_2 \) belongs to a unique triangle, say \( a_1a_2c \). The triangle \( a_1a_2c \) divides \( H \) into two near-triangulations \( H_1 \) and \( H_2 \), where the edge \( a_i c \) is incident with the unbounded face of \( H_i \). Let \( Q_1, Q_2, u_1, v_1, u_2, v_2, w_1, w_2 \) be as in the definition of \( Q_k \). By the induction hypothesis the graph \( H_i \) is isomorphic to a minor of \( Q_i \) in such a way that the vertex \( u_i \) belongs to the node of \( a_i \) and the vertex \( v_i \) belongs to the node of \( c \). It follows that \( H \) is isomorphic to \( Q_k \) in such a way that \( w_i \) belongs to the node of \( a_i \). \qed

**Lemma 2.4.** Let \( H \) be a graph that has no \( K_4 \) or \( K_{2,3} \) minor. Then there exists an integer \( k \) such that \( H \) is isomorphic to a minor of \( Q_k \).

**Proof.** It is well-known \([6, \text{Exercise 23}]\) that the hypotheses of the lemma imply that \( H \) is outerplanar. We may assume, by replacing \( H \) by a graph with an \( H \) minor, that \( H \) is a 2-connected outerplanar near-triangulation. The lemma now follows from Lemma 2.3. \qed

**Corollary 2.5.** Let \( H \) be a graph that has no \( K_4, K_{2,3}, C_{3,2}, \) or \( A \) minor. Then there exists an integer \( k \) such that \( H \) is isomorphic to a minor of \( P_k \) and \( H \) is isomorphic to a minor of \( Q_k \).

**Proof.** This follows from Lemmas 2.1, 2.2 and 2.4. \qed

**Proof of Theorem 1.4 assuming Theorem 1.3** To prove the “if” part notice that \( P_k \) and \( Q_k \) are 2-connected and have large path-width when \( k \) is large, because \( Q_k \) has a \( CT_{k-1} \) minor. There is no vertex \( v \) in \( A \) such that \( A \setminus v \) is acyclic. So, \( A \) and \( C_{3,2} \) are not minors of \( P_k \) for any \( k \). The graph \( Q_k \) is outerplanar, so \( K_4 \) and \( K_{2,3} \) are not minors of \( Q_k \) for any positive integer \( k \). This means \( g(H) \geq 3 \) for \( H \in \{K_4, K_{2,3}, C_{3,2}, A\} \). This proves the “if” part.

To prove the “only if” part, if \( H \) has no \( K_4, K_{2,3}, C_{3,2} \) or \( A \) minor, then by Corollary 2.5 \( H \) is a minor of both \( P_k \) and \( Q_k \) for some \( k \). Then \( g(H) \leq 2 \) by Theorem 1.3. \qed
3 A Special Tree-decomposition

In this section we review properties of tree-decompositions established in [3, 9, 12]. The proof of the following easy lemma can be found, for instance, in [12].

Lemma 3.1. Let \((T, Y)\) be a tree-decomposition of a graph \(G\), and let \(H\) be a connected subgraph of \(G\) such that \(V(H) \cap Y_t \neq \emptyset \neq V(H) \cap Y_{t'}\), where \(t, t' \in V(T)\). Then \(V(H) \cap Y_t \neq \emptyset\) for every \(t \in V(T)\) on the path between \(t_1\) and \(t_2\) in \(T\).

A tree-decomposition \((T, Y)\) of a graph \(G\) is said to be linked if

(W3) for every two vertices \(t_1, t_2\) of \(T\) and every positive integer \(k\), either there are \(k\) disjoint paths in \(G\) between \(Y_{t_1}\) and \(Y_{t_2}\), or there is a vertex \(t\) of \(T\) on the path between \(t_1\) and \(t_2\) such that \(|Y_t| < k\).

It is worth noting that, by Lemma 3.1, the two alternatives in (W3) are mutually exclusive. The following is proved in [12].

Lemma 3.2. If a graph \(G\) admits a tree-decomposition of width at most \(w\), where \(w\) is some integer, then \(G\) admits a linked tree-decomposition of width at most \(w\).

Let \((T, Y)\) be a tree-decomposition of a graph \(G\), let \(t_0 \in V(T)\), and let \(B\) be a component of \(T \setminus t_0\). We say that a vertex \(v \in Y_{t_0}\) is \(B\)-tied if \(v \in Y_t\) for some \(t \in V(B)\). We say that a path \(P\) in \(G\) is \(B\) confined if \(|V(P)| \geq 3\) and every internal vertex of \(P\) belongs to \(\bigcup_{t \in V(B)} Y_t - Y_{t_0}\). We wish to consider the following three properties of \((T, Y)\):

(W4) if \(t, t'\) are distinct vertices of \(T\), then \(Y_t \neq Y_{t'}\),

(W5) if \(t_0 \in V(T)\) and \(B\) is a component of \(T \setminus t_0\), then \(\bigcup_{t \in V(B)} Y_t - Y_{t_0} \neq \emptyset\),

(W6) if \(t_0 \in V(T)\), \(B\) is a component of \(T \setminus t_0\), and \(u, v\) are \(B\)-tied vertices in \(Y_{t_0}\), then there is a \(B\)-confined path in \(G\) between \(u\) and \(v\).

The following strengthening of Lemma 3.2 is proved in [9].

Lemma 3.3. If a graph \(G\) has a tree-decomposition of width at most \(w\), where \(w\) is some integer, then it has a tree-decomposition of width at most \(w\) satisfying (W1)-(W6).

We need one more condition, which we now introduce. Let \(T\) be a tree. If \(t, t' \in V(T)\), then by \(T[t, t']\) we denote the set of vertices belonging to the unique path in \(T\) from \(t\) to \(t'\). A triad in \(T\) is a triple \(t_1, t_2, t_3\) of vertices of \(T\) such that there exists a vertex \(t\) of \(T\), called the center, such that \(t_1, t_2, t_3\) belong to different components of \(T \setminus t\). Let \((T, W)\) be a tree-decomposition of a graph \(G\), and let \(t_1, t_2, t_3\) be a triad in \(T\). The torso of \((T, W)\) at \(t_1, t_2, t_3\) is the subgraph of \(G\) induced by the set \(\bigcup W_t\), the union taken over all vertices \(t \in V(T)\) such that either \(t \in \{t_1, t_2, t_3\}\), or for all \(i \in \{1, 2, 3\}\), \(t\) belongs to the component of \(T \setminus t_i\) containing the center of \(t_1, t_2, t_3\). We say that the
triad $t_1, t_2, t_3$ is $W$-separable if, letting $X = W_{t_1} \cap W_{t_2} \cap W_{t_3}$, the graph obtained from the torso of $(T, W)$ at $t_1, t_2, t_3$ by deleting $X$ can be partitioned into three disjoint non-null graphs $H_1, H_2, H_3$ in such a way that for all distinct $i, j \in \{1, 2, 3\}$ and all $t \in T[t_j, t_0]$, $|V(H_i) \cap W_t| \geq |V(H_j) \cap W_{t_j}| = |W_{t_j} - X|/2 \geq 1$. (Let us remark that this condition implies that $|W_{t_i}| = |W_{t_2}| = |W_{t_3}|$ and $V(H_i) \cap W_t = \emptyset$ for $i = 1, 2, 3$.) The last property of a tree-decomposition $(T, W)$ that we wish to consider is

(W7) if $t_1, t_2, t_3$ is a $W$–separable triad in $T$ with center $t$, then there exists an integer $i \in \{1, 2, 3\}$ with $W_{t_i} \cap W_t - (W_{t_1} \cap W_{t_2} \cap W_{t_3}) \neq \emptyset$.

The following is proven in [3].

**Theorem 3.4.** If a graph $G$ has a tree-decomposition of width at most $w$, where $w$ is some integer, then it has a tree-decomposition of width at most $w$ satisfying (W1)-(W7).

This theorem is used to prove Theorem 1.3 in Section 7.

4 Cascades

In this section we introduce “cascades”, our main tool. The main result of this section, Lemma 4.1, states that in any tree-decomposition with no duplicate bags of bounded width of a graph of big path-width there is an “injective” cascade of large height

**Lemma 4.1.** Let $p, w$ be two positive integers and let $G$ be a graph of tree-width strictly less than $w$ and path-width at least $p$. Then for every tree-decomposition $(T, X)$ of $G$ of width strictly less than $w$, the path-width of $T$ is at least $\lceil p/w \rceil$.

*Proof.* We will prove the contrapositive. Assume there exists a tree-decomposition $(T, X)$ of $G$ of width $< w$ such that the path-width of $T$ is less than $\lceil p/w \rceil$. Because the path-width of $T$ is less than $\lceil p/w \rceil$, there exists a path-decomposition $(Y_1, Y_2, \ldots, Y_s)$ of $T$ with $|Y_i| \leq \lceil p/w \rceil$ for all $i$. We will construct a path-decomposition $(Z_1, Z_2, \ldots, Z_s)$ for $G$ of width less than $p$. Set $Z_i = \bigcup_{y \in Y_i} X_y$ for every $i \in \{1, 2, \ldots, s\}$. For every vertex $v \in V(G)$, $v$ belongs to at least one set $X_t$ for some $t \in V(T)$. The vertex $t$ of the tree $T$ must be in $Y_l$ for some $l \in \{1, 2, \ldots, s\}$, so $v \in X_{t_l} \subseteq Z_l$. Therefore, $\bigcup Z_i = V(G)$. Similarly, for every edge $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in X_t$. Therefore, $u, v \in Z_l$ for some $l \in \{1, 2, \ldots, s\}$.

Now, if a vertex $v \in V(G)$ belongs to both $Z_a$ and $Z_b$ for some $a, b \in \{1, 2, \ldots, s\}, a < b$, we will show that $v \in Z_c$ for all $c$ such that $a < c < b$. Let $c$ be an arbitrary integer satisfying $a < c < b$. The fact that $v \in Z_a$ implies $v \in X_{y_1}$ for some $y_1 \in Y_a$. Similarly, $v \in X_{y_2}$ for some $y_2 \in Y_b$. Let $H$ be the set of vertices of $T$ on the path from $y_1$ to $y_2$. Since $y_1 \in Y_a$ and $y_2 \in Y_b$, $H \cap Y_a \neq \emptyset \neq H \cap Y_b$. Hence, by Lemma 3.1 with $H = T$ and $(T, Y)$ the path-decomposition $(Y_1, Y_2, \ldots, Y_s)$, we have $H \cap Y_c \neq \emptyset$. Let $t \in H \cap Y_c$, then $v \in X_t \subseteq Z_c$. So $(Z_1, Z_2, \ldots, Z_s)$ is a path-decomposition of $G$. Since the width of $(T, X)$ is less than $w$, we have $|X_y| \leq w$ for every $y \in Y_i$, where $i \in \{1, 2, \ldots, s\}$. Therefore, $|Z_i| \leq w \cdot \lceil p/w \rceil \leq p$ for every $i \in \{1, 2, \ldots, s\}$. Therefore, the width of $(Z_1, Z_2, \ldots, Z_s)$ is less than $p$, so the path-width of $G$ is less than $p$, as desired.  

\[\square\]
Let $T, T'$ be trees. A homeomorphic embedding of $T$ into $T'$ is a mapping $\eta : V(T) \to V(T')$ such that

- $\eta$ is an injection, and
- if $tt_1, tt_2$ are edges of $T$ with a common end, and $P_i$ is the unique path in $T'$ with ends $\eta(t)$ and $\eta(t_i)$, then $P_1$ and $P_2$ are edge-disjoint.

We will write $\eta : T \hookrightarrow T'$ to denote that $\eta$ is a homeomorphic embedding of $T$ into $T'$. Since $CT_a$ has maximum degree at most three, the following lemma follows from [8, Lemma 6].

**Lemma 4.2.** Let $T$ be a forest of path-width at least $a \geq 1$. Then there exists a homeomorphic embedding $CT_{a-1} \hookrightarrow T$.

For every integer $h \geq 1$ we will need a specific type of tree, which we will denote by $T_h$. The tree $T_h$ is obtained from $CT_h$ by subdividing every edge not incident with a vertex of degree one exactly once, and adding a new vertex $r'$ of degree one adjacent to the root $r$ of $CT_h$. The vertices of $T_h$ of degree three will be called major, and all the other vertices will be called minor. We say that $r$ is the major root of $T_h$ and that $r'$ is the minor root of $T_h$. Each major vertex at distance $2k$ from $r$ has height $k$, and each minor vertex at distance $2k$ from $r'$ has height $k$.

If $t$ belongs to the unique path in $T_h$ from $r'$ to a vertex $t' \in V(T_h)$, then we say that $t'$ is a descendant of $t$ and that $t$ is an ancestor of $t'$. If, moreover, $t$ and $t'$ are adjacent, then we say that $t$ is the parent of $t'$ and that $t'$ is a child of $t$. Thus every major vertex $t$ has exactly three minor neighbors. Exactly one of those neighbors is an ancestor of $t$. The other two neighbors are descendants of $t$. We will assume that one of the two descendant neighbors is designated as the left neighbor and the other as the right neighbor. Let $t_0, t_1, t_2$ be the parent, left neighbor and right neighbor of $t$, respectively. We say that the ordered triple $(t_0, t_1, t_2)$ is the trinity at $t$. In case we want to emphasize that the trinity is at $t$, we use the notation $(t_0(t), t_1(t), t_2(t))$.

Let $\eta : T \hookrightarrow T'$. We define $sp(\eta)$, the span of $\eta$, to be the set of vertices $t \in V(T')$ that lie on the path from $\eta(t_1)$ to $\eta(t_2)$ for some vertices $t_1, t_2 \in V(T)$.

Let $s > 0$ be an integer and let $(T, X)$ be a tree-decomposition of a graph $G$. By a cascade of height $h$ and size $s$ in $(T, X)$ we mean a homeomorphic embedding $\eta : T_h \hookrightarrow T$ such that $|X_{\eta(t)}| = s$ for every minor vertex $t \in V(T_h)$ and $|X_t| \geq s$ for every $t$ in the span of $\eta$.

**Lemma 4.3.** For any positive integer $h$ and nonnegative integers $a, k$, the following holds. Let $m = (a + 2)h + a$. Let $(T, X)$ be a tree-decomposition of a graph $G$ and let $\phi : CT_m \hookrightarrow T$ be a homeomorphic embedding such that $|X_t| \geq k$ for all $t \in sp(\phi)$. If for every $t \in V(CT_m)$ at height $l \leq m - a$ there exist a descendant $t'$ of $t$ at height $l + a$ and a vertex $r \in T[\phi(t), \phi(t')]$ such that $|X_r| = k$, then there exists a cascade $\eta$ of height $h$ and size $k$ in $(T, X)$.
Proof. By hypothesis there exist a vertex \( x_0 \in V(CT_m) \) at height \( a \) and a vertex \( u_0 \in V(T) \) on the path from the image under \( \phi \) of the root of \( CT_m \) to \( \phi(x_0) \) such that \( |X_{u_0}| = k \). Let \( x \) be a child of \( x_0 \), and let \( x_1 \) and \( x_2 \) be the children of \( x \). By hypothesis there exist, for \( i = 1, 2 \), a vertex \( y_i \in V(CT_m) \) at height \( 2a + 2 \) that is a descendant of \( x_i \) and a vertex \( u_i \in T[\phi(x_i), \phi(y_i)] \) such that \( |X_{u_i}| = k \). Let \( r \) be the major root of \( T \), and let \( (t_0, t_1, t_2) \) be its trinity. We define \( \eta_i : T_1 \hookrightarrow T \) by \( \eta_i(t_i) = u_i \) for \( i = 0, 1, 2 \) and \( \eta_i(r) = \phi(x) \). Then \( \eta_i \) is a cascade of height one and size \( k \) in \((T, X)\). If \( h = 1 \), then \( \eta_1 \) is as desired, and so we may assume that \( h > 1 \).

Assume now that for some positive integer \( l < h \) we have constructed a cascade \( \eta_l : T_l \hookrightarrow T \) of height \( l \) and size \( k \) in \((T, X)\) such that for every leaf \( t_0 \) of \( T_l \) other than the minor root there exists a vertex \( x_0 \in V(CT_m) \) at height \( (a + 2)l + a \) such that the image under \( \eta_l \) of every vertex on the path in \( T_l \) from the minor root to \( t_0 \) belongs to the path in \( T \) from the image under \( \phi \) of the root of \( CT_m \) to \( \phi(x_0) \). Our objective is to extend \( \eta_l \) to a cascade \( \eta_{l+1} \) of height \( l + 1 \) and size \( k \) in \((T, X)\) with the same property. To that end let \( \eta_{l+1}(t) = \eta_l(t) \) for all \( t \in V(T_l) \), let \( t_0 \) be a leaf of \( T_l \) other than the minor root and let \( x_0 \) be as earlier in the paragraph. Let \( x \) be a child of \( x_0 \), and let \( x_1 \) and \( x_2 \) be the children of \( x \). By hypothesis there exist, for \( i = 1, 2 \), a vertex \( y_i \in V(CT_m) \) at height \( (a + 2)(l + 1) + a \) that is a descendant of \( x_i \) and a vertex \( u_i \in T[\phi(x_i), \phi(y_i)] \) such that \( |X_{u_i}| = k \). Let \( r \) be the child of \( t_0 \) in \( T_{l+1} \), and let \( (t_0, t_1, t_2) \) be its trinity. We define \( \eta_{l+1}(t_i) = u_i \) for \( i = 1, 2 \) and \( \eta_{l+1}(r) = \phi(x) \). This completes the definition of \( \eta_{l+1} \).

Now \( \eta_h \) is as desired. \( \square \)

**Lemma 4.4.** For any two positive integers \( h \) and \( w \), there exists a positive integer \( p = p(h, w) \) such that if \( G \) is a graph of path-width at least \( p \), then in any tree-decomposition of \( G \) of width less than \( w \), there exists a cascade of height \( h \).

**Proof.** Let \( a_{w+1} = 0 \), and for \( k = w, w - 1, \ldots, 0 \) let \( a_k = (a_{k+1} + 2)h + a_{k+1} \), and let \( p = w(a_0 + 1) \). We claim that \( p \) satisfies the conclusion of the lemma. To see that let \((T, X)\) be a tree-decomposition of \( G \) of width less than \( w \). Let \( k \in \{0, 1, \ldots, w + 1\} \) be the maximum integer such that there exists a homeomorphic embedding \( \phi : CT_{a_k} \hookrightarrow T \) satisfying \( |X_t| \geq k \) for all \( t \in sp(\phi) \). Such an integer exists, because \( k = 0 \) satisfies those requirements by Lemmas 4.1 and 4.2, and it satisfies \( k \leq w \), because the width of \((T, X)\) is less than \( w \). The maximality of \( k \) implies that for the integers \( h, k \) and \( a_{k+1} \) the hypothesis of Lemma 4.3 is satisfied. Thus the lemma follows from Lemma 4.3. \( \square \)

Let \((T, X)\) be a tree-decomposition of a graph \( G \), and let \( \eta : T_h \hookrightarrow T \) be a cascade of height \( h \) and size \( s \) in \((T, X)\). We say that \( \eta \) is injective if there exists \( I \subseteq V(G) \) such that \(|I| < s \) and \( X_{\eta(t)} \cap X_{\eta(t')} = I \) for every two distinct vertices \( t, t' \in V(T_h) \). We call this set \( I \) the **common intersection set** of \( \eta \).

**Lemma 4.5.** Let \( a, b, s, w \) be positive integers and let \( k \) be a nonnegative integer. Let \((T, X)\) be a tree-decomposition of a graph \( G \) of width strictly less than \( w \). Let \( h = (2(a + 2)w + 2)b \). If there is a cascade \( \eta \) of height \( h \) and size \( s + k \) in \((T, X)\) such that \(|\bigcap_{t \in V(T_h)} X_{\eta(t)}| \geq k \), then either there is a cascade \( \eta' \) of height \( a \) and size \( s + k \) in \((T, X)\) such that \(|\bigcap_{t \in V(T_a)} X_{\eta'(t)}| \geq k + 1 \) or there is an injective cascade \( \eta' \) of height \( b \), size \( s + k \) and common intersection set of size \( k \) in \((T, X)\).
Proof. We may assume that

(*) there does not exist a cascade $\eta'$ of height $a$ and size $s + k$ in $(T, X)$ such that $|\bigcap_{t \in V(T_h)} X_{\eta'(t)}| \geq k + 1$.

Let $F = \bigcap_{t \in V(T_h)} X_{\eta(t)}$. By (*), $|F| = k$. We claim the following.

Claim 4.5.1. For every vertex $t \in V(T_h)$ at height $l \leq h - a - 2$ and every $u \in X_{\eta(t)} - F$ there exists a descendant $t' \in V(T_h)$ of $t$ at height at most $l + a + 2$ such that $u \not\in X_{\eta(t')}$. To prove the claim let $u \in X_{\eta(t)} - F$. By (*) in the subtree of $T_h$ consisting of $t$ and its descendants there is a vertex $t'$ of height at most $l + a + 2$ such that $u \not\in X_{\eta(t')}$. This proves the claim.

We use the previous claim to deduce the following generalization.

Claim 4.5.2. For every vertex $t \in V(T_h)$ at height $l \leq h - (a + 2)w$ there exists a descendant $t' \in V(T)$ of $t$ at height at most $l + (a + 2)w$ such that $X_{\eta(t)} \cap X_{\eta(t')} = F$. To prove the claim let $X_{\eta(t)} \cap F = \{u_1, u_2, \ldots, u_p\}$, where $p \leq w$. By Claim 4.5.1 there exists a descendant $t_1 \in V(T)$ of $t$ at height at most $l + a + 2$ such that $u_1 \not\in X_{\eta(t')}$.

By another application of Claim 4.5.1 there exists a descendant $t_2 \in V(T)$ of $t_1$ at height at most $l + 2(a + 2)$ such that $u_2 \not\in X_{\eta(t')}$.

By continuing to argue in the same way we finally arrive at a vertex $t_p$ that is a descendant of $t$ at height at most $l + (a + 2)p$ such that $X_{\eta(t)} \cap X_{\eta(t_p)} = F$. Thus $t_p$ is as desired. This proves the claim.

Let $x_0 \in V(T_h)$ be the minor root of $T_h$. By Claim 4.5.2 and (W2) there exists a major vertex $x \in V(T)$ at height at most $(a + 2)w + 1$ such that $X_{\eta(x_0)} \cap X_{\eta(x)} = F$. Let $y_1$ and $y_2$ be the children of $x$. By Claim 4.5.2 and (W2) there exists, for $i = 1, 2$, a minor vertex $x_i \in V(T_h)$ at height at most $2(a + 2)w + 2$ that is a descendant of $y_i$ and such that $X_{\eta(x_i)} \cap X_{\eta(x)} = F$. Let $r$ be the major root of $T_1$, and let $(t_0, t_1, t_2)$ be its trinity. We define $\eta_i : T_1 \hookrightarrow T$ by $\eta_i(t_i) = \eta(x_i)$ for $i = 0, 1, 2$ and $\eta(1) = \eta(x)$. Then $\eta_i$ is an injective cascade of height one and size $s + k$ in $(T, X)$ with common intersection set $F$. If $b = 1$, then $\eta_i$ is as desired, and so we may assume that $b > 1$.

Assume now that for some positive integer $l < b$ we have constructed an injective cascade $\eta_i : T_i \hookrightarrow T$ of height $l$ and size $s + k$ with common intersection set $F$ in $(T, X)$ such that for every leaf $t_0$ of $T_i$ other than the minor root there exists a vertex $x_0 \in V(T_h)$ at height $(2(a + 2)w + 2)l$ such that the image under $\eta_i$ of every vertex on the path in $T_i$ from the minor root to $t_0$ belongs to the path in $T$ from the image under $\eta_i$ of the root of $T_i$ to $\eta_i(x_0)$. Our objective is to extend $\eta_i$ to an injective cascade $\eta_{i+1}$ of height $l + 1$, size $s + k$, and common intersection set $F$ in $(T, X)$ with the same property. To that end let $\eta_{i+1}(t) = \eta_i(t)$ for all $t \in V(T_i)$, let $t_0$ be a leaf of $T_i$ other than the minor root, and let $x_0$ be as earlier in the paragraph. By Claim 4.5.2 and (W2) there exists a descendant $x$ of $x_0$ at height at least $(2(a + 2)w + 2)l + (a + 2)w + 1$ such that $x$ is major and $X_{\eta(t_0)} \cap X_{\eta(x)} = F$. Let $y_1$ and $y_2$ be the children of $x$. By Claim 4.5.2 and (W2) there exists, for $i = 1, 2$, a minor vertex $x_i \in V(T_h)$ at height at most $(2(a + 2)w + 2)(l + 1)$ that is a descendant of $y_i$ and such that $X_{\eta(x_i)} \cap X_{\eta(x)} = F$. Let $r$ be the child of $t_0$ in $T_{i+1}$,
and let \((t_0, t_1, t_2)\) be its trinity. We define \(\eta_{i+1}(t_i) = \eta(x_i)\) for \(i = 1, 2\) and \(\eta_{i+1}(r) = \eta(x)\).

This completes the definition of \(\eta_{i+1}\).

Now \(\eta_0\) is as desired.

Lemma 4.6. For any two positive integers \(h\) and \(w\), there exists a positive integer \(p = p(h, w)\) such that if \(G\) is a graph of tree-width less than \(w\) and path-width at least \(p\), then in any tree-decomposition \((T, X)\) of \(G\) that has width less than \(w\) and satisfies \((W4)\), there is an injective cascade of height \(h\).

Proof. Let \(a_w = 0\), and for \(k = w - 1, \ldots, 0\) let \(a_k = (2(a_{k+1} + 2)w + 2)h\). Let \(p\) be the integer in Lemma 4.4 for input integers \(a_0\) and \(w\). We claim that \(p\) satisfies the conclusion of the lemma. To see that let \((T, X)\) be a tree-decomposition of \(G\) of width less than \(w\) satisfying \((W4)\). By Lemma 4.4 there exists a cascade \(\eta\) of height \(a_0\) in \((T, X)\). Let \(k \in \{0, 1, \ldots, w\}\) be the maximum integer such that there exists a cascade \(\eta' : T_{a_k} \hookrightarrow T\) satisfying \(|\bigcap_{t \in V(T_{a_k})} X_{\eta'(t)}| \geq k\). Such an integer exists, because \(k = 0\) satisfies those requirements and \(k < w\) because of \((W4)\) and because the width of \((T, X)\) is less than \(w\).

The maximality of \(k\) implies that there does not exist a cascade \(\eta'' : T_{a_{k+1}} \hookrightarrow T\) satisfying \(|\bigcap_{t \in V(T_{a_{k+1}})} X_{\eta''(t)}| \geq k + 1\). Thus the lemma follows from Lemma 4.5.

5 Ordered Cascades

The main result of this section, Theorem 5.5, states that every 2-connected graph of big path-width and bounded tree-width admits a tree-decomposition of bounded width and a cascade with linkages that are minimal.

Let \((T, X)\) be a tree-decomposition of a graph \(G\), and let \(\eta\) be an injective cascade in \((T, X)\) with common intersection set \(I\). Assume the size of \(\eta\) is \(|I| + s\). Then we say \(\eta\) is ordered if for every minor vertex \(t \in V(T_h)\) there exists a bijection \(\xi_t : \{1, 2, \ldots, s\} \rightarrow X_{\eta(t)} - I\) such that for every major vertex \(t_0\) with trinity \((t_1, t_2, t_3)\), there exist \(s\) disjoint paths \(P_1, P_2, \ldots, P_s\) in \(G\backslash I\) such that the path \(P_i\) has ends \(\xi_{t_1}(i)\) and \(\xi_{t_2}(i)\), and there exist \(s\) disjoint paths \(Q_1, Q_2, \ldots, Q_s\) in \(G\backslash I\) such that the path \(Q_i\) has ends \(\xi_{t_1}(i)\) and \(\xi_{t_3}(i)\). In that case we say that \(\eta\) is an ordered cascade with orderings \(\xi_t\). We say that the set of paths \(P_1, P_2, \ldots, P_s\) is a left \(t_0\)-linkage with respect to \(\eta\), and that the set of paths \(Q_1, Q_2, \ldots, Q_s\) is a right \(t_0\)-linkage with respect to \(\eta\).

We will need to fix a left and a right \(t_0\)-linkage for every major vertex \(t_0 \in V(T_h)\); when we do so we will indicate that by saying that \(\eta\) is an ordered cascade in \((T, X)\) with orderings \(\xi_t\) and specified linkages, and we will refer to the specified linkages as the left specified \(t_0\)-linkage and the right specified \(t_0\)-linkage. We will denote the left specified \(t_0\)-linkage by \(P_1(t_0), P_2(t_0), \ldots, P_s(t_0)\) and the right specified \(t_0\)-linkage by \(Q_1(t_0), Q_2(t_0), \ldots, Q_s(t_0)\).

We say that the specified \(t_0\)-linkages are minimal if for every set of disjoint paths \(P_1, P_2, \ldots, P_s\) in \(G\backslash I\) from \(X_{\eta(t_1)} - I\) to \(X_{\eta(t_2)} - I\) such that \(\xi_{t_1}(i)\) is an end of \(P_i\) (let the other end be \(p_i\)) and every set of disjoint paths \(Q_1, Q_2, \ldots, Q_s\) in \(G\backslash I\) from \(X_{\eta(t_1)} - I\) to \(X_{\eta(t_3)} - I\) such that \(\xi_{t_3}(i)\) is an end of \(Q_i\) (let the other end be \(q_i\)) we have

\[
\left| E \left( \bigcup (x_i P_i p_i \cup x_i Q_i q_i) \right) \right| \geq \left| E \left( \bigcup (y_i P_i(t_0) \xi_{t_2}(i) \cup y_i Q_i(t_0) \xi_{t_3}(i)) \right) \right|, \tag{1}
\]

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where the unions are taken over \( i \in \{1, 2, \ldots, s\} \), \( x_i \) is the first vertex from \( \xi_{t(i)}(i) \) that \( P_i \) departs from \( Q_i \), and \( y_i \) is the first vertex from \( \xi_{t_0}(i) \) that \( P_i(t_0) \) departs from \( Q_i(t_0) \).

**Lemma 5.1.** Let \( h \) and \( s \) be two positive integers, and let \( \eta : T_h \hookrightarrow T \) be an injective cascade of height \( h \) and size \( s \) in a linked tree-decomposition \((T, X)\) of a graph \( G \). Then the cascade \( \eta \) can be turned into an ordered cascade with specified \( t_0 \)-linkages that are minimal for every major vertex \( t_0 \in V(T_h) \).

**Proof.** Let \( s' := s - |I| \). To show that \( \eta \) can be made ordered let \( r \) be the minor root of \( T_h \), let \( \xi : \{1, 2, \ldots, s'\} \to X_{\eta(r)} - I \) be arbitrary, assume that for some integer \( l \in \{0, 1, \ldots, h - 1\} \) we have already constructed \( \xi_t : \{1, 2, \ldots, s'\} \to X_{\eta(t)} - I \) for all minor vertices \( t \in V(T_h) \) at height at most \( l \), let \( t \in V(T_h) \) be a minor vertex at height exactly \( l \), let \( t_0 \) be its child, and let \( (t, t_1, t_2) \) be the trinity at \( t_0 \). By condition (W3) there exist \( s' \) disjoint paths \( P_1, P_2, \ldots, P_{s'} \) in \( G \setminus I \) from \( X_{\eta(t)} - I \) to \( X_{\eta(t_1)} - I \) and \( s' \) disjoint paths \( Q_1, Q_2, \ldots, Q_{s'} \) in \( G \setminus I \) from \( X_{\eta(t)} - I \) to \( X_{\eta(t_2)} - I \). We may assume that \( \xi_t(i) \) is an end of \( P_i \) and \( Q_i \), and we define \( \xi_{t_1}(i) \) and \( \xi_{t_2}(i) \) to be their other ends, respectively. We may also assume that these paths satisfy the minimality condition \([1]\). It follows that \( \eta \) is an ordered cascade with orderings \( \xi_t \) and specified \( t_0 \)-linkages that are minimal for every major vertex \( t_0 \in V(T_h) \). \( \square \)

Let \( h, h' \) be integers. We say that a homeomorphic embedding \( \gamma : T_{h'} \hookrightarrow T_h \) is **monotone** if

- \( t \) is a major vertex of \( T_{h'} \) with trinity \((t_1, t_2, t_3)\), then \( \gamma(t_2) \) is the left neighbor of \( \gamma(t) \) and \( \gamma(t_3) \) is the right neighbor of \( \gamma(t) \), and

- the image under \( \gamma \) of the minor root of \( T_{h'} \) is the minor root of \( T_h \).

**Lemma 5.2.** For every two integers \( a \geq 1 \) and \( k \geq 1 \) there exists an integer \( h = h(a, k) \) such that the following holds. Color the major vertices of \( T_h \) using \( k \) colors. Then there exists a monotone homeomorphic embedding \( \eta : T_a \hookrightarrow T_h \) such that the major vertices of \( T_a \) map to major vertices of the same color in \( T_h \).

**Proof.** Let \( c \) be one of the colors. We will prove by induction on \( k \) and subject to that by induction on \( b \) that there is a function \( h = g(a, b, k) \) such that there is either a monotone homeomorphic embedding \( \eta : T_a \hookrightarrow T_h \) such that the major vertices of \( T_a \) map to major vertices of the same color in \( T_h \), or a monotone homeomorphic embedding \( \eta : T_b \hookrightarrow T_h \) such that the major vertices of \( T_b \) map to major vertices of color \( c \) in \( T_h \). In fact, we will show that \( g(a, b, 1) = a \) and \( g(a, 1, k + 1) \leq g(a, a, k) \) and \( g(a, b + 1, k + 1) \leq g(a, b, k + 1) + g(a, a, k) \).

The assertion holds for \( k = 1 \) by letting \( h = a \) and letting \( \eta \) be the identity mapping. Assume the statement is true for some \( k \geq 1 \), let the major vertices of \( T_h \) be colored using \( k + 1 \) colors, and let \( c \) be one of the colors. If \( b = 1 \), then if \( T_h \) has a major vertex colored \( c \), then the second alternative holds; otherwise at most \( k \) colors are used and the assertion follows by induction on \( k \).

We may therefore assume that the assertion holds for some integer \( b \geq 1 \) and we must prove it for \( b + 1 \). To that end we may assume that \( T_h \) has a major vertex \( t_0 \) colored \( c \)
at height at most \(g(a, a, k)\), for otherwise the assertion follows by induction on \(k\). Let the trinity at \(t_0\) be \((t_1, t_2, t_3)\). For \(i = 2, 3\) let \(R_i\) be the subtree of \(T_h\) with minor root \(t_i\). If for some \(i \in \{2, 3\}\) there exists a monotone homeomorphic embedding \(T_a \hookrightarrow R_i\) such that the major vertices of \(T_a\) map to major vertices of the same color in \(T_h\), then the statement holds. We may therefore assume that for \(i \in \{2, 3\}\) there exists a monotone homeomorphic embedding \(\eta_i : T_b^i \hookrightarrow R_i\) such that the major vertices of \(T_b^i\) map to major vertices of color \(c\), the major root of \(T_{b+1}\) is \(r_0\), the trinity at \(r_0\) is \((r_1, r_2, r_3)\) and \(T_b^i\) is the subtree of \(T_{b+1} - \{r_0, r_1\}\) with minor root \(r_i\). Let \(\eta : T_{b+1} \hookrightarrow T_h\) be defined by \(\eta(t) = \eta_i(t)\) for \(t \in V(T_b^i)\), \(\eta(r_0) = t_0\) and \(\eta(r_1)\) is defined to be the minor root of \(T_h\). Then \(\eta : T_{b+1} \hookrightarrow T_h\) is as desired. This proves the existence of the function \(g(a, b, k)\).

Now \(h(a, k) = g(a, a, k)\) is as desired. \(\square\)

Let \(G\) be a graph, let \(v \in V(G)\) and for \(i = 1, 2, 3\) let \(P_i\) be a path in \(G\) with ends \(v\) and \(v_i\) such that the paths \(P_1, P_2, P_3\) are pairwise disjoint, except for \(v\). Assume that at least two of the paths \(P_i\) have length at least one. We say that \(P_1 \cup P_2 \cup P_3\) is a tripod with center \(v\) and feet \(v_1, v_2, v_3\).

Let \((T, X)\) be a tree-decomposition of a graph \(G\), and let \(\eta : T_h \hookrightarrow T\) be an injective cascade in \((T, X)\) with common intersection set \(I\). Let \(t_0 \in V(T_h)\) be a major vertex, and let \((t_1, t_2, t_3)\) be the trinity at \(t_0\). We define the \(\eta\)-torso at \(t_0\) as the subgraph of \(G\) induced by \(\bigcup X_t - I\), where the union is taken over all \(t\) in \(V(T)\) such that the unique path in \(T\) from \(t\) to \(\eta(t_0)\) do not contain \(\eta(t_1), \eta(t_2),\) or \(\eta(t_3)\) as an internal vertex.

Let \(s > 0\) be an integer. Let \((T, X)\) be a tree-decomposition of a graph \(G\), let \(\eta : T_h \hookrightarrow T\) be an ordered cascade in \((T, X)\) of size \(|I| + s\) and with orderings \(\xi_t\), where \(I\) is the common intersection set of \(\eta\). Let \(t_0 \in V(T_h)\) be a major vertex, let \((t_1, t_2, t_3)\) be the trinity at \(t_0\), let \(G'\) be the \(\eta\)-torso at \(t_0\), and let \(i, j \in \{1, 2, \ldots, s\}\) be distinct. We say that \(t_0\) has property \(A_{ij}\) in \(\eta\) if there exist disjoint triple 

\[A_{ij} = \{\xi_t(i), \xi_t(j)\}\]

for each \(y \in \{1, 2, 3\}, \{v_{i,y}, v_{j,y}\} = \{\xi_t(i), \xi_t(j)\}\)

for each \(m \in \{i, j\}\), \(L_m\) has feet \(v_{m,1}, v_{m,2}, v_{m,3}\)

\[L_i \cap L_j = c_i L_{i^3} \cap c_j L_{j^3} \] and it is a path that does not contain \(c_i, c_j\).

We say that \(t_0\) has property \(C_{ij}\) in \(\eta\) if there exist three pairwise disjoint paths \(R_i, R_j, R_{ij}\) and a path \(R\) in \(G'\) such that the ends of \(R_i\) are \(\xi_t(i)\) and \(\xi_t(j)\), the ends of \(R_j\) are \(\xi_t(i)\) and \(\xi_t(j)\), the ends of \(R_{ij}\) are \(\xi_t(i)\) and \(\xi_t(j)\), and \(R\) is internally disjoint from \(R_i, R_j, R_{ij}\) and connects two of these three paths. We will denote these paths as \(R_i(t_0), R_j(t_0), R_{ij}(t_0), R(t_0)\) when we want to emphasize they are in the torso at the major vertex \(t_0\).

We say that the path \(P_i\) of a left or right \(t_0\)-linkage is confined if it is a subgraph of the \(\eta\)-torso at \(t_0\).

Now let \(\eta : T_h \hookrightarrow T\) be an ordered cascade in \((T, X)\) with orderings \(\xi_t\) and specified linkages. Let \(t_0 \in V(T_h)\) be a major vertex with trinity \((t_1, t_2, t_3)\), and let \(P_1, P_2, \ldots, P_s\)
be the left specified \( t_0 \)-linkage. We define \( A_{t_0} \) to be the set of integers \( i \in \{1, 2, \ldots, s\} \) such that the path \( P_i \) is confined, and we define \( B_{t_0} \) in the same way but using the right specified \( t_0 \)-linkage instead. Define \( C_{t_0} \) as the set of all triples \((i, l, m)\) such that \( i \in \{1, 2, \ldots, s\} \), the path \( P_i \) is not confined and when following \( P_i \) from \( \xi_{t_1}(i) \), it exits the \( \eta \)-torso at \( t_0 \) for the first time at \( \xi_{t_2}(l) \) and re-enters the \( \eta \)-torso at \( t_0 \) for the last time at \( \xi_{t_3}(m) \). Let \( D_{t_0} \) be defined similarly, but using the right \( t_0 \)-linkage instead. We call the sets \( A_{t_0}, B_{t_0}, C_{t_0} \) and \( D_{t_0} \) the confinement sets for \( \eta \) at \( t_0 \) with respect to the specified linkages.

Let \( A_{t_0} \) and \( B_{t_0} \) be the confinement sets for \( \eta \) at \( t_0 \). We say that \( t_0 \) has property \( C \) in \( \eta \) if \( s \) is even, \( A_{t_0} \) and \( B_{t_0} \) are disjoint and both have size \( s/2 \), and there exist disjoint paths \( R_1, R_2, \ldots, R_{3s/2} \) in \( G' \) in such a way that

- each \( R_i \) is a subpath of both the left specified \( t_0 \)-linkage and the right specified \( t_0 \)-linkage,
- for \( i \in A_{t_0} \), the path \( R_i \) has ends \( \xi_{t_1}(i) \) and \( \xi_{t_2}(i) \),
- for \( i \in B_{t_0} \), the path \( R_i \) has ends \( \xi_{t_1}(i) \) and \( \xi_{t_3}(i) \), and
- for \( i = s + 1, s + 2, \ldots, 3s/2 \) the path \( R_i \) has one end \( \xi_{t_2}(k) \) and the other and \( \xi_{t_3}(l) \) for some \( k \in B_{t_0} \) and \( l \in A_{t_0} \).

Let \((T, X)\) be a tree-decomposition of a graph \( G \), let \( \eta : T_h \hookrightarrow T \) be a cascade in \((T, X)\) and let \( \gamma : T_{h'} \hookrightarrow T_h \) be a monotone homeomorphic embedding. Then the composite mapping \( \eta' := \eta \circ \gamma : T_{h'} \hookrightarrow T \) is a cascade in \((T, X)\) of height \( h' \), and we will call it a subcascade of \( \eta \).

**Lemma 5.3.** Let \((T, X)\) be a tree-decomposition of a graph \( G \), let \( \eta : T_h \hookrightarrow T \) be an ordered cascade in \((T, X)\) with orderings \( \xi_t \), specified linkages and common intersection set \( I \), let \( \gamma : T_{h'} \hookrightarrow T_h \) be a monotone homeomorphic embedding, and let \( \eta' := \eta \circ \gamma : T_{h'} \hookrightarrow T \) be a subcascade of \( \eta \) of height \( h' \). Then for every major vertex \( t_0 \in V(T_{h'}) \)

(i) \( \eta' \) is an ordered cascade with orderings \( \xi_{\gamma(t)} \) and common intersection set \( I \),

(ii) if the vertex \( \gamma(t_0) \) has property \( A_{ij} \) (\( B_{ij}, C_{ij} \), resp.) in \( \eta \), then \( t_0 \) has property \( A_{ij} \) (\( B_{ij}, C_{ij} \), resp.) in \( \eta' \).

Furthermore, the specified linkages for \( \eta' \) may be chosen in such a way that

(iii) \((A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0}) = (A_{\gamma(t_0)}, B_{\gamma(t_0)}, C_{\gamma(t_0)}, D_{\gamma(t_0)})\),

(iv) the vertex \( t_0 \) has property \( C \) in \( \eta' \) if and only if \( \gamma(t_0) \) has property \( C \) in \( \eta \), and

(v) if the specified linkages for \( \eta \) are minimal, then the specified linkages for \( \eta' \) are minimal.
Proof. For each major vertex $t \in V(T_h)$ or $t \in V(T_h)$ we denote its trinity by $(t_1(t), t_2(t), t_3(t))$. Assume $t_0$ is a major vertex of $T_h$. Let $v_0 = \gamma(t_1(t_0)), v_1, \ldots, v_k = t_1(\gamma(t_0))$ be the minor vertices on $T_h[v_0, v_k]$. Let $U$ be the union of the left (or right) linkage from $X_{\eta(v_i)} - I$ to $X_{\eta(v_{i+1})} - I$ for all $i \in \{0, 1, \ldots, k - 1\}$ depending on whether $v_{i+1}$ is a left (or right) neighbor of its parent. Let $P$ be the left specified $\gamma(t_0)$-linkage and $Q$ be the right specified $\gamma(t_0)$-linkage. Then $U \cup P$ is a left $t_0$-linkage and $U \cup Q$ is a right $t_0$-linkage. We designate $U \cup P$ to be the left specified $t_0$-linkage and $U \cup Q$ to be the right specified $t_0$-linkage. It is easy to see that this choice satisfies the conclusion of the lemma. \qed

Let $(T, X)$ be a tree-decomposition of a graph $G$, and let $\eta$ be an ordered cascade with specified linkages in $(T, X)$ of height $h$ and size $|I| + s$, where $I$ is the common intersection set. We say that $\eta$ is regular if there exist sets $A, B \subseteq \{1, 2, \ldots, s\}$, and sets $C$ and $D$ such that the confinement sets $A_{t_0}, B_{t_0}, C_{t_0}$ and $D_{t_0}$ satisfy $A_{t_0} = A$, $B_{t_0} = B$, $C_{t_0} = C$ and $D_{t_0} = D$ for every major vertex $t_0 \in V(T_h)$.

**Lemma 5.4.** For every two positive integers $a$ and $s$ there exists a positive integer $h = h(a, s)$ such that the following holds. Let $(T, X)$ be a linked tree-decomposition of a graph $G$. If there exists an injective cascade $\eta$ of height $h$ in $(T, X)$, then there exists a regular cascade $\eta' : T_a \rightarrow T$ of height $a$ in $(T, X)$ with specified $t_0$-linkages that are minimal for every major vertex $t_0 \in V(T_a)$ such that $\eta'$ has the same size and common intersection set as $\eta$.

Proof. Let $\eta$ be an injective cascade of size $|I| + s$ and height $h$ in $(T, X)$, where we will specify $h$ in a moment. By Lemma 5.1 $\eta$ can be turned into an ordered cascade with specified $t_0$-linkages that are minimal for every major vertex $t_0 \in V(T_h)$. For every major vertex $t_0 \in V(T_h)$, the number of possible quadruples $(A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0})$ is a finite number $k = k(s)$ that depends only on $s$.

Consider each choice of $(A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0})$ as a color; then by Lemma 5.2 there exists a positive integer $h = h(a, k)$ such that there exists a monotone homeomorphic embedding $\gamma : T_a \rightarrow T_h$ such that the quadruple $(A_{\gamma(t)}, B_{\gamma(t)}, C_{\gamma(t)}, D_{\gamma(t)})$ for $\eta$ is the same for every $t \in V(T_a)$. Now, let $\eta' = \eta \circ \gamma : T_a \rightarrow T$. Then $\eta'$ is as desired by Lemma 5.3. \qed

The following is the main result of this section.

**Theorem 5.5.** For any two positive integers $a$ and $w$, there exists a positive integer $p = p(a, w)$ such that the following holds. Let $G$ be a 2-connected graph of tree-width less than $w$ and path-width at least $p$. Then $G$ has a tree-decomposition $(T, X)$ such that:

- $(T, X)$ has width less than $w$,
- $(T, X)$ satisfies (W1)–(W7), and
- for some $s$, where $2 \leq s \leq w$, there exists a regular cascade $\eta : T_a \rightarrow T$ of height $a$ and size $s$ in $(T, X)$ with specified $t_0$-linkages that are minimal for every major vertex $t_0 \in V(T_a)$. 

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Proof. Given positive integers $a$ and $w$ let $h$ be as in Lemma 5.4 and let $p = p(h, w)$ be as in Lemma 4.6. We claim that $p$ satisfies the conclusion of the theorem. To see that let $G$ be a graph of tree-width less than $w$ and path-width at least $p$. By Theorem 3.4 $G$ admits a tree-decomposition $(T, X)$ of width less than $w$ satisfying (W1)–(W7). By Lemma 4.6 there is an injective cascade of height $h$ in $(T, X)$. Let $s$ be the size of this cascade, then $s \leq w$. If $G$ is 2-connected, then $s \geq 2$. The last conclusion of the theorem follows from Lemma 5.4.

\section{Taming Linkages}

Lemma 6.6, the main result of this section, states that there are essentially only two types of linkage.

Let $s > 0$ be an integer. Let $(T, X)$ be a tree-decomposition of a graph $G$, let $\eta : T_0 \hookrightarrow T$ be an ordered cascade in $(T, X)$ of size $|I| + s$ and with orderings $\xi_t$, where $I$ is the common intersection set of $\eta$. Let $t_0 \in V(T_h)$ be a major vertex, let $(t_1, t_2, t_3)$ be the trinity at $t_0$, let $G'$ be the $\eta$-torso at $t_0$, and let $i, j \in \{1, 2, \ldots, s\}$ be distinct. We say that $t_0$ has property $AB_{ij}$ in $\eta$ if there exist disjoint paths $L_i, L_j$ and disjoint paths $R_i, R_j$ in $G'$ such that the two ends of $L_m$ are $\xi_{t_1}(m)$ and $\xi_{t_2}(m)$ for each $m \in \{i, j\}$ and the two ends of $R_m$ are $\xi_{t_1}(m)$ and $\xi_{t_3}(m)$ for each $m \in \{i, j\}$.

If $P$ is a path and $u, v \in V(P)$, then by $uPv$ we denote the subpath of $P$ with ends $u$ and $v$.

\begin{lemma}
Let $(T, X)$ be a tree-decomposition of a graph $G$. Let $\eta : T_1 \hookrightarrow T$ be an ordered cascade in $(T, X)$ with orderings $\xi_t$ of height one and size $s + |I|$, where $I$ is the common intersection set. Let $t_0$ be the major vertex in $T_1$, and let $i, j \in \{1, 2, \ldots, s\}$ be distinct. If $t_0$ has property $AB_{ij}$ in $\eta$, then $t_0$ has either property $A_{ij}$ or property $B_{ij}$ in $\eta$.
\end{lemma}

\begin{proof}
Let $(t_1, t_2, t_3)$ be the trinity at $t_0$. Let $G'$ be the $\eta$-torso at $t_0$. Since $t_0$ has property $AB_{ij}$ in $\eta$, there exist disjoint paths $L_i, L_j$ and disjoint paths $R_i, R_j$ in $G'$ such that two endpoints of $L_m$ are $\xi_{t_1}(m)$ and $\xi_{t_2}(m)$ for all $m \in \{i, j\}$, and two endpoints of $R_m$ are $\xi_{t_1}(m)$ and $\xi_{t_3}(m)$ for each $m \in \{i, j\}$.

We may choose $L_i, L_j, R_i, R_j$ such that $|E(L_i) \cup E(L_j) \cup E(R_i) \cup E(R_j)|$ is as small as possible.

Let $x_k = \xi_{t_1}(k)$ and $z_k = \xi_{t_3}(k)$ for $k \in \{i, j\}$. Starting from $z_i$, let $a$ be the first vertex where $R_i$ meets $L_i \cup L_j$, and starting from $z_j$, let $b$ be the first vertex where $R_j$ meets $L_i \cup L_j$. If $a$ and $b$ are not on the same path (one on $L_i$ and the other on $L_j$), then by considering $L_i, L_j$ and the parts of $R_i$ and $R_j$ from $z_i$ to $a$ and from $z_j$ to $b$ we see that $t_0$ has property $A_{ij}$ in $\eta$.

If $a$ and $b$ are on the same path, then we may assume they are on $L_i$. We may also assume that $a \in L_i[y_i, b]$. Then following $R_i$ from $a$ away from $z_i$, the paths $R_i$ and $L_i$ eventually split; let $c$ be the vertex where the split occurs. In other words, $c$ is such that $aL_i c \cap aR_i c$ is a path and its length is maximum. Let $d$ be the first vertex on $cR_i x_i \cap (L_i \cup L_j) = \{c\}$ when traveling on $R_i$ from $c$ to $x_i$. If $d \in V(L_i)$, then by replacing
from (W6).

Thus \( P \) from Lemma 3.1 applied to the path \( x \) is the neighbor of \( P \) and that vertex is its end. Let \( \eta \) be the one of those paths that ends in \( x \) and let \( x, y \in X_\eta(v) \). Then there exists a path of length at least two with ends \( x \) and \( y \) and every internal vertex in \( V(H) - X_\eta(v) \).

**Lemma 6.2.** Let \( (T, X) \) be a tree-decomposition satisfying (W6) of a graph \( G \) and let \( \eta : T_h \hookrightarrow T \) be an ordered cascade in \( (T, X) \) of height \( h \) and size \( |I| + s \), where \( I \) is the common intersection set. Let \( v \) be a vertex of \( T_h \) and let \( Y \) consist of \( \eta(v) \) and the vertex-sets of all components of \( T \setminus \eta(v) \) that do not contain the image under \( \eta \) of the minor root of \( T_h \). Let \( H \) be the subgraph of \( G \) induced by \( \bigcup_{t \in Y} X_t - I \). We will call \( H \) the outer graph at \( v \).

**Proof.** Let \( v_0 \) be the child of \( v \), let \( v_1 \) be a child of \( v_0 \), and let \( B \) be the component of \( T - \eta(v) \) that contains \( \eta(v_1) \). We show that \( x \) is \( B \)-tied. This is obvious if \( x \in I \), and so we may assume that \( x \notin I \). Since \( \eta \) is ordered, there exist \( s \) disjoint paths from \( X_{\eta(v)} - I \) to \( X_{\eta(v_1)} - I \) in \( G \setminus I \). It follows that each of the paths uses exactly one vertex of \( X_{\eta(v)} - I \), and that vertex is its end. Let \( P \) be the one of those paths that ends in \( x \), and let \( x' \) be the neighbor of \( x \) in \( P \). The vertex \( x' \) exists, because \( X_{\eta(v)} \cap X_{\eta(v_1)} = I \). By (W1) there exists a vertex \( t \in V(T) \) such that \( x, x' \in X_t \). Since \( P - x \) is disjoint from \( X_{\eta(v)} \), it follows from Lemma 3.1 applied to the path \( P - x \) and vertices \( t \) and \( \eta(v_1) \) of \( T \) that \( t \in V(B) \). Thus \( x \) is \( B \)-tied and the same argument shows that so is \( y \). Hence the lemma follows from (W6).

We will refer to a path as in Lemma 6.2 as a W6-path.

Let \( h, h' \) be integers. We say that a homeomorphic embedding \( \gamma : T_{h'} \hookrightarrow T_h \) is weakly monotone if for every two vertices \( t, t' \in V(T_{h'}) \)

- if \( t' \) is a descendant of \( t \) in \( T_{h'} \), then the vertex \( \gamma(t') \) is a descendant of \( \gamma(t) \) in \( T_h \)
- if \( t \) is a minor vertex of \( T_{h'} \), then the vertex \( \gamma(t) \) is minor in \( T_h \).

Let \( (T, X) \) be a tree-decomposition of a graph \( G \), let \( \eta : T_h \hookrightarrow T \) be a cascade in \( (T, X) \) and let \( \gamma : T_{h'} \hookrightarrow T_h \) be a weakly monotone homeomorphic embedding. Then the composite mapping \( \eta' := \eta \circ \gamma : T_{h'} \hookrightarrow T \) is a cascade in \( (T, X) \) of height \( h' \), and we will call it a weak subcascade of \( \eta \).

**Lemma 6.3.** Let \( s \geq 2 \) be an integer, let \( (T, X) \) be a tree-decomposition of a graph \( G \) satisfying (W6), and let \( \eta : T_5 \hookrightarrow T \) be a regular cascade in \( (T, X) \) of height five and size \( |I| + s \) with specified linkages that are minimal, where \( I \) is the common intersection set of \( \eta \). Then either there exists a weak subcascade \( \eta' : T_1 \hookrightarrow T \) of \( \eta \) of height one such that the unique major vertex of \( T_1 \) has property \( A_{ij} \) or \( B_{ij} \) in \( \eta' \) for some distinct integers \( i, j \in \{1, 2, \ldots, s\} \), or the major root of \( T_5 \) has property \( C \) in \( \eta \).
Proof. We will either construct a weakly monotone homeomorphic embedding \( \gamma : T_1 \hookrightarrow T_5 \) such that in \( \eta' = \eta \circ \gamma \) the major root of \( T_1 \) will have property \( AB_{ij} \) for some distinct \( i, j \in \{1, 2, \ldots, s\} \), or establish that the major root of \( T_5 \) has property \( C \) in \( \eta \). By Lemma 6.1 this will suffice.

Since \( \eta \) is regular, there exist sets \( A, B, C, D \) as in the definition of regular cascade. Let \( t_0 \) be the unique major vertex of \( T_1 \) and let \((t_1, t_2, t_3)\) be its trinity. Let \( u_0 \) be the major root of \( T_5 \) and let \((v_1, v_2, v_3)\) be its trinity. Let \( u_1 \) be the major vertex of \( T_5 \) of height one that is adjacent to \( v_3 \) and let \((v_3, v_4, v_5)\) be its trinity. Let us recall that for a major vertex \( u \) of \( T_5 \) we denote the paths in the specified left \( u \)-linkage by \( P_i(u) \) and the paths in the specified right \( u \)-linkage by \( Q_i(u) \). If there exist two distinct integers \( i, j \in A \cap B \), then the paths \( P_i(u_0), P_j(u_0), Q_i(u_0), Q_j(u_0) \) show that \( u_0 \) has property \( AB_{ij} \) in \( \eta \). Let \( \gamma : T_1 \hookrightarrow T_5 \) be the homeomorphic embedding that maps \( t_0, t_1, t_2, t_3 \) to \( u_0, v_1, v_2, v_3 \), respectively. Then \( \eta' = \eta \circ \gamma \) is as desired. We may therefore assume that \( |A \cap B| \leq 1 \).

For \( i \in \{1, 2, \ldots, s\} - A \) the path \( P_i(u_0) \) exits and re-enters the \( \eta \)-torso at \( u_0 \), and it does so through two distinct vertices of \( X_{\eta(v_4)} \). But \( |X_{\eta(v_3)} - I| = s \), and hence \( |A| \geq s/2 \). Similarly \( |B| \geq s/2 \). By symmetry we may assume that \( |B| \geq |A| \). It follows that \( |A| = \lceil s/2 \rceil \), and hence for \( i \in \{1, 2, \ldots, s\} - A \) and every major vertex \( w \) of \( T_5 \) the path \( P_i(w) \) exits and re-enters the \( \eta \)-torso at \( w \) exactly once. The set \( C \) includes an element of the form \((i, l, m)\), which means that the vertices \( \xi_{w_1}(i), \xi_{w_3}(l), \xi_{w_3}(m), \xi_{w_2}(i) \) appear on the path \( P_i(w) \) in the order listed. Let \( l_1 := l, m_1 := m, x_i(w) := \xi_{w_3}(l), y_i(w) := \xi_{w_3}(m), X_i(w) := \xi_{w_3}(i)P_i(w)x_i(w) \) and \( Y_i(w) := y_i(w)P_i(w)\xi_{w_3}(i) \). Thus \( X_i(w) \) and \( Y_i(w) \) are subpaths of the \( \eta \)-torso at \( w \). We distinguish two main cases.

Main case 1: \( |A \cap B| = 1 \). Let \( j \) be the unique element of \( A \cap B \). We claim that \( B - A \neq \emptyset \). To prove the claim suppose for a contradiction that \( B \subseteq A \). Thus \( |B| = 1 \), and since \( |B| \geq |A| \) we have \( |A| = 1 \), and hence \( s = 2 \). We may assume, for the duration of this paragraph, that \( A = B = \{1\} \). The paths \( P_1(u_0), X_2(u_0), Y_2(u_0) \) are pairwise disjoint, because they are subgraphs of the specified left \( u_0 \)-linkage. The path \( Q_2(u_0) \) is unconfined, and hence it has a subpath \( R \) joining \( \xi_{v_2}(1) \) and \( \xi_{v_2}(2) \) in the outer graph at \( v_2 \). It follows that \( P_1(u_0) \cup R \cup Y_2(u_0) \) and \( X_2(u_0) \) are disjoint paths from \( X_{\eta(v_1)} \) to \( X_{\eta(v_3)} \), and it follows from the minimality of the specified \( u_0 \)-linkage that they form the specified right \( u_0 \)-linkage, contrary to \( 1 \in A \). This proves the claim that \( B - A \neq \emptyset \), and so we may select an element \( i \in B - A \).

![Figure 1: First case of the construction of the path \( R \).](image-url)
Let us assume as a case that either $l_i \in A$ or $l_i \notin B$. In this case we let $\gamma$ map $t_0, t_1, t_2, t_3$ to $u_0, v_1, v_2, v_5$, respectively, and we will prove that $t_0$ has property $AB_{ij}$ in $\eta'$. To that end we need to construct two pairs of disjoint paths. The first pair is $Q_i(u_0) \cup Q_i(u_1)$ and $Q_j(u_0) \cup Q_j(u_1)$. The second pair will consist of $P_j(u_0)$ and another path from $\xi_{u_1}(i)$ to $\xi_{u_2}(i)$ which is a subgraph of a walk that we are about to construct. It will consist of $X_i(u_0) \cup Y_i(u_0)$ and a walk $R$ in the outer graph of $v_3$ with ends $x_i(u_0)$ and $y_i(u_0)$. To construct the walk $R$ we will construct paths $R_1, R_2$ and a walk $R_3$, whose union will contain the desired walk $R$. If $l_i \in A$, then we let $R_1 := P_i(u_1)$. If $l_i \notin B$, then the path $Q_i(u_1)$ is unconfined, and hence includes a subpath $R_1$ from $x_i(u_0)$ to $X_{\eta(v_4)}$ that is a subgraph of the $\eta$-torso at $u_1$. We need to distinguish two subcases depending on whether $m_i \in B$. Assume first that $m_i \notin B$ and refer to Figure 1. Then similarly as above the path $Q_{m_i}(u_1)$ is unconfined, and hence includes a subpath $R_3$ from $y_i(u_0)$ to $X_{\eta(v_4)}$ that is a subgraph of the $\eta$-torso at $u_1$, and we let $R_2$ be a W6-path in the outer graph at $v_4$ joining the ends of $R_1$ and $R_3$ in $X_{\eta(v_4)}$. This completes the subcase $m_i \notin B$, and so we may assume that $m_i \in B$. In this subcase we define $R_3 := Y_i(u_1) \cup Q_{m_i}(u_1)$ and we define $R_2$ as above. See Figure 2. This completes the case that either $l_i \in A$ or $l_i \notin B$.

Next we consider the case $l_i \in B$ and $m_i \notin A - B$. We proceed similarly as in the previous paragraph, but with these exceptions: the homeomorphic embedding $\gamma$ will map $t_3$ to $v_4$, rather than $v_5$, the first pair of disjoint paths will now be $Q_i(u_0) \cup P_i(u_1)$ and $Q_j(u_0) \cup P_j(u_1)$, and for the second pair we define $R_1 = Q_i(u_1)$, $R_3 = X_{m_i}(u_1)$ if $m_i \notin A$ and $R_3 = Q_{m_i}(u_1)$ if $m_i \in B$, and $R_2$ will be a W6-path in the outer graph of $v_5$ joining the ends of $R_1$ and $R_3$.

Therefore assume that $l_i \in B - A$ and $m_i \in A - B$ for every $i \in B - A$. Let $u_2$ be the major vertex of $T_5$ at height two whose trinity includes $v_5$ and assume its trinity is $(v_5, v_6, v_7)$. Let $u_3$ be the major vertex of $T_5$ at height three whose trinity includes $v_7$ and assume its trinity is $(v_7, v_8, v_9)$. Let $\gamma$ map $t_0, t_1, t_2, t_3$ to $u_0, v_1, v_2, v_8$, respectively. Then $t_0$ also has property $AB_{ij}$ in $\eta'$. To see that the first pair of disjoint paths is $Q_i(u_0) \cup Q_i(u_1) \cup P_i(u_3)$ and $Q_j(u_0) \cup Q_j(u_1) \cup Q_j(u_2) \cup P_j(u_3)$. The first path of the second pair is $P_j(u_0)$. Let $R_1 = Y_i(u_0) \cup Q_{m_i}(u_1) \cup P_{m_i}(u_2)$, $R_2 = P_j(u_2) \cup Q_j(u_2) \cup Q_j(u_3)$, and $R_3 = X_i(u_0) \cup Q_i(u_1) \cup X_i(u_2) \cup X_i(u_3)$. Then the second path of the second pair is a path from $\xi_{u_1}(i)$ to $\xi_{u_2}(i)$ that is a subgraph of $R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5$, where $R_4$ is a W6-path in the outer graph of $v_6$ joining the ends of $R_1$ and $R_2$, and $R_5$ is a W6-path in

![Figure 2: Second case of the construction of the path R.](image-url)
the outer graph of $v_0$ joining the ends of $R_2$ and $R_3$. See Figure 3. This completes main case 1.

**Main case 2:** $A \cap B = \emptyset$. It follows that $s$ is even and $|A| = |B| = s/2$. Assume as a case that for some integer $i \in B$ either $l_i, m_i \in A$ or $l_i, m_i \in B$. But the integers $l_i, m_i$ are pairwise distinct, and so if $l_i, m_i \in A$, then there exists $j \in B$ such that $l_j, m_j \in B$, and similarly if $l_i, m_i \in B$. We may therefore assume that $l_i, m_i \in A$ and $l_j, m_j \in B$ for some distinct $i, j \in B$. Let us recall that for every $\gamma$ map $t_0, t_1, t_2, t_3$ to $u_0, v_1, v_2, v_6$, respectively, and we will prove that $t_0$ has property $AB_{ij}$ in $\eta'$. To that end we need to construct two pairs of disjoint paths. The first pair is $Q_i(u_0) \cap Q_i(u_1) \cap P_i(u_2)$ and $Q_j(u_0) \cap Q_j(u_1) \cap P_j(u_2)$. The first path of the second pair will consist of the union of $X_i(u_0)$ with a subpath of $Q_l_i(u_1)$ from $X_{\eta(v_3)}$ to $X_{\eta(v_4)}$, and $Y_i(u_0)$ with a subpath of $Q_m_i(u_1)$ from $X_{\eta(v_3)}$ to $X_{\eta(v_4)}$, and a suitable $W_6$-path in the outer graph of $v_4$ joining their ends, and the second path will consist of the union of $X_j(u_0) \cup Q_l_j(u_1) \cup Q_l_j(u_2)$ and $Y_j(u_0) \cup Q_m_j(u_1) \cup Q_m_j(u_2)$ and a suitable $W_6$-path in the outer a graph of $v_7$ joining their ends. See Figure 3. This completes the case that for some integer $i \in B$ either $l_i, m_i \in A$ or $l_i, m_i \in B$.

We may therefore assume that for every $i \in B$ one of $l_i, m_i$ belongs to $A$ and the other belongs to $B$. Let us recall that for every $i \in B$ a subpath of $P_i(u_0)$ joins $\xi_{v_3}(i)$ to $\xi_{v_3}(m_i)$ in the outer graph at $v_3$ and is disjoint from the $\eta$-torso at $u_0$, except for its ends. Let $J$ be the union of these subpaths; then $J$ is a linkage from $\{\xi_{v_3}(i) : i \in A\}$ to $\{\xi_{v_3}(i) : i \in B\}$. For $i \in B$ the path $Q_i(u_0)$ is a subgraph of the $\eta$-torso at $u_0$. For $i \in A$ the intersection of the path $Q_i(u_0)$ with the $\eta$-torso at $u_0$ consists of two paths, one from $X_{\eta(v_3)}$ to $X_{\eta(v_2)}$, and the other from $X_{\eta(v_2)}$ to $X_{\eta(v_3)}$. Let $L$ denote the union of these subpaths over all $i \in A$. It follows that $J \cup L \cup \bigcup_{i \in B} Q_i(u_0)$ is a linkage from $X_{\eta(v_3)}$ to $X_{\eta(v_2)}$, and so by the minimality of the specified $u_0$-linkages, it is equal to the specified left $u_0$-linkage. It follows that $u_0$ has property $C$ in $\eta$. □
Lemma 6.4. Let \((T, X)\) be a tree-decomposition of a graph \(G\) satisfying (W6) and (W7). If there exists a regular cascade \(\eta : T_3 \hookrightarrow T\) with orderings \(\xi_i\) in which every major vertex has property C, then there is a weak subcascade \(\eta'\) of \(\eta\) of height one such that the major vertex in \(\eta'\) has property \(C_{ij}\) for some \(i, j\).

Proof. Let the common confinement sets for \(\eta\) be \(A, B, C, D\). For a major vertex \(w \in V(T_3)\) with trinity \((v_1, v_2, v_3)\) there are disjoint paths in the \(\eta\)-torso at \(w\) as in the definition of property C. For \(a \in A\) and \(b \in B\) let \(R_a(w)\) denote the path with ends \(\xi_{v_1}(a)\) and \(\xi_{v_2}(a)\), let \(R_b(w)\) denote the path with ends \(\xi_{v_1}(b)\) and \(\xi_{v_2}(b)\), and let \(R_{ab}(w)\) denote the path with ends \(\xi_{v_2}(b)\) and \(\xi_{v_3}(a)\).

Assume the major root of \(T_3\) is \(u_0\) and its trinity is \((v_1, v_2, v_3)\), and let \(I\) be the common intersection set of \(\eta\). Then \(\eta(v_1), \eta(v_2), \eta(v_3)\) is a triad in \(T\) with center \(\eta(u_0)\) and for all \(i \in \{1, 2, 3\}\) we have \(X_{\eta(v_i)} \cap X_{\eta(u_0)} = I = X_{\eta(v_1)} \cap X_{\eta(v_2)} \cap X_{\eta(v_3)}\), and hence the triad is not \(X\)-separable by (W7). Thus by Lemma 3.1 there is a path \(R(u_0)\) connecting two of the three sets of disjoint paths in the \(\eta\)-torso at \(u_0\). Assume without loss of generality that one end of \(R(u_0)\) is in a path \(R_i(u_0)\), where \(i \in A\). Then the other end of \(R(u_0)\) is either in a path \(R_j(u_0)\), where \(j \in B\); or in a path \(R_{aj}(u_0)\), where \(j \in B\) and \(a \in A\). In the former case we define \(a \in A\) to be such that \(R_{aj}(u_0)\) is a path in the family.

Let the major root of \(T_1\) be \(t_0\) and its trinity be \((t_1, t_2, t_3)\). Let \(\gamma(t_0) = u_0\), \(\gamma(t_1) = v_1\), \(\gamma(t_2) = v_2\). Let the major vertex that is the child of \(v_3\) be \(u_1\), and the trinity of \(u_1\) be \((v_3, v_4, v_5)\). Let \(\gamma(t_3) = v_5\). We will prove that \(t_0\) has property \(C_{ij}\) in \(\eta' = \eta \circ \gamma\). Let \(b \in B\) be such that \(R_{ab}(u_1)\) is a member of the family of the disjoint paths in the \(\eta\)-torso at \(u_1\) as in the definition of property C. By Lemma 5.2 there exists a \(W_6\)-path \(P\) in the outer graph at \(v_4\) joining \(\xi_{v_4}(a)\) and \(\xi_{v_4}(b)\). By considering the paths \(R_a(u_0), R_j(u_0) \cup R_j(u_1), R_{aj}(u_0) \cup R_a(u_1) \cup P \cup R_{ab}(u_1)\) and \(R(u_0)\) we find that \(t_0\) has property \(C_{ij}\) in \(\eta'\), as desired.

Lemma 6.5. Let \(s \geq 2\) be an integer and let \((T, X)\) be a tree-decomposition of a graph \(G\) satisfying (W6). Let \(\eta : T_3 \hookrightarrow T\) be an ordered cascade in \((T, X)\) of height three and size \(|I| + s\) with orderings \(\xi_i\) and common intersection set \(I\) such that every major vertex of \(T_3\)
has property $C_{ij}$ for some distinct $i, j \in \{1, 2, \ldots, s\}$. Then there exists a weak subcascade $\eta' : T_1 \leftrightarrow T$ of $\eta$ of height one such that the unique major vertex of $T_1$ has property $B_{ij}$ in $\eta'$.

Proof. Assume that the three major vertices at height zero and one of $T_3$ are $u_0, u_1, u_2$. Let the trinity at $u_0$ be $(v_1, v_2, v_3)$, the trinity at $u_1$ be $(v_2, v_4, v_5)$, and the trinity at $u_2$ be $(v_3, v_6, v_7)$. Assume the major vertex of $T_1$ is $t_0$, and its trinity is $(t_1, t_2, t_3)$. For a major vertex $w \in V(T_3)$ let $R_i(w), R_j(w), R_{ij}(w)$ and $R(w)$ be as in the definition of property $C_{ij}$.

We need to find a weakly monotone homeomorphic embedding $\gamma : T_1 \leftrightarrow T_3$ such that $\eta' = \eta \circ \gamma$ satisfies the requirement. Set $\gamma(t_0) = u_0$ and $\gamma(t_1) = v_1$. Our choice for $\gamma(t_2)$ will be $v_4$ or $v_5$, depending on which two of the three paths $R_i(u_1), R_j(u_1), R_{ij}(u_1)$ in the torso at $u_1$ the path $R(u_1)$ is connecting. If $R(u_1)$ is between $R_i(u_1)$ and $R_j(u_1)$, then choose either $v_4$ or $v_5$ for $\gamma(t_2)$. If $R(u_1)$ is between $R_i(u_1)$ and $R_{ij}(u_1)$, then set $\gamma(t_2) = v_4$, and if it is between $R_j(u_1)$ and $R_{ij}(u_1)$, then set $\gamma(t_2) = v_5$. Do this similarly for $\gamma(t_3)$. Then $\eta' = \eta \circ \gamma$ will satisfy the requirement. In fact, we will prove this for the case when $R(u_1)$ is between $R_i(u_1)$ and $R_{ij}(u_1)$ and $R(u_2)$ is between $R_j(u_2)$ and $R_{ij}(u_2)$. See Figure 5.

The other five cases are similar.

Figure 5: The case when $R(u_1)$ is between $R_i(u_1)$ and $R_{ij}(u_1)$ and $R(u_2)$ is between $R_j(u_2)$ and $R_{ij}(u_2)$.

In this case, our choice is $\gamma(t_0) = u_0, \gamma(t_1) = v_1, \gamma(t_2) = v_4, \gamma(t_3) = v_7$. Assume the two endpoints of $R(u_1)$ are $x$ and $y$ and the two endpoints of $R(u_2)$ are $w$ and $z$. By Lemma 6.2 there exists a W6-path $P_1$ between $\xi_v(i)$ and $\xi_v(j)$ in the outer graph at $v_5$ and a W6-path $P_2$ between $\xi_v(i)$ and $\xi_v(j)$ in the outer graph at $v_6$. Now let

$$P = yR_{ij}(u_1)\xi_v(i) \cup P_1 \cup R_j(u_1) \cup R_{ij}(u_0) \cup R_i(u_2) \cup P_2 \cup \xi_v(j)R_{ij}(u_2)w,$$

$$L_i = R_i(u_0) \cup R_i(u_1) \cup R(u_1) \cup P \cup wR_{ij}(u_2)\xi_v(i)$$

and

$$L_j = R_j(u_0) \cup R_j(u_2) \cup R(u_2) \cup P \cup yR_{ij}(u_1)\xi_v(j).$$

The tripods $L_i$ and $L_j$ show that the major vertex of $\eta' = \eta \circ \gamma : T_1 \leftrightarrow T$ has property $B_{ij}$.

\qed
Lemma 6.6. For every positive integers $h'$ and $w \geq 2$ there exists a positive integer $h = h(h', w)$ such that the following holds. Let $s$ be a positive integer such that $2 \leq s \leq w$. Let $(T, X)$ be a tree-decomposition of a graph $G$ of width less than $w$ and satisfying (W6) and (W7). Assume there exists a regular cascade $\eta : T_h \hookrightarrow T$ of size $|T| + s$ with specified linkages that are minimal, where $I$ is its common intersection set. Then there exist distinct integers $i, j \in \{1, 2, \ldots, s\}$ and a weak subcascade $\eta' : T_{h'} \hookrightarrow T$ of $\eta$ of height $h'$ such that

- every major vertex of $T_{h'}$ has property $A_{ij}$ in $\eta'$, or
- every major vertex of $T_{h'}$ has property $B_{ij}$ in $\eta'$

Proof. Let $h(a, k)$ be the function of Lemma 5.2, let $a_3 = 3h'$, $a_2 = h(a_3, 2(\binom{n}{2}))$, $a_1 = 5a_2$ and $h = h(a_1, 2)$. Consider having property $C$ or not having property $C$ as colors, then by Lemma 5.2 there exists a monotone homeomorphic embedding $\gamma : T_{a_1} \hookrightarrow T_h$ such that either $\gamma(t)$ has property $C$ in $\eta$ for every major vertex $t \in V(T_{a_1})$ or $\gamma(t)$ does not have property $C$ in $\eta$ for every major vertex $t \in V(T_{a_1})$. By Lemma 5.3, $\eta_1 = \eta \circ \gamma : T_{a_1} \hookrightarrow T$ is still a regular cascade with specified linkages that are minimal. Also, either $t$ has property $C$ in $\eta_1$ for every major vertex $t \in V(T_{a_1})$ or $t$ does not have property $C$ in $\eta_1$ for every major vertex $t \in V(T_{a_1})$.

If $t$ has property $C$ in $\eta_1$ for every major vertex $t \in V(T_{a_1})$, then by Lemma 6.4 there exists a weak subcascade $\eta_2$ of $\eta_1$ of height $a_2$ such that every major vertex of $T_{a_2}$ has property $C_{ij}$ in $\eta_2$ for some distinct $i, j \in \{1, 2, \ldots, s\}$. Consider each choice of pair $i, j$ as a color; then by Lemma 5.2 there exists a monotone homeomorphic embedding $\gamma_1 : T_{a_3} \hookrightarrow T_{a_2}$ such that for some distinct $i, j \in \{1, 2, \ldots, s\}$, $\gamma_1(t)$ has property $C_{ij}$ in $\eta_2$ for every major vertex $t \in V(T_{a_3})$. Let $\eta_3 = \eta_2 \circ \gamma_1$. Then by Lemma 5.3 this implies $t$ has property $C_{ij}$ in $\eta_3$ for every major vertex $t \in V(T_{a_3})$. Then by Lemma 6.5 there exists a weak subcascade $\eta_4 : h' \hookrightarrow a_3$ of $\eta_3$ such that every major vertex of $T_{h'}$ has property $B_{ij}$ in $\eta_4$. Hence $\eta_4$ is as desired.

If $t$ does not have property $C$ in $\eta_1$ for every major vertex $t \in V(T_{a_1})$, then by Lemma 6.3 there exists a weak subcascade $\eta_2$ of $\eta_1$ of height $a_2$ such that every major vertex of $T_{a_2}$ has property $A_{ij}$ or $B_{ij}$ for some distinct $i, j \in \{1, 2, \ldots, s\}$. Consider each property $A_{ij}$ or $B_{ij}$ as a color; then by Lemma 5.2 there exists a monotone homeomorphic embedding $\gamma_1 : T_{h'} \hookrightarrow T_{a_2}$ such that for some distinct $i, j \in \{1, 2, \ldots, s\}$, either $\gamma_1(t)$ has property $A_{ij}$ in $\eta_2$ for every major vertex $t \in V(T_{h'})$ or $\gamma_1(t)$ has property $B_{ij}$ in $\eta_2$ for every major vertex $t \in V(T_{h'})$. Let $\eta_3 = \eta_2 \circ \gamma_1$. Then $t$ has property $A_{ij}$ in $\eta_3$ for every major vertex $t \in V(T_{h'})$ or $t$ has property $B_{ij}$ in $\eta_3$ for every major vertex $t \in V(T_{h'})$ by Lemma 5.3. Hence $\eta_3$ is as desired.

7 Proof of Theorem 1.3

By Lemmas 2.2 and 2.4 Theorem 1.3 is equivalent to the following theorem.

Theorem 7.1. For any positive integer $k$, there exists a positive integer $p = p(k)$ such that for every 2-connected graph $G$, if $G$ has path-width at least $p$, then $G$ has a minor isomorphic to $P_k$ or $Q_k$. 

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We need the following lemma.

**Lemma 7.2.** Let \((T, X)\) be a tree-decomposition of a graph \(G\), let \(\eta : T_h \hookrightarrow T\) be an ordered cascade in \((T, X)\) with orderings \(\xi_i\) of height \(h\) and size \(s + 1\), where \(I\) is the common intersection set, and let \(i, j \in \{1, 2, \ldots, s\}\) be distinct and such that every major vertex of \(T_h\) has property \(B_{ij}\) in \(\eta\). Let \(t\) be the minor root of \(T_h\), and let \(w_1w_2\) be the base edge of \(Q_h\). Then \(G\) has a minor isomorphic to \(Q_h - w_1w_2\) in such a way that \(\xi_t(i)\) belongs to the node of \(w_1\) and \(\xi_t(j)\) belongs to the node of \(w_2\).

**Proof.** We proceed by induction on \(h\). Let \(t_0\) be the major root of \(T_h\), let \((t_1, t_2, t_3)\) be its trinity, and let \(L_i\) and \(L_j\) be the tripods in the \(\eta\)-torso at \(t_0\) as in the definition of property \(B_{ij}\). The graph \(L_i \cup L_j\) contains a path \(P\) joining \(\xi_t(i)\) to \(\xi_t(j)\), which shows that the lemma holds for \(h = 1\).

We may therefore assume that \(h > 1\) and that the lemma holds for \(h - 1\). For \(k \in \{2, 3\}\) let \(R_k\) be the subtree of \(T_h\) rooted at \(t_k\), let \(\eta_k\) be the restriction of \(\eta\) to \(R_k\), and let \(G_k\) be the subgraph of \(G\) induced by \(\bigcup \{X_r : r \in sp(\eta_k)\}\). By the induction hypothesis applied to \(\eta_k\) and \(G_k\), the graph \(G_k\) has a minor isomorphic to \(Q_{h-1} - u_1u_2\) in such a way \(\xi_{t_k}(i)\) belongs to the node of \(u_1\) and \(\xi_{t_k}(j)\) belongs to the node of \(u_2\), where \(u_1u_2\) is the base edge of \(Q_{h-1}\). By using these two minors, the path \(P\) and the rest of the tripods \(L_i\) and \(L_j\) we find that \(G\) has the desired minor.

We deduce Theorem 7.1 from the following lemma.

**Lemma 7.3.** Let \(k\) and \(w\) be positive integers. There exists a number \(p = p(k, w)\) such that for every 2-connected graph \(G\), if \(G\) has tree-width less than \(w\) and path-width at least \(p\), then \(G\) has a minor isomorphic to \(P_k\) or \(Q_k\).

**Proof.** Let \(h' = 2k + 1\), let \(h = h(h', w)\) be the number as in Lemma 6.6 and let \(p\) be as in Theorem 5.5 applied to \(a = h\) and \(w\). We claim that \(p\) satisfies the conclusion of the lemma. By Theorem 5.5 there exists a tree-decomposition \((T, X)\) of \(G\) such that:

- \((T, X)\) has width less than \(w\),
- \((T, X)\) satisfies (W1)-(W7), and
- for some \(s\), where \(2 \leq s \leq w\), there exists a regular cascade \(\eta : T_h \hookrightarrow T\) of height \(h\) and size \(s\) in \((T, X)\) with specified \(t_0\)-linkages that are minimal for every major vertex \(t_0 \in V(T_h)\).

Let \(I\) be the common intersection set of \(\eta\), let \(\xi_i\) be the orderings, and let \(s_1 = s - |I|\). Then \(s_1 \geq 1\) by the definition of injective cascade.

Assume first that \(s_1 = 1\). Since \(s \geq 2\), it follows that \(I \neq \emptyset\). Let \(x \in I\). Let \(R\) be the union of the left and right specified \(t\)-linkage with respect to \(\eta\), over all major vertices \(t \in V(T_h)\) at height at most \(h - 2\). The minimality of the specified linkages implies that \(R\) has a subtree isomorphic to a subdivision of \(CT_{[(h-1)/2]}\). Let \(t\) be a minor vertex of \(T_h\) at height \(h - 1\). By Lemma 6.2 there exists a W6-path with ends \(\xi_t(1)\) and \(x\) and every
internal vertex in the outer graph at \( t \). The union of \( R \) and these W6-paths shows that \( G \) has a \( P_k \) minor, as desired.

We may therefore assume that \( s_1 \geq 2 \). By Lemma 6.6 there exist distinct integers \( i, j \in \{1, 2, \ldots, s\} \) and a subcascade \( \eta': T_{h'} \hookrightarrow T \) of \( \eta \) of height \( h' \) such that

- every major vertex of \( T_{h'} \) has property \( A_{ij} \) in \( \eta' \), or
- every major vertex of \( T_{h'} \) has property \( B_{ij} \) in \( \eta' \)

Assume next that every major vertex of \( T_{h'} \) has property \( A_{ij} \) in \( \eta' \), and let \( R \) be the union of the corresponding tripods, over all major vertices \( t \in V(T_{h'}) \) at height at most \( h' - 2 \). It follows that \( R \) is the union of two disjoint trees, each containing a subtree isomorphic to \( CT_{(h'-1)/2} \). Let \( t \) be a minor vertex of \( T_{h'} \) at height \( h' - 1 \). By Lemma 6.2 there exists a W6-path with ends \( \xi_t(i) \) and \( \xi_t(j) \) in the outer graph at \( t \). By contracting one of the trees comprising \( R \) and by considering these W6-paths we deduce that \( G \) has a \( P_k \) minor, as desired.

We may therefore assume that every major vertex of \( T_{h'} \) has property \( B_{ij} \) in \( \eta' \). It follows from Lemma 7.2 that \( G \) has a minor isomorphic to \( Q_{h'-1} \), as desired.

**Proof of Theorem 7.1.** Let a positive integer \( k \) be given. By Theorem 1.1 there exists an integer \( w \) such that every graph of tree-width at least \( w \) has a minor isomorphic to \( P_k \). Let \( p = p(k, w) \) be as in Lemma 7.3. We claim that \( p \) satisfies the conclusion of the theorem. Indeed, let \( G \) be a 2-connected graph of path-width at least \( p \). By Theorem 1.1 if \( G \) has tree-width at least \( w \), then \( G \) has a minor isomorphic to \( P_k \), as desired. We may therefore assume that the tree-width of \( G \) is less than \( w \). By Lemma 7.3 \( G \) has a minor isomorphic to \( P_k \) or \( Q_k \), as desired.

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