CLIQUE-SUMS, TREE-DECOMPOSITIONS AND
COMPACTNESS

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We develop a technique for extending excluded minor theorems to infinite graphs, and in
particular we answer a question of Neil Robertson.

1. Introduction

The aim of this paper is to extend certain results of finite graph theory to
infinite graphs and to show a limitation of this.

Recall that a graph $G$ is a minor of a graph $H$ if $G$ can be obtained from a
subgraph of $H$ by contraction of edges. There are several so-called excluded
minor theorems in finite graph theory, i.e. statements describing finite graphs
without minors isomorphic to members of a given list of finite graphs. The
celebrated Kuratowski’s theorem [4] is also of this form: Finite graphs without
minors isomorphic to $K_{3,3}$ and $K_5$ are planar.

Note that this theorem is exact in the sense that it in fact characterizes the class
of graphs with no $K_{3,3}$ and $K_5$ minor. Some other excluded minor theorems are
listed in Table 1 below; the first six of them are exact. (For the definition of
$k$-sums see Section 2.4.)

These statements have a common feature: The describing structure involves
clique-sums of graphs from a certain basic class $I$. We develop a general technical
tool to treat this situation, the concept of a $k$-decomposition over a class $I$ (for
definition see 2.3). This concept covers the notion of tree-width introduced by
Robertson and Seymour [6] in the sense that graphs of tree-width $\leq k$ are exactly
those admitting a $k$-decomposition over

$$I_k = \{(V, E) \mid |V| \leq k + 1\}.$$

The first idea of such a kind was probably due to Wagner [13].

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Table 1.

<table>
<thead>
<tr>
<th>Excluded minor(s)</th>
<th>Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_3$</td>
<td>1-sums of graphs with $\leq 2$ vertices (graphs of tree-width $= 1$)</td>
</tr>
<tr>
<td>$K_4$</td>
<td>2-sums of graphs with $\leq 3$ vertices (graphs of tree-width $= 2$)</td>
</tr>
<tr>
<td>$K_5$</td>
<td>3-sums of planar graphs and the four rung Möbius ladder [12]</td>
</tr>
<tr>
<td>$K_{3,3}$ minus one edge</td>
<td>2-sums of planar graphs and $K_5$ [13]</td>
</tr>
<tr>
<td>All finite 4-connected graphs</td>
<td>2-sums of the five-sided prism, the four rung Möbius ladder and graphs with $\leq 4$ vertices (see [11])</td>
</tr>
<tr>
<td>Arbitrary $H$ finite planar</td>
<td>$k$-sums of graphs with $\leq k + 1$ vertices (graphs of tree-width $\leq k$) for some $k$ depending on $H$ [6]</td>
</tr>
<tr>
<td>Arbitrary $H$ finite</td>
<td>$k$-sums of graphs which are $(p, q, h)$-representable over some surface $\Sigma$ in which $H$ cannot be embedded (for some $k, p, q, h$ depending on $H$) [7]</td>
</tr>
</tbody>
</table>

Let us briefly recall the definition of tree-width and the basic results. We start with a

**Theorem 1.1.** Let $G$ be a finite graph and let $k > 0$. Then the following conditions are equivalent:

(i) $k$ is the least integer such that $G$ can be constructed by repeated clique-sums, starting from graphs in $\Gamma_k$.

(ii) $k$ is the least integer such that $G$ admits a $k$-decomposition over $\Gamma_k$.

(iii) $k$ is the least integer such that $G$ is a subgraph of a chordal graph not containing $K_{k+2}$.

**Definition 1.2.** The number $k$ from the above theorem is called the **tree-width** of $G$.

It can be shown, for instance, that the graphs of tree-width $\leq 1$ are precisely the forests, and the graphs of tree-width $\leq 2$ are the series-parallel graphs. The complete graph $K_n$ has tree-width $n - 1$ and the $n \times n$ grid (adjacency graph of the $n \times n$ chessboard) has tree-width $n$. For tree-width the following compactness theorem holds.

**Theorem 1.3** [9]. If every finite subgraph of $G$ has tree-width $\leq k$, then $G$ has tree-width $\leq k$; in other words, $G$ admits a $k$-decomposition over $\Gamma_k$ iff each finite subgraph of $G$ does.

A simpler proof of 1.3 can be found in [10] or [3]. In this paper we prove a generalization of Theorems 1.1, 1.3 with $\Gamma_k$ replaced by an arbitrary class $\Gamma'$.
satisfying certain natural assumptions (see 2.2, 3.5, 3.9). As an immediate consequence we obtain infinite versions of the theorems of Table 1.

Our results apply only to the case of $k$-sums for fixed $k$. In Section 4 we present a counterexample showing that the technique cannot be extended to the case of unbounded clique-sums, that is, Proposition 3.4 does not hold for $|V(K_1) \cap V(K_2)| \leq k$ replaced by $|V(K_1) \cap V(K_2)| < \omega$. The counterexample presented also shows that 1.1 fails for $k = \omega$. More exactly, we construct a graph $G$ such that $k = \omega$ satisfies 1.1(iii) while the $k$ satisfying 1.1(ii) is arbitrarily large.

2. Definitions

2.1. A graph may be infinite, and may have multiple edges but may not have loops. A clique in a graph is a complete subgraph. If $M$ is a set, then $K(M)$ denotes the complete graph on $M$. A graph is chordal if every cycle with at least four vertices has a chord.

If $G$ is a graph and $A \subseteq V(G)$, then $G \upharpoonright A$ means the graph induced by $A$ in $G$. A cut in $G$ is a set $C \subseteq V(G)$ such that $G$ becomes disconnected after deleting vertices from $C$. A $k$-cut in $G$ is a cut $C$ such that $|C| \leq k$ and $G \upharpoonright C$ is complete. If $G_a = (V_a, E_a)$ are graphs, then $\bigcap G_a$, $\bigcup G_a$ are the graphs $(\bigcap V_a, \bigcap E_a)$ and $(\bigcup V_a, \bigcup E_a)$, respectively.

2.2. Let $\Gamma$ be a class of graphs with the following properties.

(Γ1) If $G$ is a graph and $e_1, e_2 \in E(G)$ are two distinct edges with the same endpoints and if $G \upharpoonright e_i \in \Gamma$ then $G \in \Gamma$.

(Γ2) If $G \in \Gamma$ and $H$ is a $k$-subsimplex of $G$, then $H \in \Gamma$.

(Γ3) If $G$ is such that for every finite subgraph $H$ of $G$ there exists a finite subgraph $H' \in \Gamma$ of $G$ containing $H$, then $G \in \Gamma$.

2.3. Let $G$ be a graph. A tree-decomposition of $G$ is a pair $(T, X)$, where $T$ is a tree and $X = (X^t : t \in V(T))$ is such that

(T1) $\bigcup_{t \in V(T)} X^t = V(G)$,

(T2) every edge $e$ of $G$ has both endpoints in some $X^t$,

(T3) if $t, t', t'' \in V(T)$ and $t'$ is on the path between $t'$ and $t''$, then $X^t \cap X^{t'} \subseteq X^{t''}$.

We say that a tree-decomposition $(T, X)$ is a $k$-decomposition of $G$ over $\Gamma$, if

(T4) $|X^t \cap X^{t'}| \leq k$ for every $(t, t') \in E(T)$,

(T5) $G^* \upharpoonright X^t \in \Gamma$ for every $t \in V(T)$,

where $G^*$ is the graph obtained from $G$ by joining every edge which has both its endpoints in some $X^t \cap X^{t'}$ for $(t, t') \in E(T)$.

2.4. Let $G_1$, $G_2$ be two graphs such that $G_2 \upharpoonright (V(G_1) \cap V(G_2))$ is edge-less and let $H_t$ be the graph obtained from $G_t$ by adding new edges joining every pair of vertices in $V(G_1) \cap V(G_2)$. We say $G_1 \cup G_2$ is the clique-sum of $H_t$ and $H_2$ and if
\(|V(G_1) \cap V(G_2)| \leq k\) we also say it is the \(k\)-sum of \(H_1\) and \(H_2\). We say that a graph \(G\) is \(k\)-summable over \(\Gamma\) if there exists a transfinite sequence \(\{G_\alpha\}_{\alpha < \lambda}\) of graphs such that \(G_0 \in \Gamma, G_1 = G, G_{\alpha + 1}\) is a \(k\)-sum of \(G_\alpha\) and some graph from \(\Gamma\) and for \(\alpha\) a limit ordinal
\[(1) \quad V(G_\alpha) = \bigcup_{\beta < \alpha} V(G_\beta)\]
\[(2) \quad E(G_\alpha) = \bigcup_{\beta < \alpha} \cap_{\gamma > \beta} E(G_\gamma).\]
The least \(\lambda\) with this property is called the \textit{rank} of \(G\). Relations (1) and (2) will be abbreviated \(G_\alpha = \liminf_{\beta < \alpha} G_\beta\).

\(2.5\). A graph \(G\) is called a \(k\)-simplex if it does not contain a \(k\)-cut. If \(G\) is a graph, then each induced subgraph \(H\) of \(G\), which is a \(k\)-simplex, is called a \(k\)-subsimplex. A graph \(G\) is a \(k\)-complex over \(\Gamma\) if every finite \(k\)-subsimplex belongs to \(\Gamma\).

\(3.\) The main results

\textbf{Lemma 3.1.} Let \(G\) be a graph and \(S_1, S_2\) two distinct maximal \(k\)-subsimplices. Then \(S_1 \cap S_2\) is a clique of size at most \(k\).

\textbf{Proof.} Suppose not. Let \(K\) be a clique in \(G\) of size \(\leq k\). Then \((S_1 \cap S_2) \setminus V(K) \neq \emptyset\) and both \(S_1 \setminus V(K), S_2 \setminus V(K)\) are connected. Hence \((S_1 \cup S_2) \setminus V(K)\) is connected and thus \(S_1 \cup S_2\) is a \(k\)-simplex, contrary to the maximality of \(S_1\) and \(S_2\). \(\square\)

\textbf{Proposition 3.2.} Let \(G\) be a \(k\)-simplex and \(G_0\) a finite subgraph. Then there exists a finite \(k\)-subsimplex of \(G\), which contains \(G_0\).

\textbf{Proof.} Assume we have already constructed a finite induced subgraph \(G_n\) of \(G\) such that \(G_n\) contains \(G_0\) and each of its \(k\)-cuts has size at least \(n\). Let \(C_1, \ldots, C_s\) be its \(k\)-cuts. Since \(G\) is a \(k\)-simplex, there is an induced finite subgraph of \(G\), which contains \(G_n\) and is such that for no \(i\), \(C_i\) is a cut of it. Let \(G_{n+1}\) denote a minimal such subgraph. If \(C\) is a \(k\)-cut of \(G_{n+1}\), it follows by minimality that \(C \cap V(G_n)\) is a \(k\)-cut of \(G_n\) and thus \(C \supseteq C_i\) for some \(i = 1, \ldots, s\). Hence every \(k\)-cut of \(G_{n+1}\) is of size at least \(n + 1\). Now \(G_{k+1}\) is the desired graph. \(\square\)

\textbf{Corollary 3.3.} If \(G\) is a \(k\)-complex over \(\Gamma\), then every \(k\)-subsimplex belongs to \(\Gamma\).

\textbf{Proof.} Let \(S\) be a \(k\)-subsimplex of \(G\) and \(S'\) a finite subgraph of \(S\). By Proposition 3.2 there is a finite \(k\)-subsimplex \(S''\) of \(G\) which contains \(S'\). Then \(S'' \in \Gamma\), and hence \(S \in \Gamma\) by (T3). \(\square\)
Proposition 3.4. Let $G$ be a chordal graph such that $|V(K_1) \cap V(K_2)| \leq k$ for any two distinct maximal cliques $K_1, K_2$ of $G$. Then $G$ admits a tree-decomposition $(T, X)$ such that $G \upharpoonright X^t$ is a maximal clique in $G$ for any $t \in V(T)$.

Proof. We just sketch the proof, and for the remaining details we refer to [3]. Consider the complete edge-valued graph $\Delta(G)$ on the set of all maximal cliques of $G$ with the value of edge $\{K_1, K_2\}$ equal to $|V(K_1) \cap V(K_2)|$. Construct a "maximal" skeleton $T$ of $\Delta(G)$ by adding at each step an edge of the maximal possible value, not to obtain a cycle. Put $X = \{t \mid t \in V(T)\}$. Then $(T, X)$ is the desired tree-decomposition. □

Theorem 3.5. The following are equivalent for every integer $k \geq 2$:

(i) $G$ is $k$-summable over $\Gamma$.

(ii) $G$ is a subgraph of a $k$-complex over $\Gamma$.

(iii) $G$ admits a $k$-decomposition over $\Gamma$.

Proof. (i)→(ii). We proceed by induction on rank of $G$. First we show that if $G_{\alpha+1}$ is a $k$-sum of $G_\alpha$ and $G \in \Gamma$ and $H_\alpha$, $H$ are $k$-complexes over $\Gamma$ containing $G_\alpha$, $G$, respectively, then there exists a $k$-complex $H_{\alpha+1}$ over $\Gamma$ such that $H_{\alpha+1}$ contains $G_{\alpha+1}$ and $H_{\alpha+1} \upharpoonright V(G_\alpha) = H_\alpha$. Indeed, it is sufficient to put $H_{\alpha+1}$ to be the graph obtained from $H_\alpha \cup H$ by deleting those edges from $H$ which join vertices in $V(H_\alpha) \cap V(H)$. Now if $\alpha$ is a limit ordinal and $G_\alpha = \operatorname{liminf}_{\beta < \alpha} G_\beta$, let $H_\beta$ be $k$-complexes over $\Gamma$ defined as above and put $H_\alpha = \bigcup_{\beta < \alpha} H_\beta$. It is easily seen that $H_\alpha$ is a $k$-complex over $\Gamma$ which contains $G_\alpha$.

(ii)→(iii). Let $H$ be a $k$-complex over $\Gamma$ which contains $G$. We put

$$H^* := \bigcup \{K(V(S)) \mid S \text{ is a } k\text{-subsimplex of } H\}.$$ 

Since every chordless cycle of $H$ of length at least four is a $k$-subsimplex of $H$, we see that $H^*$ is chordal. We claim

(*) If $K^*$ is a maximal clique in $H^*$, then $K = H \upharpoonright V(K^*)$ is a $k$-subsimplex of $H$.

Choose a minimal system $\{S_\alpha\}$ of $k$-subsimplexes of $H$ with the property that $K^*$ is a subgraph of $K \cup \bigcup_\alpha K(V(S_\alpha))$. We shall show that $S := \bigcup_\alpha S_\alpha \cup K$ is a $k$-subsimplex of $H$. Let $V$ be a clique in $S$ of size $\leq k$. We shall show that $S \setminus V$ is connected. Let first $u, v \in K \setminus V$. If $u, v$ are not joined by an edge in $S$, then, since they are joined in $K^*$, there exists an $\alpha$ such that $u, v \in V(S_\alpha)$. Hence they are joined by a path in $S_\alpha \setminus V$.

Now let $u \in V(S_\beta) \setminus V$. Since $\{S_\alpha\}$ is assumed minimal, it follows that $K \upharpoonright V(S_\beta)$ is not a clique. Hence $(V(S_\beta) \cap V(K)) \setminus V \neq \emptyset$. Let $v \in (V(S_\beta) \cap V(K)) \setminus V$; now $u, v$ are joined by a path in $S_\beta \setminus V$. This shows that $S \setminus V$ is connected, and hence $S$ is a $k$-simplex. Now the maximality of $K^*$ implies that $K = S$, proving (*).

If $K_1^*, K_2^*$ are distinct maximal cliques in $H^*$, then $K_1 = H \upharpoonright V(K_1^*)$, $K_2 = H \upharpoonright
$H \upharpoonright V(K^*_2)$ are maximal $k$-subsimplices of $H$ by $(\ast)$, and hence $|V(K^*_2) \cap V(K^*_2)| = |V(K_1) \cap V(K_2)| \leq k$ by Lemma 3.1. Thus, the assumptions of Proposition 3.4 are satisfied.

Let $(T, X)$ be the tree-decomposition of $H^*$ from Proposition 3.4. If $(t, t') \in E(T)$, then $H \upharpoonright (X' \cap X'')$ is the intersection of two maximal $k$-subsimplices of $H$ by $(\ast)$. Hence by Lemma 3.1 $|X' \cap X''| \leq k$ and $H \upharpoonright (X' \cap X'')$ is a clique. From this and Corollary 3.3 it follows that if $G^*$ is as in $(T5)$ then $G^* \upharpoonright X'$ is a subgraph of $H \upharpoonright X' \in \Gamma$. Hence $(T, X)$ can be converted to a $k$-decomposition over $\Gamma$ of $G$ by adding a leaf $r(t, e)$ with $X'(t, e) = e$ for every $e \in E(H \upharpoonright X') \setminus E(G^*)$.

(iii)$\Rightarrow$(i). Let $(T, X)$ be a $k$-decomposition over $\Gamma$ of $G$. There are subtrees $(T_\alpha)_{\alpha < \kappa}$ of $T$ such that $T_0 = (\{t_0\}, \emptyset)$, the one-point tree, $T_1 = T$, $T_{\alpha + 1}$ is obtained from $T_\alpha$ by joining a vertex $t_{\alpha + 1}$ of degree one and $T_\alpha = \bigcup_{\beta < \alpha} T_\beta$ for $\alpha$ a limit ordinal. Let $G^*$ be the graph obtained from $G$ by joining all edges

$$e^G_{(t, t')} \mid (t, t') \in E(T), u, v \in X' \cap X'', u \neq v,$$

where $e^G_{(t, t')} \upharpoonright (t, t') \in E(T)$, $u, v \in X' \cap X''$, $u \neq v)$, where $e^G_{(t, t')} \upharpoonright (t, t') \in E(T)$, $u, v \in X' \cap X''$, $u \neq v$.

It is easily seen that $G_\alpha \in \Gamma$, $G_\alpha = G$, $G_{\alpha + 1}$ is the $k$-sum of $G_\alpha$ and $G_\alpha \in \Gamma$ and that for a limit ordinal $\alpha$ $G_\alpha = \liminf_{\beta < \alpha} G_\beta$. Hence $G$ is $k$-summable over $\Gamma$. □

**Corollary 3.6.** Let $\Gamma$ satisfy the following condition:

Let $G$ be a graph and $e \in E(G)$. If $G \setminus e \in \Gamma$ then $G \in \Gamma$. Then (i), (ii), (iii) of the preceding theorem are equivalent to:

(iv) $G$ is a subgraph of a chordal graph $H$ such that every clique of $H$ belongs to $\Gamma$ and any two distinct maximal cliques of $H$ have at most $k$ vertices in common.

**Remark 3.7.** The method of proof (ii)$\Rightarrow$(iii) in Theorem 3.5 gives another existence theorem for prime graph decompositions of infinite graphs. To make this statement precise let us call a graph $G$ prime if there is no complete subgraph of $G$ which is a cut. A prime graph decomposition is a tree-decomposition $(T, X)$ of $G$ such that $G \upharpoonright X'$ is prime for any $t \in V(T)$ and $G \upharpoonright (X' \cap X'')$ is complete for any $(t, t') \in E(T)$. An infinite graph need not have a prime decomposition, but every graph without infinite cliques does (see [2]). Thus, the following corollary is another theorem of this kind.

**Corollary 3.8.** Let $H$ be a graph and $k$ an integer such that

$$|V(G_1) \cap V(G_2)| \leq k$$

for any two distinct maximal prime induced subgraphs $G_1$, $G_2$ of $H$. Then $H$ admits a prime graph decomposition.
Proof. We use the proof of (ii)→(iii) from 3.5 with "k-simplex" replaced by "prime" and "k-subsimplex" replaced by "induced prime subgraph", we also drop the restriction on the size of V and instead of Lemma 3.1 we use our assumption. Then the tree-decomposition (T, X) of H thus produced is as desired, because for t ∈ V(T), H □ X′ is maximal prime by (*) and for {t, t′} ∈ E(T), H □ (X′ ∩ X′′) is complete, being the intersection of two maximal prime graphs. □

Theorem 3.9. Let G be such that every finite subgraph of G is a subgraph of a k-complex over Γ. Then G is a subgraph of a k-complex over Γ.

Proof. We shall prove the theorem for graphs without multiple edges for simplicity. The general case is then trivial.

For each α ∈ (V(G)) we introduce a logical variable Aα. Consider the system of formulas

(1) Aα
(2) \( \bigvee_{\alpha \in E} \neg A_{\alpha} \vee \bigwedge_{\alpha \in (\ell)} A_{\alpha} \) for every finite graph (V, E) which is a k-simplex not belonging to Γ such that \( V \subseteq V(G), E \subseteq (\ell) \).

A valuation v: \( \{ A_{\alpha} \mid \alpha \in (V(G)) \} \rightarrow 2 \) satisfies (1), (2) iff the graph with vertex set V(G) and edges \( \{ \alpha \mid A_{\alpha}[v] = 1 \} \) is a k-complex over Γ which contains G. By assumption, each finite subsystem of (1), (2) can be fulfilled. Use Logical Compactness. □

3.10. Now we are able to extend the results of Table 1 to infinite graphs. The last statement answers a question of Robertson [8] as to whether the excluded minor theorem from [7] can be extended to the infinite case. As in the finite case, the first six results are exact.

Table 2.

<table>
<thead>
<tr>
<th>Excluded minor(s)</th>
<th>Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>K_3</td>
<td>tree-width ≤1</td>
</tr>
<tr>
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</tr>
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<td>K_5</td>
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</tr>
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</tr>
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</tr>
<tr>
<td>Arbitrary finite H</td>
<td>k-summable over graphs with each finite subgraph (p, q, h)-representable over some Σ H cannot be drawn on.</td>
</tr>
</tbody>
</table>
4. An example

4.1. Theorem 1.1 does not hold for infinite \( k \). In fact for any cardinal \( \kappa \) there exists a chordal graph \( G \) with each clique at most countable such that every tree-decomposition \( (T, X) \) contains an \( X' \) of cardinality \( \gtrsim k \).

Let the vertices of \( G \) be all sequences of elements of \( \kappa \) of length \( \leq \omega \) including the empty sequence and let \( \{u, v\} \in E(G) \) if \( u \) is a proper initial segment of \( v \). Clearly \( G \) is a chordal graph in which all cliques are at most countable. We prove the following

**Theorem 4.2.** Any tree-decomposition \( (T, X) \) of \( G \) contains an \( X' \) of cardinality \( \gtrsim k \).

**Proof.** Suppose the contrary. Sequences of ordinals will be indicated by juxtaposition. We shall construct finite sequences \( \lambda_0, \ldots, \lambda_n, \ldots \in V(G) \) and distinct vertices \( t_0, \ldots, t_n, \ldots \in V(T) \) such that for all \( n \)

(i) there exist ordinals \( \alpha_0, \ldots, \alpha_n, \ldots < \kappa \) such that \( \lambda_n = \alpha_0 \cdot \alpha_1 \cdots \alpha_{n-1} \),

(ii) \( X' \) contains \( \lambda_0, \ldots, \lambda_n \) but does not contain any \( \lambda \in V(G) \) with \( \lambda_{n+1} \) as an initial segment,

(iii) if \( t \) is on the path between \( t_n \) and \( t_{n+1} \) and \( X' \supseteq \{\lambda_0, \ldots, \lambda_{n+1}\} \), then \( t = t_{n+1} \).

Let \( \lambda_0 \) be the empty sequence and choose \( t_0 \) such that \( \lambda_0 \in X^0 \). Now assume we have already constructed \( \lambda_0, \ldots, \lambda_n, t_0, \ldots, t_n \). Since \( |X^n| < \kappa \), there exists an ordinal \( \alpha_n \in \kappa \) such that \( \lambda \notin X^\alpha \) whenever \( \lambda_{n+1} = \lambda_n \alpha_n \) is the initial segment of \( \lambda \).

Now choose \( t_{n+1} \in V(T) \) such that \( \lambda_0, \ldots, \lambda_n, \lambda_{n+1} \in X^{n+1} \) (this is possible since \( G \upharpoonright \{\lambda_0, \ldots, \lambda_{n+1}\} \) is a complete graph) and such that the distance of \( t_{n+1} \) from \( t_n \) in \( T \) is the least possible. Obviously this choice of \( t_{n+1} \) implies (iii). This completes the construction.

We claim that for any \( n \in \omega \), \( t_{n+1} \) lies on the path between \( t_n \) and \( t_{n+2} \). Should this not be the case, we have the following possibilities:

(a) \( t_n \) is on the path between \( t_{n+1} \) and \( t_{n+2} \). Then

\[
\lambda_{n+1} \in X^{n+1} \cap X^{n+2} \subseteq X^n
\]

by (T3), a contradiction.

(b) \( t_n \) is not on the path between \( t_{n+1} \) and \( t_{n+2} \). Then denote by \( t \) that vertex of \( T \), which belongs to the path between \( t_{n+1} \) and \( t_{n+2} \) and whose distance from \( t_n \) is the least possible. By the assumptions, \( t \notin \{t_n, t_{n+1}\} \) and

\[
\{\lambda_0, \ldots, \lambda_{n+1}\} \subseteq X^{n+1} \cap X^{n+2} \subseteq X',
\]

by (T3), a contradiction to (iii) (realize that \( t \) lies on the path between \( t_n, t_{n+1} \)).

It follows from the claim that there is an infinite path \( s_0, s_1, \ldots \) in \( T \) containing \( t_0, t_1, \ldots \) in this order. Let, say \( s_n = t_n \). Then \( i_1 < i_2 < \cdots \). Let \( \lambda \in V(G) \) be the infinite sequence with initial segments \( \lambda \). Since \( G \upharpoonright \{\lambda_0, \ldots, \lambda_n, \lambda\} \) is a
complete graph, we have for each \( n \) a vertex \( u_n \in V(T) \) such that
\[
\lambda_0, \ldots, \lambda_n, \lambda \in X^{u_n}.
\]
Let \( j_n \) be such that the distance between \( u_n \) and \( s_n \) is the least possible. Now we distinguish two cases.

If \( j_{n+1} > i_n \) for any \( n \), take \( n \) such that \( i_n > j_1 \). Then we have \( j_{n+1} > i_n > j_1 \) and hence \( s_{i_n} \) lies on the path between \( u_n, u_{n+1} \). By (T3) we conclude
\[
\lambda \in X^{u_n} \cap X^{s_{i_n}} \subseteq X^{s_{i_n}} \cap X^{s_{i+1}} \subseteq X^{s_n} = X^{i_n},
\]
contradicting (ii).

If \( j_{n+1} \leq i_n \) for some \( n \), then by (T3)
\[
\lambda_{n+1} \in X^{u_{n+1}} \cap X^{s_{i_n}} = X^{s_{i_n}} \cap X^{s_{i+1}} \subseteq X^{s_n} = X^{i_n},
\]
again contradicting (ii). \( \square \)

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References
