NON-EMBEDDABLE EXTENSIONS OF EMBEDDED MINORS[1]

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Abstract

A graph G is weakly 4-connected if it is 3-connected, has at least five vertices, and for every pair (A, B) such that $A \cup B = V(G)$, $|A \cap B| = 3$ and no edge has one end in $A - B$ and the other in $B - A$, one of the induced subgraphs $G[A], G[B]$ has at most four edges. We describe a set of constructions that starting from a weakly 4-connected planar graph G produce a finite list of non-planar weakly 4-connected graphs, each having a minor isomorphic to G, such that every non-planar weakly 4-connected graph H that has a minor isomorphic to G has a minor isomorphic to one of the graphs in the list. Our main result is more general and applies in particular to polyhedral embeddings in any surface.

1 Introduction

We begin with some basic notation and ingredients needed to state the main result of this paper. Graphs are finite and simple (i.e., they have no loops or multiple edges). *Paths* and *cycles* have no "repeated" vertices. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. For a graph G and an edge e in G, $G \backslash e$ an G/e are the graphs obtained from G by respectively deleting and contracting the edge e. A graph is a *subdivision* of another if the first can be obtained from the second by replacing each edge by a non-zero length path with the same ends, where the paths are disjoint, except possibly for shared ends. The replacement paths are called *segments*, and their ends are called *branch-vertices*. A graph is a *topological minor* of another if a subdivision of the first is a subgraph of the second.

Let a non-planar graph H have a subgraph isomorphic to a subdivision of a planar graph G. For various problems in Graph Structure Theory it is useful to know the minimal subgraphs of H that are isomorphic to a subdivision of G and are non-planar. In other words, one wants to know what more does H contain on account of its non-planarity. In [7] it is shown that under some mild connectivity assumptions these "minimal non-planar enlargements" of G are quite nice. In the applications of the result, G is explicitly known, whereas H is not, and the enlargement operations would furnish an explicit list of graphs such that (i) H has a subgraph isomorphic to a subdivision of one of the graphs on the list, and (ii) each graph on the list is a witness both to the fact that G

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is a topological minor of H, and that H is, in addition, non-planar. (The minimality of the graphs in the list is required to avoid redundancy.) Before we state that result, we need a few definitions.

For $Z \subseteq V(G)$, $G[Z]$ denotes the subgraph *induced* by Z, that is, the subgraph consisting of Z and all edges with both ends in Z. A subgraph of G is said to be *induced* if it is induced by its vertex set.

A *separation* of a graph G is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$, and there is no edge between $A - B$ and $B - A$. The *order* of (A, B) is $|A \cap B|$. The separation is called *non-trivial* if both A and B are proper subsets of $V(G)$. A graph G is *weakly 4-connected* if G is 3-connected, has at least five vertices, and for every separation (A, B) of G of order at most three, one of the graphs $G[A]$, $G[B]$ has at most four edges.

A cycle C in a graph G is called *peripheral* if it is induced and $G \backslash V(C)$ is connected. It is well-known [9, 10] that the peripheral cycles in a 3-connected planar graph are precisely the cycles that bound faces in some (or, equivalently, every) planar embedding of G.

Let S be a subgraph of a graph H. An *S-path* in H is a path with both ends in S, and otherwise disjoint from S. Let C be a cycle in S, and let P_1 and P_2 be two disjoint S-paths in H with ends u_1, v_1 and u_2, v_2, respectively, such that u_1, u_2, v_1, v_2 belong to $V(C)$ and occur on C in the order listed. In those circumstances we say that the pair P_1, P_2 is an *S-cross in H*. We also say that it is an *S-cross on C*. We say that u_1, v_1, u_2, v_2 are the *feet* of the cross. We say that the cross P_1, P_2 is *free* if

(F1) for $i = 1, 2$ no segment of S includes both ends of P_i, and

(F2) no two segments of S that share a vertex include all the feet of the cross.

The following was proved in [7].

Theorem 1.1. *Let G be a weakly 4-connected planar graph, and let H be a weakly 4-connected non-planar graph such that a subdivision of G is isomorphic to a subgraph of H. Then there exists a subgraph S of H isomorphic to a subdivision of G such that one of the following conditions holds:*

1. *there exists an S-path in H such that its ends belong to no common peripheral cycle in S, or*
2. *there exists a free S-cross in H on some peripheral cycle of S.*

This theorem has been used in [3, 8], and its extension has been used in [6]. However, in more complicated applications it is more efficient to work with minors, rather than topological minors. We sketch one such application in Section 8. For any fixed graph G, there exists a finite and explicitly constructible set $\{G_1, G_2, \ldots, G_t\}$ of graphs such that a graph H has a minor isomorphic to G if and only if it has a topological minor isomorphic to one of the graphs G_i. Thus one can apply Theorem 1.1 t times to deduce the desired conclusion about G, but it would be nicer to have a more direct route to the result that involves less potential duplication. Furthermore, if the outcome is allowed to be a minor of H rather than a topological minor, then the outcomes (i) and (ii) above can be strengthened to require that the ends of the paths involved are branch-vertices of S, as we shall see.

It turns out that Theorem 1.1 is not exclusively about face boundaries of planar graphs, but that an appropriate generalization holds under more general circumstances. Thus rather than working with peripheral cycles in planar graphs we will introduce an appropriate set of axioms for a set of cycles of a general graph. We do so now in order to avoid having to restate our definitions later when we present the more general form of our results.

A *segment* in a graph G is a maximal path such that its internal vertices all have degree in G exactly two. If a graph G has no vertices of degree two, then the segments of a subdivision of G defined earlier coincide with the notion just defined. Since we will not consider subdivisions of graphs with vertices of degree two there is no danger of confusion. A *cycle double cover* in a graph G is a set $\mathcal{D}$ of distinct cycles of G, called *disks*, such that

(D1) each edge of G belongs to precisely two members of $\mathcal{D}$.

A cycle double cover $\mathcal{D}$ is called a *disk system* in G if

(D2) for every vertex v of G, the edges incident with v can be arranged in a cyclic order such that for every pair of consecutive edges in this order, there is precisely one disk in $\mathcal{D}$ containing that pair of edges, and

(D3) the intersection of any two distinct disks in $\mathcal{D}$ either has at most one vertex or is a segment.

A cycle double cover satisfying (D3) is called a *weak disk system*. It is easy to see that if a connected graph has a disk system, then it is a subdivision of a 3-connected graph. Also, note that in a 3-connected graph, Axiom (D3) is equivalent to the requirement that every two distinct disks intersect in a complete subgraph on at most two vertices. The peripheral cycles of a 3-connected planar graph form a disk system. More generally, if G is a subdivision of a 3-connected graph embedded in a surface Σ in such a way that every homotopically nontrivial closed curve intersects the graph at least three times (a "polyhedral embedding"), then the face boundaries of this embedding form a disk system in G. Conversely, it can be shown that a disk system in a graph is the set of face boundaries of a polyhedral embedding of the graph in some surface. Weak disk systems correspond to face boundaries of embeddings into pseudosurfaces (surfaces with "pinched" points).

Let G be a graph with a cycle double cover $\mathcal{D}$. Two vertices or edges of G are said to be *confluent* if there is a disk containing both of them. If $\mathcal{D}$ is a cycle double cover in a graph G and S is a subdivision of G, then $\mathcal{D}$ induces a cycle double cover $\mathcal{D}'$ in S in the obvious way, and vice versa. We say that $\mathcal{D}'$ is the *cycle double cover induced in S by $\mathcal{D}$*.

Let v be a vertex of a graph G with degree at least 4. Partition the set of its neighbors into two disjoint sets N_1 and N_2, with at least two vertices in each set. Let G' be obtained from G by replacing the vertex v with two adjacent vertices v_1, v_2, with v_i adjacent to the vertices in N_i for $i = 1, 2$. The graph G' is said to be obtained from G by *splitting* the vertex v. It is easy to see that if G is 3-connected, then so is G'. The vertices v_1 and v_2 are called the *new vertices* of G' and the edge v_1v_2 of G' is called the *new edge* of G'.

Suppose a graph G has a cycle double cover $\mathcal{D}$. The above splitting operation on a vertex v of G is said to be a *conforming split* (with respect to $\mathcal{D}$) if

(S1) among the disks that use the vertex v, there are exactly two, say D_1 and D_2, that use one vertex each from N_1 and N_2, and

(S2) D_1 and D_2 intersect precisely in the vertex v.

The split is then said to be *along* D_1 (and along D_2). A split that is not conforming as above is said to be a *non-conforming split*.

Let G, G' be as in the above paragraph. If G is a 3-connected planar graph, then G' is planar if and only if the split is conforming with respect to the disk system of peripheral cycles of G. More generally, to each cycle C of G there corresponds a unique cycle C' of G', and so to $\mathcal{D}$ there corresponds a uniquely defined set of cycles $\mathcal{D}'$ of G'. If $\mathcal{D}$ is a weak disk system, then so is $\mathcal{D}'$, and if $\mathcal{D}$ is a disk system, then so is $\mathcal{D}'$. We call $\mathcal{D}'$ the (weak) disk system *induced* in G' by $\mathcal{D}$. This is the purpose of conditions (S1) and (S2). If $\mathcal{D}$ is a disk system, then an equivalent way to define a conforming split of a vertex v is to say that both N_1 and N_2 form contiguous intervals in the cyclic order induced on the neighborhood of v by $\mathcal{D}$. Similarly, an equivalent condition for a split to be non-conforming with respect to a disk system is the existence of vertices $a, c \in N_1$ and $b, d \in N_2$ such that a, b, c and d appear in the cyclic order listed around v (as given by $\mathcal{D}$ in (D2)). The reason we use the definition above is that it applies more generally to weak disk systems.

A graph G' obtained from a graph G by repeatedly splitting vertices of degree at least four is said to be an *expansion* of G. In particular, each graph is an expansion of itself. Each split leading to an expansion of G has exactly one new edge; the set of these edges are the *new edges of the expansion of G*. The new edges form a forest in G'. If G has a cycle double cover $\mathcal{D}$, the expansion is called a *conforming* expansion if each of the splits involved in it is conforming (with respect to $\mathcal{D}$). If at least one of the splits involved is not conforming, then the expansion is called *non-conforming*. From the above discussion, it is clear that a disk system in G induces a unique disk system in a conforming expansion.

We now describe seven enlargement operations. Let G be a graph with a cycle double cover $\mathcal{D}$, and let G^+ be the graph obtained from G by applying one of the operations described below.

1. (non-conforming jump) G^+ is obtained from G by adding an edge uv where u and v are non-confluent vertices of G.

2. (cross) Let a, b, c, d be vertices appearing on a disk of G in that cyclic order. Add the edges ac and bd to obtain G^+.

3. (non-conforming split) G^+ is obtained from G by performing a non-conforming split of a vertex of G.

4. (split + non-conforming jump) Let u, v be non-adjacent vertices on some disk $C \in \mathcal{D}$. Perform a conforming split of v into v_1, v_2 such that u and v_2 are non-confluent vertices. (In particular, the split is not along C.) Now add the edge uv_2 to obtain G^+.

5. (double split + non-conforming jump) Let u, v be adjacent vertices and C_1, C_2 be the two disks containing the edge uv. Make a conforming split of u into u_1, u_2 along C_1 and a conforming split of v into v_1, v_2 along C_2 such that both splits are conforming and u_1 and v_1 are adjacent in the resulting graph. Now add the edge u_2v_2 to obtain G^+.

6. (split + cross) Let u, v, w be vertices on a disk C such that u is not adjacent to v or w. Perform a conforming split of u into u_1, u_2, along C, with u_1, u_2, v, w in that cyclic order on the new disk corresponding to C. Now add the edges u_1v and u_2w to obtain G^+.

7. (double split + cross) Let u, v be non-adjacent vertices on a disk C. Perform conforming splits of u and v, into u_1, u_2 and v_1, v_2, respectively such that both splits are along C. Let u_1, u_2, v_1, v_2 appear in that cyclic order on the new disk corresponding to C. Now add the edges u_1v_1 and u_2v_2 to obtain G^+.

If G^+ is obtained as in paragraph i above, then we say that G^+ is an *i-enlargement of G with respect to $\mathcal{D}$*. When the disk system $\mathcal{D}$ is implied by context, we may simply refer to an *i-enlargement* of G. We are now ready to state a preliminary form of our main result, a counterpart of Theorem 1.1, with minors instead of topological minors. A graph is a *prism* if it has exactly six vertices and its complement is a cycle on six vertices.

Theorem 1.2. *Let G be a weakly 4-connected planar graph that is not a prism, let H be a weakly 4-connected non-planar graph such that G is isomorphic to a minor of H, and let $\mathcal{D}$ be the disk system in G consisting of all peripheral cycles. Then there exists an integer $i \in \{1, 2, \ldots, 7\}$ such that H has a minor isomorphic to an i-enlargement of G with respect to $\mathcal{D}$.*

Theorem 1.1 is definitely easier to state than Theorem 1.2. So what are the advantages of the latter result? First, in the applications one is usually concerned with minors rather than topological minors, and so Theorem 1.2 gives a more direct route to the desired results. Second, while the number of types of outcome is larger in Theorem 1.2, in most cases the actual number of cases needed to examine will be smaller. (Notice that, for instance, in Theorem 1.1 one must examine all S-paths between non-confluent ends, whereas in Theorem 1.2 one is only concerned with those between non-confluent branch-vertices.)

Third, Theorem 1.1 allows as an outcome an S-cross on a cycle consisting of three segments. That is a drawback, which essentially means that in order for the theorem to be useful the graph G should have no triangles. On the other hand, Theorem 1.2 does not suffer from this shortcoming and gives useful information even when G has triangles.

Fourth, while a graph listed as an outcome of Theorem 1.1 may fail to be weakly 4-connected (and may do so in a substantial way), an i-enlargement of a weakly 4-connected graph is again weakly 4-connected. This has two advantages. In the applications we are often seeking to prove that weakly 4-connected graphs, with a minor isomorphic to some weakly 4-connected graph embeddable in a surface Σ, that themselves do not embed into Σ have a minor isomorphic to a member of a specified list $\mathcal{L}$ of graphs. In order to get a meaningful result we would like each member of $\mathcal{L}$ to satisfy the same connectivity requirement imposed on the input graphs.

From a more practical viewpoint, the advantage of maintaining the same connectivity in the outcome graph is that the theorem can then be applied repeatedly. That will become important when we consider a generalization to arbitrary surfaces (that is, in the context of theorems 2.1 and 7.5). While a weakly 4-connected graph G has at most one planar embedding, it may have several embeddings in a non-planar surface Σ. Now one application of the generalization of Theorem 1.2 will dispose of one embedding into Σ, but some other embedding might extend naturally to those outcome graphs. So it may be necessary to apply the theorem in turn to those outcome graphs in place of G. It will be important that the outcomes of (the generalization of) Theorem 1.2 satisfy the same requirement as the input graph. We can then apply such a theorem repeatedly till we get a list of graphs that no longer embed in Σ — in other words, we would have obtained the non-embeddable extensions of G. This will be illustrated in Section 8.

2 Main Theorem

Our main theorem applies to arbitrary disks systems, at the expense of having to add two outcomes. We also add a third additional outcome in order to allow G to be a prism. The extra outcomes are the following. As before, let G be a graph with a cycle double cover $\mathcal{D}$, and let G^+ be obtained by one of the operations below.

8. (non-separating triad) Let x_1, x_2, x_3 be three vertices of G such that (i) they are pairwise confluent, but not all contained in any single disk, and (ii) $\{x_1, x_2, x_3\}$ is independent, and does not separate G. To obtain G^+, add a new vertex to G adjacent to x_1, x_2 and x_3.

9. (non-conforming T-edge) Let a vertex u and an edge xy be such that (i) u is not confluent with the edge xy, but is confluent with both x and y, (ii) u is not adjacent to either x or y, and (iii) $\{u, x, y\}$ does not separate G. Subdivide the edge xy and join u to the new vertex, to obtain G^+.

10. (enlargement of a prism) Let G be a prism, and let G^+ be obtained from G by selecting two edges of G that do not belong to a common peripheral cycle but both belong to a triangle, subdividing them, and joining the two new vertices by an edge.

As before, if G^+ is obtained as in paragraph i above, then we say that G^+ is an *i-enlargement of G with respect to $\mathcal{D}$*. Thus if G is not a prism, then it has no 10-enlargement, and if G is a prism, then its 10-enlargement is unique, up to isomorphism. The unique 10-enlargement of the prism is known as V_8.

We also need to define an appropriate analogue of being non-planar in the context of cycle double covers. That is the objective of this paragraph and the next. Let S be a subgraph of a graph H. An *S-bridge of H* is a subgraph B of H such that either B consists of a unique edge of $E(H) - E(S)$ and its ends, where the ends belong to S, or B consists of a component J of $H \backslash V(S)$ together with all edges from $V(J)$ to $V(S)$ and all their ends. For an S-bridge B, the vertices of $B \cap S$ are called the *attachments* of B. Let $\mathcal{D}$ be a cycle double cover in S. We say that $\mathcal{D}$ is *locally planar* in H if the following conditions are satisfied:

(i) for every S-bridge B of H there exists a disk $C_B \in \mathcal{D}$ such that all the attachments of B lie on C_B, and

(ii) for every disk $C \in \mathcal{D}$ the subgraph $\bigcup B \cup C$ of H has a planar drawing with C bounding the unbounded face, where the big union is taken over all S-bridges B of H with $C_B = C$.

Let G have a weak disk system $\mathcal{D}$ and H have a minor isomorphic to G. It is easy to see that there is an expansion G' of G, such that G' is a topological minor of H. We say that $\mathcal{D}$ has a *locally planar extension* into H if:

(i) there exists a *conforming* expansion G' of G such that a subdivision of G' is a isomorphic to a subgraph S of H, and

(ii) the weak disk system $\mathcal{D}'$ induced in S by $\mathcal{D}$ is locally planar in H.

We are now ready to state the main result.

Theorem 2.1. *Let G and H be weakly 4-connected graphs such that H has a minor isomorphic to G. Let G have a disk system $\mathcal{D}$ that has no locally planar extension into H. Then H has a minor isomorphic to an i-enlargement of G, for some $i \in \{1, \dots, 10\}$.*

Let us deduce Theorem 1.2 from Theorem 2.1.

Proof of Theorem 1.2, assuming Theorem 2.1. Let G be as in Theorem 1.2, and let $i \in \{8, 9, 10\}$. By Theorem 2.1 it suffices to show that G has no i-enlargement with respect to the disk system consisting of all peripheral cycles of G. This is clear when $i = 10$, because G is not a prism. Thus we may assume for a contradiction that $i \in \{8, 9\}$ and that such an i-enlargement exists. Let u, x, y be the three vertices of G as in the definition of i-enlargement. Since every pair of vertices among u, x, y are confluent, it follows that $G \backslash \{u, x, y\}$ is disconnected, a contradiction. □

3 Outline of Proof

The purpose of this section is to outline the proof of the main theorem. Our main tool for the proof of Theorem 2.1 will be its counterpart for subdivisions, proved in [7]. Before we can state it we need one more definition. Let S be a subgraph of a graph H, and let $\mathcal{D}$ be a cycle double cover in S. Let $x \in V(H) - V(S)$ and let x_1, x_2, x_3 be distinct vertices of S such that every two of them are confluent, but no disk of S contains all three. Let L_1, L_2, L_3 be three paths such that (i) they share a common end x, (ii) they share no internal vertex among themselves or with S, and (iii) the other end of L_i is x_i, for $i = 1, 2, 3$. The paths L_1, L_2, L_3 are then said to form an *S-triad*. The vertices x_1, x_2, x_3 are called the *feet* of the triad. We are now ready to state our tool. It is an immediate corollary of [7, Theorem (4.6)].

Theorem 3.1 ([7]). *Let G be a graph with no vertices of degree two that is not the complete graph on four vertices, let H be a weakly 4-connected graph, let $\mathcal{D}$ be a weak disk system in G, and let a* *subdivision of G be isomorphic to a subgraph of H. Then there exists a subgraph S of H isomorphic to a subdivision of G such that, letting $\mathcal{D}'$ denote the weak disk system induced in S by $\mathcal{D}$, one of the following conditions holds:*

1. *there exists an S-path in H such that its ends are not confluent in S, or*
2. *there exists a free S-cross in H on some disk of S, or*
3. *the graph H has an S-triad, or*
4. *the weak disk system $\mathcal{D}'$ is locally planar in H.*

Now let G, $\mathcal{D}$ and H be as in Theorem 2.1. It is easy to see that there exists an expansion G' of G such that a subdivision of G' is isomorphic to a subgraph S of H. (If G itself is a topological minor of H, then $G' = G$.) In Lemma 4.4 we prove that if G' is a nonconforming expansion, then there exists a 3-enlargement of G that is isomorphic to a minor of H. Thus from now on we may assume that G' is a conforming expansion of G. By Lemma 3.1 applied to S and H we deduce that one of the outcomes of that lemma holds. Notice that those outcomes correspond to 1-enlargement, 2-enlargement and 8-enlargement, respectively, except that in the enlargements the vertices in question are required to be branch-vertices of S, whereas in Lemma 3.1 they are allowed to be interior vertices of segments. We deal with this in Section 5 by showing that each of the outcomes mentioned leads to a suitable enlargement of G'. To be precise, at this point we settle for what we call weak 8- and weak 9-enlargements, and in Section 6 show that these weak enlargements can be replaced by ordinary enlargements, possibly of a different expansion of G and of a different kind. Finally, in Section 7 we complete the proof of Theorem 2.1 by showing that the expansion G' can be chosen to be equal to G.

4 Preliminaries

Let G' be an expansion of a graph G. Then every vertex v of G corresponds to a connected subgraph T_v of G'. We call $V(T_v)$ the *branch-set corresponding to v*.

Lemma 4.1. *Let G' be an expansion of a graph G, let $u, v \in V(G)$ be distinct, and let T_u, T_v be the corresponding subgraphs of G'. Then T_u and T_v are induced subtrees of G'. If u is adjacent to v then exactly one edge of G' has one end in $V(T_u)$ and the other in $V(T_v)$, and if u is not adjacent to v, then no such edge exists.*

An expansion of a weakly 4-connected graph may fail to be weakly 4-connected, but only in a limited way. The next definition and lemma make that precise. Let (A, B) be a nontrivial separation of order three in a graph G. We say that (A,B) is *degenerate* if the vertices in $A\cap B$ can be numbered v_1,v_2,v_3 such that either

(1) $|A-B|=1$ and $A\cap B$ is an independent set, or

(2) there exists a triangle $u_1u_2u_3$ in $G[A]$ such that for $i=1,2,3$ the vertices u_i and v_i are either adjacent or equal, $A\subseteq\{u_1,u_2,u_3,v_1,v_2,v_3\}$, and each edge of $G[A]$ is of the form u_iv_i for $1\le i\le 3$ or u_iu_j for $1\le i<j\le 3$.

The following two lemmas are routine, and we omit the straightforward proofs.

Lemma 4.2. *Let G be an expansion of a weakly 4-connected graph. Then G is 3-connected, and if it is not a prism, then for every nontrivial separation (A,B) of G of order three, exactly one of (A,B), (B,A) is degenerate.*

Lemma 4.3. *Let G' be expansion of a weakly 4-connected graph G, let (A,B) be a degenerate separation of G of order three satisfying condition (2) of the definition of degenerate separation, and let u_1,u_2,u_3,v_1,v_2,v_3 be as in that condition. Then for at least two integers $i\in\{1,2,3\}$ either $u_i=v_i$ or u_iv_i is a new edge of G'.*

We now show that a non-conforming expansion of G must have a minor isomorphic to a 3-enlargement of G.

Lemma 4.4. *Let $\mathcal{D}$ be a disk system in a graph G, and let G' be a non-conforming expansion of G. Then G' has a minor isomorphic to a 3-enlargement of G.*

Proof: We may assume that for every new edge e of G' the graph G'/e is a conforming expansion of G. We shall refer to this as the *minimality* of G'. We will prove that G' is a 3-enlargement of G.

Let $\widehat{G}$ be an expansion of G such that G' is obtained from $\widehat{G}$ by splitting a vertex v into v_1 and v_2. By the minimality of G' this split is non-conforming, and $\widehat{G}$ is a conforming expansion of G. If $G=\widehat{G}$, then G' is a 3-enlargement of G, and so we may assume that $G\neq\widehat{G}$. Let e be a new edge of $\widehat{G}$. If e is not incident with v, then G'/e is a non-conforming expansion of $\widehat{G}/e$, contrary to the minimality of G'. Now let us consider e as an edge of G'. From the symmetry between v_1 and v_2 we may assume that e is incident with v_2 in G'; let v_3 be its other end. The split of the vertex v of the graph $\widehat{G}$ into v_1 and v_2 violates (S1) or (S2). But it does not violate (S1), for otherwise the same violation occurs in the analogous split of $\widehat{G}/e$, contrary to the minimality of G'. Thus the split of the vertex v of the graph $\widehat{G}$ into v_1 and v_2 satisfies (S1); let D_1 and D_2 be the corresponding disks. It follows that the disks violate (S2), but they do not do so for the corresponding split in $\widehat{G}/e$. It follows that $e \in E(D_1) \cap E(D_2)$. Thus the split that creates $\widehat{G}$ from $\widehat{G}/e$ is also along D_1 and D_2. Let f be an edge incident with v_2 in G' that is not e or the new edge v_1v_2 of G'. It follows from (D2) by considering the edge f that either the split that creates $\widehat{G}$ from $\widehat{G}/e$ or the split that creates G'/e from $\widehat{G}/e$ is non-conforming, contrary to the minimality of G'. □

The following lemma will be useful.

Lemma 4.5. *Let G' be a conforming expansion of a graph G with respect to a weak disk system $\mathcal{D}$, and let $\mathcal{D}'$ be the weak disk system induced in G' by $\mathcal{D}$. Let qr be a new edge of G', and let the vertex $p \in V(G') - \{q, r\}$ share distinct disks D_q, D_r of G' with q and r, respectively, such that D_r does not contain q. Then p is adjacent to r and the disks D_q, D_r both contain the edge pr.*

Proof: The disks of G/qr that correspond to D_q and D_r share p and the new vertex of G/qr, say w. By (D3) p is adjacent to w in G/qr and the edge pw belongs to both those disks. By Lemma 4.1 the vertex p is adjacent to exactly one of q, r. But $q \notin V(D_r)$, and hence p is adjacent to r and D_q, D_r both contain the edge pr, as desired. □

We end this section with a lemma about fixing separations in weakly 4-connected graphs, a special case of a lemma from [5]. First some additional notation: when a graph G is a minor of a graph H, we say that an *embedding* η of G into H is a mapping with domain $V(G) \cup E(G)$ as follows. η maps vertices $v \in G$ to connected subgraphs $\eta(v)$ of H, with distinct vertices being mapped to disjoint vertex-disjoint subgraphs. Further, η maps edges uv of G to paths $\eta(uv)$ in H with one end in $\eta(u)$ and the other in $\eta(v)$, and otherwise disjoint from $\eta(w)$ for any vertex w of G. Also, for edges $e \neq e'$ of G, if $\eta(e)$ and $\eta(e')$ share a vertex, then it must be an end of both the paths.

Lemma 4.6. *Let G_1 be a graph isomorphic to a minor of a weakly 4-connected graph H. Let $P = \{p_1, p_2\}$, $Q = \{q_1, q_2, q_3\}$ and R be such that (P, Q, R) is a partition of $V(G_1)$, and G_1 has all possible edges between P and Q, and no edge with both ends in Q. Further, suppose R has at least two vertices, and that $(P \cup Q, Q \cup R)$ is a (non-trivial) 3-separation of G_1. Then H has a minor isomorphic to a graph G_1^+ that is obtained from G_1 by*

1. *adding an edge between p_i and r for some $i \in \{1, 2\}$ and $r \in R$, or*

2. *splitting q_j for some $j \in \{1, 2, 3\}$ into vertices q_j^1 and q_j^2 such that q_j^1 is adjacent to p_1 and q_j^2 is adjacent to p_2*

Proof: Call an embedding η of G_1 into H *minimal* if for every embedding η' of G into H,

$$\sum_{j=1}^{3}|E(\eta(q_j))| \le \sum_{j=1}^{3}|E(\eta'(q_j))|$$

In particular, if η is minimal, $\eta(q_j)$ is a tree for every j. Further, we say that a vertex q_j is *good* for η if the paths $\eta(p_1q_j)$ and $\eta(p_2q_j)$ are vertex-disjoint (in other words, their ends in $\eta(q_j)$ are distinct).

Consider a minimal embedding η of G_1 into H. Suppose there exists a q_j that is good for η. For $i = 1, 2$, let p'_i be the endpoint of $\eta(p_iq_j)$ in $\eta(q_j)$. Let e be an edge in the unique path between p'_1 and p'_2 in $\eta(q_j)$, and let T_1, T_2 be the two subtrees obtained by deleting e from $\eta(q_j)$, such that $p'_i \in T_i$ for $i = 1, 2$. For $i = 1, 2$, define N_i to be the set of neighbors $r \in R$ of q_j in G that $\eta(rq_j)$ has an endpoint in T_i. Now N_1, N_2 are non-empty by the minimality of η. (If, say, N_1 were empty, then we could replace $\eta(q_j)$ by T_2 and modify $\eta(p_1q_j)$ accordingly to get a better embedding η', a contradiction.) It is easy to see that conclusion 2 of the lemma is satisfied, with the neighborhoods of q_j^1 and q_j^2 being $N_1 \cup \{p_1\}$ and $N_2 \cup \{p_2\}$, respectively.

Hence we may assume that there is no minimal embedding of G into H with a vertex in Q being good for it. Let η be an embedding of G into H. For $j = 1, 2, 3$, there exist vertices t_j such that both $\eta(p_1q_j)$ and $\eta(p_2q_j)$ have t_j as an end. Define J_1 as the union of $\eta(p_i)$, $i = 1, 2$ and of $\eta(e)$ for all edges e with at least one end in P. Define J_2 as the union of $\eta(v)$ for $v \in Q \cup R$ and of $\eta(e)$ for every edge e of G with both ends in $Q \cup R$. Now $V(J_1) \cap V(J_2) = \{t_1, t_2, t_3\}$. Since H is weakly 4-connected, there is a path in H with ends $a \in V(J_1) \setminus V(J_2)$ and $b \in V(J_2) \setminus V(J_1)$, and otherwise disjoint from $J_1 \cup J_2$. If b belongs to $\eta(q_j) \setminus t_j$ for some j, then we can modify η to get a minimal embedding where q_j is a good vertex, which is a contradiction. Thus b belongs to $\eta(r)$ for some $r \in R$ or b is an internal vertex of $\eta(e)$ for an edge e of G that has an end in R (recall that Q is an independent set). In either case, it is easy to see that conclusion 1 holds. □

5 The Enlargements of an Expansion of G

Let G and H be as in Theorem 2.1. In order to apply Theorem 3.1 we select an expansion G' of G such that a subdivision of G' is isomorphic to a subgraph of H. By Lemma 4.4 we may assume that G' is a conforming expansion. In this section we prove three lemmas, one corresponding to each of the first three outcomes of Theorem 3.1. The lemmas together almost imply that the conclusion of Theorem 2.1 holds for G'. The reason for the word almost is that for convenience we allow a weaker form of 8-enlargements and 9-enlargements.

The weaker form of 9-enlargements is defined as follows. Let G be a graph with a cycle double cover $\mathcal{D}$, and let $u, x, y \in V(G)$, where x and y are adjacent and u is not confluent with the edge xy. Let G^+ be obtained from G by subdividing the edge xy and adding an edge joining the new vertex to u. We say that G^+ is a weak 9-enlargement of G. Later, in Lemma 6.3, we show how to move from a weak 9-enlargement to a 9-enlargement or another useful outcome. Our first lemma deals with the first outcome of Theorem 3.1.

Lemma 5.1. *Let G, H be graphs such that G is connected, has at least five vertices and no vertices of degree two. Let $\mathcal{D}$ be a weak disk system in G, let S be a subgraph of H isomorphic to a subdivision of G, and let P be an S-path in H such that its ends are not confluent in the weak disk system $\mathcal{D}'$ induced in S by $\mathcal{D}$. Then H has a minor isomorphic to a 1-enlargement, 3-enlargement or a weak 9-enlargement of G.*

Proof: Let s, t be the ends of P. If both s and t are branch-vertices in S, then H has a minor isomorphic to a 1-enlargement of G, and we are done. If one of s and t is a branch-vertex and the other is an internal vertex of a segment of S, then H has a minor isomorphic to a weak 9-enlargement of G, as desired.

Thus we may assume that s and t are internal vertices of two different segments Q_1 and Q_2 of S, respectively. Let Q_1 correspond to an edge $uv \in E(G)$, and let Q_2 correspond to an edge $xy \in E(G)$. Now, if u is not confluent with the edge xy, then H has a minor isomorphic to a weak 9-enlargement of G, and the lemma holds. Thus, we may assume that u shares a disk D_1 with the edge xy. By symmetry, we get a disk D_2 shared by v and the edge xy, and disks D_3, D_4 that the edge uv shares with vertices x and y respectively. (The disks D_i may not be pairwise distinct.)

The disks D_1 and D_3, however, must be distinct, since the vertices s, t are not confluent. Notice, however, that they share the vertices u and x. It follows that u, v, x, y are pairwise distinct, for if $v = y$, say, then $u, v = y, x$ all belong to $V(D_1 \cap D_3)$, and hence $D_1 = D_3$ by (D3), a contradiction. By (D3) this implies that u is adjacent to x in G and the intersection of D_1 and D_3 is precisely the edge ux. In other words, the vertices u and x must be adjacent in G, and D_1, D_3 are precisely the two disks containing the edge ux. By a similar argument, it follows that u and y are adjacent, and D_1, D_4 are precisely the two disks containing the edge uy. Thus u is adjacent to each of v, x, y in G, and the edges uv, ux, uy are pairwise confluent.

By symmetry, we get similar conclusions about the vertices v, x, y. Thus $G[u, v, x, y]$ is a detached K_4 subgraph of G. Since G has at least five vertices and is connected, we may assume, without loss of generality, that u has a neighbor in G outside of the set $\{v, x, y\}$. Let N be the set of all such neighbors of u. But then delete the edges of the segment of S corresponding to the edge ux and contract the edges of the subpath of Q_2 between t and the end corresponding to x. It follows that H has a minor isomorphic to a graph obtained from G by splitting u corresponding to the partition $\{\{v,x\}, N \cup \{y\}\}$ of its neighbors. This split is non-conforming since the disks D_1 and D_4 violate condition (S2) in the definition of a conforming split. Hence H has a minor isomorphic to a 3-enlargement of G. □

Lemma 5.2. *Let G, H be graphs such that H is weakly 4-connected, and G is connected, has at least 5 vertices and has no vertices of degree two. Let $\mathcal{D}$ be a weak disk system in G, let S be a subgraph of H isomorphic to a subdivision of G, such that $\mathcal{D}$ induces the weak disk system $\mathcal{D}'$ in S. Further, let there exist a free S-cross on some disk of S. Then H has a minor isomorphic to a 2-enlargement or a 3-enlargement or a weak 9-enlargement of G.*

Proof: Let the free cross consist of paths P_1, P_2, in a disk C' of S, that corresponds to a disk C of G. We shall call the paths P_1, P_2 the *legs* of the cross. Recall that the ends of P_1, P_2 are called the *feet* of the cross.

If C has at least four vertices, then we claim that H has a minor isomorphic to a 2-enlargement of G. We define an auxiliary bipartite graph B, with the vertex set being the set of feet of the cross and the set of branch-vertices of S that belong to C'. A foot f and a branch-vertex b are adjacent if one of the subpaths of C' with ends f and b includes no feet or branch-vertices in its interior. Since the cross is free, it follows from Hall's bipartite matching theorem that B has a complete matching from the set of feet to the set of branch vertices (in other words, one that matches each of the feet). By contracting the edges of the paths that correspond to this matching, we deduce that H has a minor isomorphic to a 2-enlargement of G, as desired.

Hence we may assume that C is in fact a triangle on vertices u_1, u_2 and u_3, say. For $i = 1, 2, 3$, if u_i has degree 3 in G, then define v_i to be its third neighbor (that is, the neighbor not in C). Otherwise, define $v_i = u_i$. Let $u'_1, u'_2, u'_3, v'_1, v'_2, v'_3$ be the corresponding vertices of S. Let Q_i denote the segment of S corresponding to the edge u_iv_i if $u_i \neq v_i$ and let Q_i be the null graph otherwise, and let $A = V(C' \cup P_1 \cup P_2)$ and $B = (V(S) - V(C' \cup Q_1 \cup Q_2 \cup Q_3)) \cup \{v'_1, v'_2, v'_3\}$. There exist three vertex-disjoint paths in H linking $\{u'_1, u'_2, u'_3\}$ to $\{v'_1, v'_2, v'_3\}$. Since H is weakly 4-connected, it follows that there is no 3-cut in H separating A from B. Hence, by a variant of Menger's theorem, H contains four vertex-disjoint paths $L_1, \ldots L_4$ linking $\{v'_1, v'_2, v'_3, y\}$ to $\{u'_1, u'_2, u'_3, x\}$ (not necessarily in that order), where $x \in A$ and $y \in B$. We assume the numbering of the paths is such that for $i = 1, 2$ and 3, L_i has end $v'_i \in B$. (The remaining path L_4 then has end $y \in B$.) We wish to define a suitable vertex $w \in V(G)$. If y is a branch-vertex, then let w be the corresponding vertex of G; otherwise y is an internal vertex of a segment of H. By Lemmas 4.1 and 4.3 at least one end of that segment, say w', does not belong to $\{v_1, v_2, v_3\}$, and we let w be the vertex of G that corresponds to w'.

We may assume that $x \in V(C')$. If not, then we may contract edges suitably in P_1 or P_2 such that the vertex corresponding to x, after the contraction, lies on C'. (Note that this contraction does not affect the graph S, neither does it destroy the cross.)

Relabel the vertices u_1', u_2', u_3', x as a, b, c, d, in the order in which they appear on C' (in some orientation), such that L_4 joins d to y. (Note that d need not be the same as x.) Let (d, a, b) denote the interior vertices of the subpath of C' with ends d and b that includes a in its interior, and let (d, c, b) be defined analogously.

We claim that there is a leg of the cross with feet f, g such that $f \in (d, a, b)$ and $g \in (d, c, b)$. Since the cross is free, there exists a leg with foot in (d, a, b). We may assume the other foot of this leg does not belong to (d, c, b), but then the other leg of the cross satisfies the claim.

Choose a leg as above such that there is no foot between f and a, and no foot between g and c. (Such a choice must be possible, due to the freeness of the cross.) Let the other leg of the cross have feet h, i, such that b and h are joined by a subpath of the cycle C' that is disjoint from $\{f, g\}$. By contracting disjoint subpaths of C' with ends (a, f), (c, g), and (b, h) respectively, it follows that H has a minor isomorphic to the graph G' obtained from G by adding a new vertex z adjacent to u_1, u_2, u_3 and w.

If w is not confluent with the edge u_1u_2 then $G' \backslash u_1u_2 \backslash zu_3$ is isomorphic to a weak 9-enlargement of G, and we are done. Thus we may assume that w is confluent with the edge u_1u_2, and by symmetry, with the edges u_2u_3 and u_1u_3 as well. It follows similarly as in the proof of Lemma 5.1 that $G[u_1, u_2, u_3, w]$ is a detached K_4 subgraph of G. Since G is connected, and $|V(G)| \geq 5$, we may assume, without loss of generality, that u_1 has a neighbor in G outside that set. It follows that a graph obtained from G by a non-conforming split of u_1 is isomorphic to a minor of H. $\square$

We now define the weaker form of 8-enlargements. Let G be a graph with a cycle double cover $\mathcal{D}$, and let x_1, x_2, x_3 be vertices of G such that no disks contains all three. Let G^+ be obtained from G by adding a vertex with neighborhood $\{x_1, x_2, x_3\}$. We say that G^+ is a weak 8-enlargement of G. Our third lemma deals with the third outcome of Theorem 3.1.

Lemma 5.3. *Let G, H be graphs such that G is connected, has at least five vertices and no vertices of degree two. Let $\mathcal{D}$ be a weak disk system in G, let S be a subgraph of H isomorphic to a subdivision of G, and let there exist an S-triad in H. Then H has a minor isomorphic to an i-enlargement of G for $i = 1$ or 3, or a weak i-enlargement of G for $i = 8$ or 9.*

Proof: We proceed by induction of $|E(H)|$. Let the S-triad be L_1, L_2, L_3, and let its feet be x_1, x_2, x_3. If each x_i is a branch-vertex of S, then $S \cup L_1 \cup L_2 \cup L_3$ gives rise to a minor of H isomorphic to a weak 8-enlargement, as desired. We may therefore assume that x_3 is an internal vertex of a segment Q_3 of S with ends u_3 and v_3. Let f be an edge of Q_3. By induction applied to G, H/f, and S/f, we may assume that f is incident with x_3 and one end of Q_3, say u_3, and that there exists a disk D_1 in G containing x_1, x_2, u_3. Similarly, we may assume that there exists a disk D_2 in G containing x_1, x_2, v_3. Then $D_1 \neq D_2$, because otherwise $D_1 = D_2$ includes the segment Q_3 by (D3), and hence each of x_1, x_2, x_3, a contradiction. Since x_1 and x_2 belong to $D_1 \cap D_2$, it follows from (D3) that x_1 and x_2 belong to a common segment Q of S.

Let S' be obtained from S by replacing $Q[x_1, x_2]$ by $L_1 \cup L_2$. Applying Lemma 5.1 to G, H, S' and the S'-path L_3, the lemma now follows. □

Using Theorem 3.1 we can summarize Lemmas 5.1–5.3 as follows.

Lemma 5.4. *Let G, H be weakly 4-connected graphs, let G have a disk system $\mathcal{D}$ with no locally planar extension into H, and let a subdivision of G be isomorphic to a subgraph of H. Then H has a minor isomorphic to*

(i) an i-enlargement of G for some $i \in \{1, 2, 3\}$, or

(ii) a weak i-enlargement of G for some $i \in \{8, 9\}$.

Proof: Let G, H and $\mathcal{D}$ be as stated. By Theorem 3.1 we deduce that there exists a subgraph S of H isomorphic to a subdivision of G such that the induced disk system in S satisfies one of the outcomes of Theorem 3.1. But then (i) or (ii) of this lemma holds by Lemmas 5.1–5.3. □

6 From Weak Enlargements to Enlargements

The purpose of this section is to replace weak enlargements by enlargements in Lemma 5.4(ii). We start with a special case of weak 9-enlargements.

Lemma 6.1. *Let G be a graph with a cycle double cover $\mathcal{D}$, let G^+ be a weak 9-enlargement of G, and let u, x, y be as in the definition of weak 9-enlargement. If $G \backslash \{u, x, y\}$ is connected, then G^+ has a minor isomorphic to an i-enlargement of G for some $i \in \{1, 3, 9\}$.*

Proof: Let z be the new vertex of G^+ that resulted from the subdivision of the edge xy. If u and x are not confluent, then contracting the edge xz of G^+ produces a 1-enlargement of G, and so the lemma holds. Thus we may assume that u and x are confluent, and, by symmetry, we may assume that u and y are confluent. If u is not adjacent to x or y, then G^+ is a 9-enlargement of G, and the lemma holds. Thus, from the symmetry, we may assume that u is adjacent to x. The edges xy and xu are not confluent, for otherwise u is confluent with the edge xy, contrary to what a weak 9-enlargement stipulates. But then deleting the edge xu from G^+ yields a graph isomorphic to a 3-enlargement of G — more specifically, a graph obtained by a non-conforming split of the vertex x. □

Lemma 6.2. *Let G be an expansion of a weakly 4-connected graph, let H be a weakly 4-connected graph, let $\mathcal{D}$ be a weak disk system in G, let v be a vertex of G of degree three and let u, x, y be the neighbors of v. Let G^+ be the graph obtained from G by adding a new vertex z adjacent to u, x, y and deleting all edges with both ends in $\{u, x, y\}$. If H has a minor isomorphic to G^+, then H has a minor isomorphic to an i-enlargement of G for some $i \in \{1, 2, 3, 4, 9, 10\}$.*

Proof: Since G is an expansion of a weakly 4-connected graph, Lemma 4.2 implies that at most one edge of G has both ends in $\{u, x, y\}$. Thus we may assume that u is not adjacent to x or y. Since v has degree three, it follows from (D1) and (D3) that if x is adjacent to y, then the triangle vxy is a disk in G. We can apply Lemma 4.6 to $G^+ = G_1$ and H, with $P = \{v, z\}$, $Q = \{u, x, y\}$ and $R = V(G^+) - (P \cup Q)$. From the lemma, using the symmetry between x and y, and the symmetry among x, y and u if x is not adjacent to y, we get the following three cases:

Case 1: H has a minor isomorphic to a graph G^{++} that is obtained from G^+ by adding an edge between a vertex $p \in P$ and a vertex $r \in R$. Note that the vertices v and z are symmetric for the application of Lemma 4.6. Hence we may assume that $p = v$. Now if r is not confluent with v in G, then G^{++} above has a minor isomorphic to a 1-enlargement of G. Thus we may assume that r is confluent with v in G. Furthermore, we may assume, without loss of generality, that the disk D_3 shared by r and v contains the edges vu and vy. (Note that v has degree 3 in G.) On the disk D_3, the vertices u, v, y and r occur in that cyclic order. Now in G^{++}, contracting the edge yz gives a cross in the disk D_3 with arms uy and rv. In other words, G^{++}, and hence H, has a minor isomorphic to a 2-enlargement of G, as desired.

Case 2: The vertices x and y are adjacent in G and H has a minor isomorphic to a graph G^{++} that is obtained from G^+ by splitting the vertex x into x_1 and x_2, with x_1 adjacent to v and x_2 adjacent to z. Let N_i be the neighbors of x_i in G^{++} other than v, z, x_1, x_2. The neighborhood of x in G is thus $N_1 \cup N_2 \cup \{v, y\}$. In G, let D_4 be the disk that contains the edge xy, other than the triangle vxy. The disk D_4 must contain a vertex in either N_1 or N_2, and from the symmetry between v and z we may assume that it contains a vertex in N_1. Then, in G^{++}, delete the edge uz and contract the edge x_2z. This gives a graph that is a 3-enlargement of G (non-conforming split of x, with the disks vxy and D_4 violating condition (S2) in the definition of a conforming split), as desired.

Case 3: H has a minor isomorphic to a graph G^{++} that is obtained from G^+ by splitting the vertex u into u_1 and u_2, with u_1 adjacent to v and u_2 adjacent to z. Let N_i be the set of neighbors of u_i other than v, z, u_1, u_2. Thus in G, the neighborhood of u is $N_1 \cup N_2 \cup \{v\}$.

Let D_1 be the disk in G shared by the edges xv and vu, and D_2 be the disk in G shared by the edges yv and vu. The disks D_1 and D_2 both contain exactly one vertex each from $N_1 \cup N_2$. Let us assume first that $|N_2| \geq 2$. Contract the edge xz in G^{++}, and if x is not adjacent to y in G, then delete also the resulting edge xy to obtain a graph G_1, and let G_2 be the graph obtained from G_1 by further deleting the edge u_2x. Now G_2 is isomorphic to a graph obtained from G by splitting the vertex u into u_1 and u_2. If this split is non-conforming, then G_2 is a 3-enlargement of G, and we are done. Otherwise, the split is not along D_1 or D_2, and from the symmetry we may assume it is not along D_1. Thus G_1 is a 4-enlargement of G. (Note that in G, u and x are non-adjacent, and hence non-consecutive on D_1.) This completes the case when $|N_2| \geq 2$.

From the symmetry we may therefore assume that $|N_1| = |N_2| = 1$. Thus the degree of u in G is three. For $i = 1, 2$ let $N_i = \{n_i\}$. We may assume that the edge un_i belongs to the disk D_i. It follows that the vertex x and edge un_2 are not confluent in G, for if some disk D contained both of them, then the intersection $D \cap D_1$ would violate (D3), because u is not adjacent to x. The graph G_1 from the previous paragraph is a weak 9-enlargement of G, and so by Lemma 6.1 we may assume that $G \backslash \{x, u, n_2\}$ is disconnected. Since u has degree three, the weak 4-connectivity of G implies that n_1 has degree three and its neighbors are x, u, n_2. Since $G \backslash \{n_2, y\}$ is connected, we deduce that G is isomorphic to the prism, and G^{++} is isomorphic to a 10-enlargement of G, as desired. □

Now we are ready to eliminate weak 9-enlargements.

Lemma 6.3. *Let G be an expansion of a weakly 4-connected graph, let $\mathcal{D}$ be a weak disk system in G, and let G^+ be a weak 9-enlargement of G such that G^+ is isomorphic to a minor of a weakly 4-connected graph H. Then H has a minor isomorphic to an i-enlargement of G for some $i \in \{1, 2, 3, 4, 9, 10\}$.*

Proof: Let u, x, y be as in the definition of weak 9-enlargement. By Lemma 6.1 we may assume that $G \backslash \{u, x, y\}$ is disconnected. Since x is adjacent to y and G is an expansion of a weakly 4-connected graph, Lemma 4.2 implies that the neighborhood of some vertex v of G is precisely the set $\{u, x, y\}$. Thus G^+ is as described in Lemma 6.2, and the conclusion follows from that lemma. □

We now turn to weak 8-enlargements. In order to save effort we prove a weaker analogue of Lemma 6.3, the following.

Lemma 6.4. *Let G_1 be an expansion of a weakly 4-connected graph G, let $\mathcal{D}$ be a weak disk system in G, and let G^+ be a weak 8-enlargement of G_1 such that G^+ is isomorphic to a minor of a weakly 4-connected graph H. Then there exists an expansion G_2 of G obtained from G_1 by contracting a possibly empty set of new edges such that H has a minor isomorphic to an i-enlargement of G_2 for some $i \in \{1, 2, 3, 4, 8, 9, 10\}$.*

Proof: We proceed by induction on $|E(G_1)|$. Let G^+ be obtained from G_1 by adding a vertex joined to v_1, v_2, v_3. If some edge of G_1 has both ends in the set $\{v_1, v_2, v_3\}$, then by deleting that edge we obtain a graph isomorphic to a weak 9-enlargement of G_1, and the lemma follows from Lemma 6.3. Thus we may assume that $\{v_1, v_2, v_3\}$ is an independent set in G_1. We may also assume that every pair of vertices in $\{v_1, v_2, v_3\}$ is confluent, for otherwise G^+ has a minor isomorphic to a 1-enlargement of G, and the lemma holds. Thus we may assume that $G\backslash\{v_1, v_2, v_3\}$ is disconnected, for otherwise G^+ is an 8-enlargement of G_1.

Let (A, B) be a non-trivial separation of G with $A \cap B = \{v_1, v_2, v_3\}$. By Lemma 4.2 we may assume that (A, B) is degenerate. If $|A - B| = 1$, then the lemma follows from Lemma 6.2. Thus we may assume that $|A - B| \geq 2$. Let $v_1, v_2, v_3, u_1, u_2, u_3$ be as in the definition of degenerate. Since $\{v_1, v_2, v_3\}$ is independent, we may assume from the symmetry that $u_1 \neq v_1$ and $u_2 \neq v_2$. Now one of u_1v_1, u_2v_2 is a new edge of G_1, and so we may assume the former is. Thus G^+/u_1v_1 is a weak 8-enlargement of G_1/u_1v_1, and hence the lemma follows by the induction hypothesis applied to the graph G_1/u_1v_1. □

The lemmas of this section allow us to upgrade Lemma 5.4 to the following.

Lemma 6.5. *Let G, H be weakly 4-connected graphs, let G have a disk system $\mathcal{D}$ with no locally planar extension into H, and let G' be a conforming expansion of G such that a subdivision of G' is isomorphic to a subgraph of H. Then there exists a conforming expansion G'' of G obtained from G' by contracting a possibly empty set of new edges such that, letting $\mathcal{D}''$ denote the weak disk system induced in G'' by $\mathcal{D}$, the graph H has a minor isomorphic to an i-enlargement of G'' with respect to $\mathcal{D}''$ for some $i \in \{1, 2, 3, 4, 8, 9, 10\}$.*

Proof: By Lemma 5.4 we may assume that a weak 8-enlargement or a weak 9-enlargement of G' is isomorphic to a minor of H. By Lemmas 6.3 and 6.4 there exists a required conforming expansion G'' of G such that H has a minor isomorphic to an i-enlargement of G'' for some $i \in \{1, 2, 3, 4, 8, 9, 10\}$. □

7 Proof of the Main Theorem

Lemma 6.5 gives an i-enlargement of an expansion G'' of G. Our final objective is to show that we can choose $G'' = G$. We break the proof into several lemmas depending on the value of i.

Lemma 7.1. *Let G and H be weakly 4-connected graphs, and let $\mathcal{D}$ be a weak disk system in G with no locally planar extension into H. Let G' be a conforming expansion of G such that H has a minor isomorphic to a 1-enlargement of G'. Then H has a minor isomorphic to an i-enlargement of G for some $i \in \{1, 3, 4, 5\}$.*

Proof: We may assume that G' is as stated in the lemma, and subject to that, it is minor-minimal. By hypothesis, H has a minor isomorphic to G^+, a graph obtained from G' by adding an edge between two vertices x and y that are not confluent. Let e be a new edge of G'. By the minimality of G', it follows that

(i) one end of e must be in $\{x, y\}$, and

(ii) the other end of e must be confluent with the vertex in $\{x, y\}$ other than the one above.

Recall that branch-sets of an expansion were defined at the beginning of Section 4. Thus all branch sets that are disjoint from $\{x, y\}$ are singleton sets. Let T_p and T_q be the branch sets corresponding to vertices $p, q \in V(G)$ such that they contain x and y respectively (p and q may be identical). We claim that the degree of x in the branch set containing it is at most one (that is, x is a leaf of the tree $G'[T_p]$). Suppose not; hence x has (at least) two neighbors x_1 and x_2 in T_p. By (ii) above, y shares disks D_1 and D_2 of G' with x_1 and x_2 respectively. Then $x \notin V(D_1 \cup D_2)$, for x, y are not confluent. It follows that $D_1 \neq D_2$, for otherwise D_1 is not a cycle in $G'/x_1x/x_2x$, and yet D_1 corresponds to a disk in G. Also, y is not adjacent to both x_1 and x_2, by Lemma 4.1. But then contracting edges xx_1 and xx_2 violates Axiom (D3) in G. This proves the claim. Thus x, and by symmetry y, are leaf vertices in $G'[T_p]$ and $G'[T_q]$ respectively.

If $p = q$, then it follows that $T_p = T_q$ must be a path of length 2, with a middle vertex z. Let $D'_1, \mathcal{D}'_2$ be the two disks in G' that include the edge xz, and let D'_3, D'_4 be the two disks that include the edge yz. Note that, since x and y are not confluent in G', all four disks above are distinct. Let D_1, D_2, D_3, D_4 be the corresponding disks in G. Let N_1, N_2 be the partition of the set of neighbors of p in G, corresponding to the partition $\{x, y\}, \{z\}$ of $V(T_p)$. Clearly, N_1 has at least two vertices, but so does N_2, by Axiom (D3) applied to $\widetilde{D}_1, \widetilde{D}_2$. In G^+ (which has the edge xy), contract the edge xy. This gives a graph G^{++} that can be obtained from G by splitting p with respect to the partition N_1, N_2 of its neighbors. This split is non-conforming, since the disks $D_1, \ldots, D_4$ violate condition (S1) in the definition of a conforming split. Thus G^{++} is a 3-enlargement of G, as desired.

If $p \neq q$, then from (i) and (ii) above, T_p is either $\{x\}$ or $\{x, x_1\}$. By symmetry, T_q is either $\{y\}$ or $\{y, y_1\}$. If T_p and T_q are both singletons, then clearly $G' = G$ and we are done.

Suppose exactly one of the two branch sets, say T_q, is a singleton, and T_p consists of $\{x, x_1\}$, where x_1 shares a disk D with y in G'. If x_1 and y are not adjacent, then G^+ is a 4-enlargement of G, and we are done. Thus we may assume that x_1 and y are adjacent, and hence by Axiom (D3), they are consecutive in D. Let D_1, D_2 be the two disks in G' containing the edge xx_1. They are both distinct from D, since x and y are not confluent in G'. By Axiom (D3) applied to D_1 and D_2, the vertex x_1 has at least two neighbors in G' other x and y. Now in G^+ (which contains the edge xy), delete the edge x_1y. This gives a graph $\widetilde{G}$ obtained from G by splitting p in the same way as in G', except that y is adjacent to x rather than x_1. Further, it is a non-conforming split, as the disks D, D_1 and D_2 violate condition (S1) in the definition of a conforming split. Thus $\widetilde{G}$, which is isomorphic to a minor of H, is a 3-enlargement of G, and we are done.

Finally, suppose $T_p = \{x, x_1\}$ and $T_q = \{y, y_1\}$, where x shares a disk D_1' with y_1 and y shares a disk D_1' with x_1. Let D_1, D_2 be the corresponding disks in G. Since x and y are not confluent in G', D_1' does not contain y and D_2' does not contain x. (In particular, D_1' and D_2' are distinct.) Apply Lemma 4.5 to $\widehat{G} = G'/xx_1$, with the vertices p, y, y_1 in that graph corresponding to p, q, r in the lemma. Thus the (conforming) split of the vertex q in G that produces $\widehat{G}$ is along D_2, and D_2 is one of the disks containing the edge pq in G. Also, since x and y are not confluent in G', the (conforming) split of p in $\widehat{G}$ that produces G' must be along D_1, and D_1 is the other disk in G containing pq. It now follows that G^+ is a 5-enlargement of G. This finishes the proof of the lemma. □

Lemma 7.2. *Let G and H be weakly 4-connected graphs, and let $\mathcal{D}$ be a weak disk system in G with no locally planar extension into H. Let G' be a conforming expansion of G such that H has a minor isomorphic to a 2-enlargement of G'. If $G' \neq G$, then there exists a conforming expansion G'' of G obtained from G' by contracting at least one new edge such that H has a minor isomorphic to an i-enlargement or a weak 9-enlargement of G'' for some $i \in \{2, 6, 7\}$.*

Proof: We may assume that G' is as stated in the lemma, and subject to that, it is minor-minimal. By hypothesis, there are vertices u, v, x, y appearing on a disk C' in G', in that cyclic order, such that H has a minor isomorphic to a graph obtained from G' by adding the edges ux and vy. Let C be the cycle in G corresponding to C'. The minimality of G' implies that every new edge of G' has both ends in $\{u, v, x, y\}$, and hence it belongs to C' by (D3). We may therefore assume that uv is a new edge of G'. We claim that if v is adjacent to x, then the lemma holds. To prove this claim suppose that v and x are adjacent in G', and let $G_1 = G^+ \backslash vx$. If v has degree three in G', then G_1 is isomorphic to a weak 9-enlargement of G'/uv (the new edge is yv; notice that y

is not confluent with the edge of G'/uv that is being subdivided by (D3)), and hence the lemma holds. Thus we may assume that v has degree at least four in G'. In that case G_1 is isomorphic to a 4-enlargement of G'/uv, for a graph isomorphic to G_1 can be obtained by a conforming split of the new vertex of G'/uv, not along C', and joining one of the new vertices to y. This proves our claim, and hence we may assume that v is not adjacent to x. By symmetry we may also assume that u is not adjacent to y.

If uv is the only new edge of G', then G' is a 6-enlargement of G, and the lemma holds. Thus we may assume that G' has another new edge, and so that edge must be xy and there are no other new edges. It follows that G' is a 7-enlargement of G, and so the lemma holds. $\square$

Lemma 7.3. *Let G and H be graphs, let $\mathcal{D}$ be a weak disk system in G, and let G' be a conforming expansion of G such that H has a minor isomorphic to a 9-enlargement G^+ of G'. If $G' \neq G$, then there exists a conforming expansion G'' obtained from G' by contracting at least one new edge such that H has a minor isomorphic to a 3-enlargement or a weak 9-enlargement of G''.*

Proof: Let $u, x, y \in V(G')$ be such that G^+ is obtained from G' by subdividing the edge xy and joining the new vertex to u, and let f be a new edge of G'. Then $f \neq xy$, for otherwise Lemma 4.5 implies that u is confluent with the edge xy, a contradiction. We may assume that f is incident with u, and that contracting f makes the new vertex confluent with the edge xy, for otherwise G^+/f is a weak 9-enlargement of G'/f, and the lemma holds. Hence the other end v of f must share a disk D_1 with the edge xy. Since u is not confluent with xy, D_1 does not contain u. Let D_2 and D_3 be disks shared by u and x, and by u and y, respectively. These three disks are pairwise distinct, since u is not confluent with the edge xy in G'. Now apply Lemma 4.5 with x as the vertex p, and u, v as the vertices q, r respectively. It follows that v and x are adjacent, and that D_1 and D_2 are the two disks containing the edge vx. Apply Lemma 4.5 again, this time with y in place of x. It follows that the edges vu, vx and vy are covered twice each by the three disks D_1, D_2 and D_3. In particular, D_1 is a triangle.

If $f' \neq f$ is a new edge of G', then by what we have shown about f it follows that f' is incident with u and its other end belongs to a disk D_1' that contains the edge xy. Since D_1 is a triangle consisting of x, y and an end of f, we see that $D_1' \neq D_1$. But the disks that correspond to D_1 and D_1' in $G'/f/f'$ have three vertices in common, contrary to (D3). Thus f is the only new edge of G', and hence $G = G'/f$. Let p be the new vertex of $G = G'/f$.

Since G^+ is a 9-enlargement of G', the graph $G' \backslash \{u, x, y\}$ is connected, and hence v has a neighbor outside $\{u, x, y\}$. (In fact, it must then have at least three neighbors outside $\{u, x, y\}$.) Let z be the new vertex of G^+ created by subdividing the edge xy. The graph $G^+ \backslash vx/xz$ is isomorphic to a graph obtained from G by splitting p into two vertices. This split is non-conforming, since the two disks in G that contain py violate condition (S2) in the definition of a conforming split. Thus H has a minor isomorphic to a 3-enlargement of G. This finishes the proof of the lemma. $\square$

Lemma 7.4. *Let G and H be graphs, let $\mathcal{D}$ be a disk system in G, and let G' be a conforming expansion of G such that H has a minor isomorphic to an 8-enlargement G^+ of G'. If $G' \neq G$, then there exists a conforming expansion G'' obtained from G' by contracting at least one new edge such that H has a minor isomorphic to a 3-enlargement or a weak 8-enlargement of G''.*

Proof: Let G^+ be obtained from G' by adding a vertex adjacent to x_1, x_2, x_3, and let f be a new edge of G'. We may assume that upon contracting f the vertices that correspond to x_1, x_2, x_3 belong to a common disk, for otherwise G^+/f is a weak 8-enlargement of G'/f, and the lemma holds. Thus f is incident with at least one of x_1, x_2, x_3, say x_1, and there exists a disk $\mathcal{D}$ in G' that includes y, x_2, x_3, where y is the other end of f.

Apply Lemma 4.5 twice, once with x_2 as the vertex p, and next with x_3 as the vertex p. In both applications, let x_1 and y be the vertices q and r respectively. It follows that y is adjacent to x_2 and x_3, and that $yx_1 \in E(D_2 \cap D_3)$, $yx_2 \in E(D \cap D_3)$ and $yx_3 \in E(D \cap D_2)$. Since G^+ is a 8-enlargement of G' the graph $G' \backslash \{x_1, x_2, x_3\}$ is connected, and hence y has degree at least four. Let N be the neighbors of y in G' other than x_1, x_2, x_3. Let G' be obtained from G by splitting x_1 in such a way that the neighborhood of one of the new vertices is N. Then G' is isomorphic to a minor of G^+, and it is a 3-enlargement of G'. Thus the lemma follows from Lemma 4.4. $\square$

We are finally ready to state and prove Theorem 2.1, which we restate.

Theorem 7.5. *Let G and H be weakly 4-connected graphs such that H has a minor isomorphic to G. Let G have a disk system $\mathcal{D}$ that has no locally planar extension into H. Then H has a minor isomorphic to an i-enlargement of G, for some $i \in \{1, 2, \ldots, 10\}$.*

Proof: There exists an expansion of G whose subdivision is isomorphic to a subgraph of H. If this expansion is not conforming, then the theorem holds by Lemma 4.4, and so we may assume that the expansion is conforming. By Lemma 6.5 there exists a conforming expansion G' of G such that H has a minor isomorphic to an i-enlargement G^+ of G' for some $i \in \{1, 2, 3, 4, 8, 9, 10\}$. We may choose G' and G^+ such that $|E(G')|$ is minimum. If $i \in \{1, 4\}$, then G^+ is isomorphic to a 1-enlargement of a conforming expansion of G', and the theorem holds by Lemma 7.1. If $i = 3$, then the theorem holds by Lemma 4.4. If $i = 10$, then the minimality of G' implies that $G = G'$, and if $i \in \{2, 8, 9\}$, then the same conclusion follows from Lemmas 7.2, 7.4 and 7.3, respectively, using Lemmas 6.3 and 6.4. Thus the theorem holds. $\square$

8 An Application

In this section, we illustrate an application of Theorem 2.1. Archdeacon [1, 2] proved that a graph H does not embed in the projective plane if and only if it has a minor isomorphic to some graph in an explicitly constructed list of 35 graphs. One might hope that if we assume that H is sufficiently connected, then the list may be shortened. Mohar and Thomas (work in progress) developed a strategy for a proof, but it will be a lengthy project with several intermediate steps. Here we complete one such step: under the assumptions that H is weakly 4-connected and has a minor isomorphic to the Petersen graph, Theorem 8.1 below gives a list of eight forbidden minors, each of which are weakly 4-connected.

Figure 1 shows these eight graphs (with a vertex-labeling for each of them). All of these graphs, with the exception of F_1' and D_3', appear in the list of 35 forbidden minors for the projective plane. F_1' and D_3', however, are obtained from two graphs in that list (F_1 and D_3, respectively) by splitting exactly one vertex. (The reason we list F_1', D_3' instead of F_1, D_3 is that the latter two graphs are not weakly 4-connected.)

Theorem 8.1. *Let H be a weakly 4-connected graph that has a minor isomorphic to the Petersen graph. Then H does not embed in the projective plane if and only if it has a minor isomorphic to one of the eight graphs F_1', F_4, D_3', E_{22}, E_{20}, C_3, E_2, or E_{18} shown in Figure 1.*

Before we derive Theorem 8.1 from Theorem 2.1, we describe some notation that will be convenient in the proof.

Let P_{10} denote a labeling of the Petersen graph as shown in Figure 2. In fact, Figure 2 shows an embedding of P_{10} in the projective plane. The disk system $\mathcal{D}$ associated with this embedding consists of the 5-cycles 6-9-7-10-8, 1-5-10-7-2, 4-3-8-10-5, 2-1-6-8-3, 5-4-9-6-1, and 3-2-7-9-4.

P_{10} has exactly one other embedding in the projective plane. This embedding is distinct from the above embedding, but is isomorphic to it. (An isomorphism of embeddings is an isomorphism τ of the underlying graphs such that a cycle C is facial in one embedding if and only if $\tau(C)$ is facial in the other.) The disk system $\mathcal{D}'$ associated with the second embedding consists of the 5-cycles 1-2-3-4-5, 6-9-4-3-8, 7-10-5-4-9, 8-6-1-5-10, 9-7-2-1-6, and 10-8-3-2-7.

We now describe notation that will let us denote specific enlargements of a (labeled) graph as given by Theorem 2.1. Recall the operations 1–9 and the definition of a split, as described in Sections 1 and 2.

Let G be a graph whose vertices are labeled $1, \dots, n$. For vertices u, v, the graph $G+(u, v)$ denotes the graph obtained from G by adding an edge joining u and v (if none existed before). Also, the graph $G * v(N_1)$ denotes the graph obtained by splitting the vertex v, where N_1 is as in

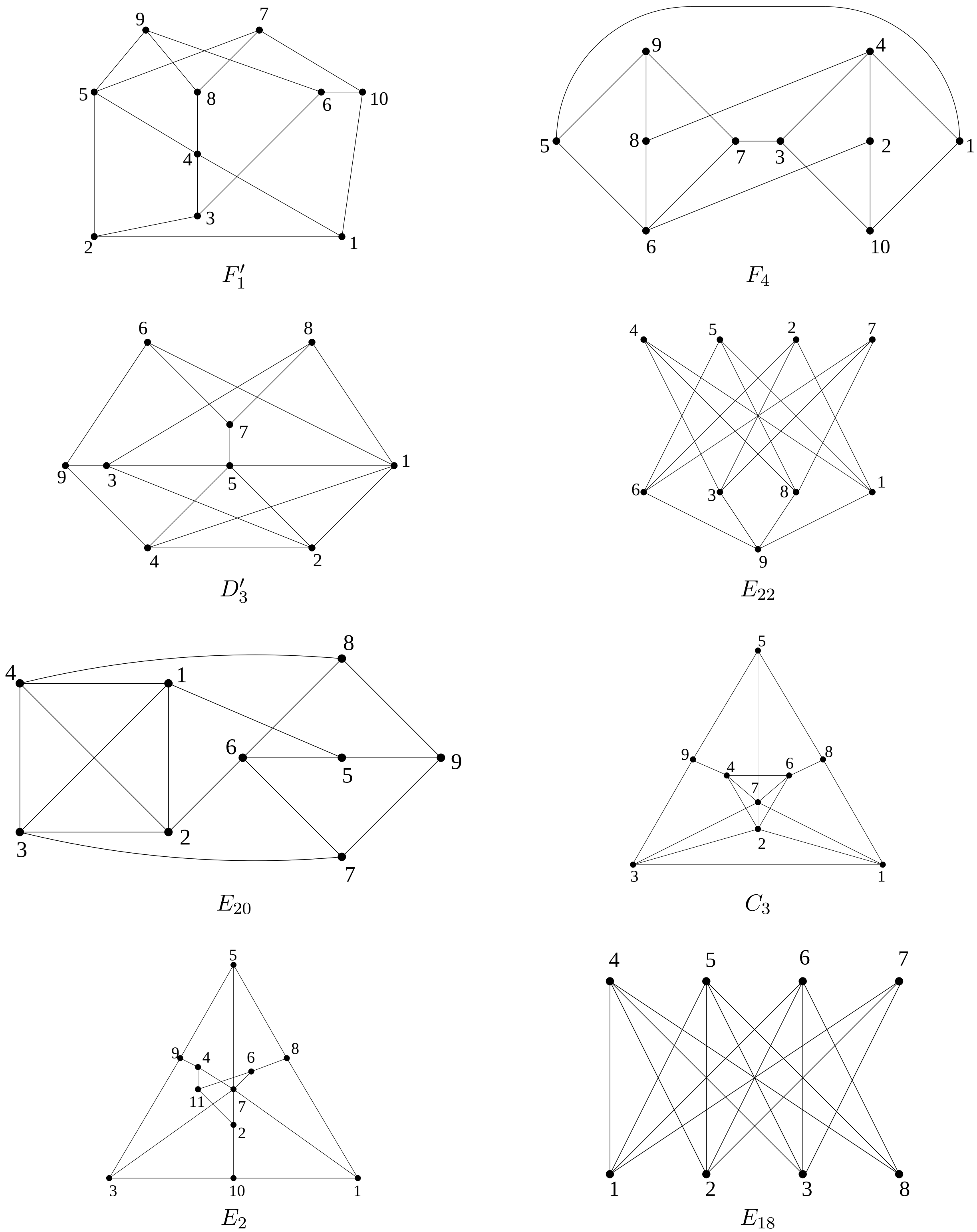

Figure 1: The eight graphs of Theorem 8.1

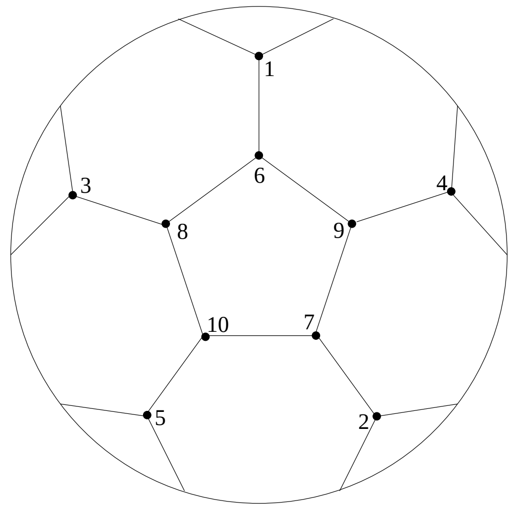

Figure 2: One of the two projective-planar embeddings of the Petersen graph

the definition of a split. We follow the convention that the vertex v_1 retains the same label as v, while v_2 is assigned the label $n+1$.

Since operations 1–7 are defined in terms of vertex splits and edge additions, the above notation lets us specify i-enlargements for $i = 1, \ldots, 7$. An 8-enlargement of G is specified as $G+(x_1, x_2, x_3)$, where the vertices x_i are as in the definition of operation 8. The new vertex x gets the label $n+1$.

Finally, a 9-enlargement of G is specified as $G+(u, x-y)$, where u, x, y are as in the definition of operation 9. The new vertex obtained by subdividing the edge xy gets the label $n+1$.

8.1 Proof of Theorem 8.1.

For the backward implication of Theorem 8.1, recall that each of the eight graphs specified is either isomorphic to one of the 35 forbidden minors of [2] or is obtained from one of them by splitting a vertex. In particular, none of these eight graphs embed in the projective plane, and so H does not embed either.

For the forward implication, H, by hypothesis, does not embed in the projective plane, and has a minor isomorphic to P_{10}. Clearly, the disk system $\mathcal{D}$ of P_{10} has no locally planar extension to H. Applying Theorem 2.1 to $P_{10}, \mathcal{D}$ and H, it is easy to check that H has a minor isomorphic to one of three enlargements, up to isomorphism:

1. a 2-enlargement $Q_1 = P_{10} + (7,8) + (9,10)$
2. an 8-enlargement $Q_2 = P_{10} + (2,4,6)$
3. a 9-enlargement $Q_3 = P_{10} + (1, 3-4)$

Q_2 has a minor isomorphic to E_{18}, as witnessed by the branch sets $\{1, 5\}$, $\{3, 8\}$, $\{7, 9\}$, $\{2\}$, $\{4\}$, $\{6\}$, $\{10\}$, and $\{11\}$. (The order of the branch sets follows that of the corresponding vertex labels in E_{18}, as shown in Figure 1.)

Thus we may assume that H has a minor isomorphic to Q_1 or Q_3. The disk system $\mathcal{D}'$ of P_{10} extends in a natural way to disk systems $\mathcal{D}_1, \mathcal{D}_3$ in the enlargements Q_1, Q_3. Thus Q_1, Q_3 each embed (*uniquely*) in the projective plane. The embeddings are shown in Figure 3.

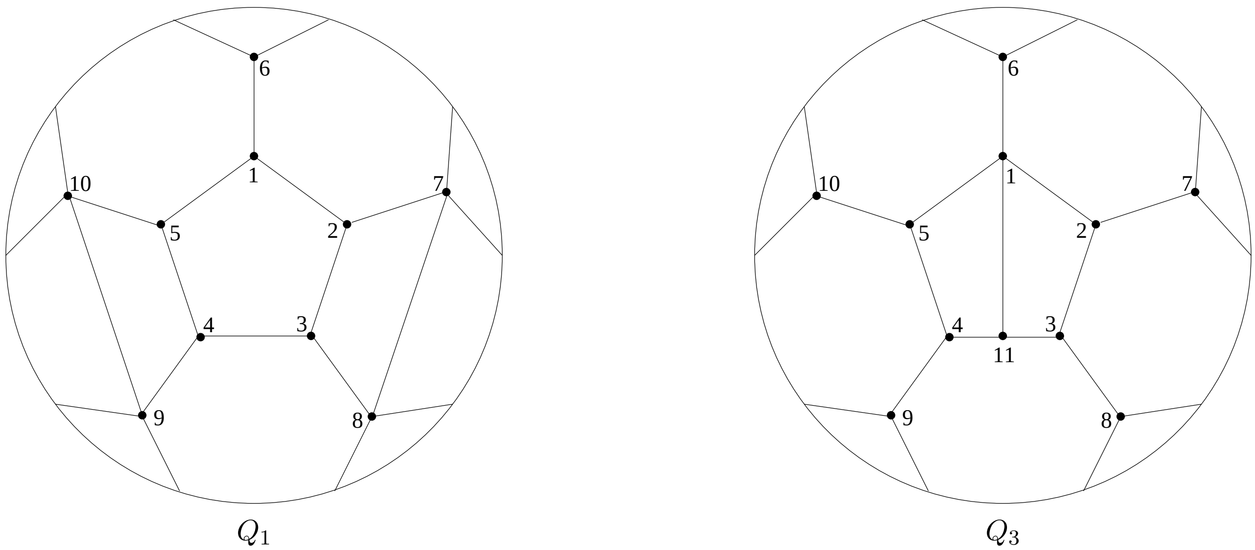

Figure 3: The graphs Q_1 and Q_3

We now apply Theorem 2.1 to $Q_1, \mathcal{D}_1, H$ and $Q_3, \mathcal{D}_3, H$ and deduce Theorem 8.1. This involves a fair amount of case-checking, which is summarized in Tables 1 and 2. Each row in the tables lists an enlargement of Q_1 or Q_3, along with one of the eight graphs from the list that is a minor of the enlargement. The branch sets in the rightmost column follow the order of the vertex labels of the corresponding graph in the preceding column. For clarity, singleton sets are not enclosed in braces.

Tables 1 and 2 respectively list all possible enlargements of Q_1 and Q_3 up to isomorphism, with the exception of 8-enlargements and 9-enlargements of Q_1, and 8-enlargements of Q_3. Every 8-enlargement of Q_1 with respect to $\mathcal{D}_1$ has a subgraph isomorphic to Q_2, and thus has a minor isomorphic to E_{18}. Every 8-enlargement of Q_3 with respect to $\mathcal{D}_3$ either has a minor isomorphic to Q_2, or is isomorphic to the 8-enlargement listed in Table 2. Finally, every 9-enlargement of Q_1 with respect to $\mathcal{D}_1$ is either isomorphic to the 9-enlargement listed in Table 1 or is isomorphic to a 2-enlargement of Q_3 with respect to $\mathcal{D}_3$ (and is thus listed in Table 2 instead). This finishes the proof of Theorem 8.1. □

Table 1: Applying Theorem 2.1 to Q_1

Type	Enlargement	Minor	Branch sets of the minor
1	$Q_1+(2,10)$	D_3'	$\{2,3\},7,9,8,10,1,5,4,6$
	$Q_1+(3,10)$	F_1'	$8,7,2,3,10,1,9,4,5,6$
2	$Q_1+(2,8)+(3,7)$	E_{20}	$2,7,3,8,\{1,6\},9,4,10,5$
	$Q_1+(2,4)+(3,5)$	E_{22}	$2,3,5,4,7,8,10,9,\{1,6\}$
	$Q_1+(1,4)+(3,5)$	F_4	$2,4,5,1,7,9,10,6,8,3$
	$Q_1+(1,4)+(2,5)$	F_4	$1,3,5,2,6,8,10,7,9,4$
	$Q_1+(3,9)+(4,8)$	C_3	$3,4,1,10,7,9,2,5,\{6,8\}$
	$Q_1+(3,9)+(4,6)$	C_3	$3,4,1,10,7,9,2,5,\{6,8\}$
	$Q_1+(4,6)+(8,9)$	E_{20}	$8,7,10,9,3,\{1,2\},5,6,4$
	$Q_1+(2,9)+(6,7)$	D_3'	$\{1,2\},7,10,6,9,3,4,5,8$
	$Q_1+(1,9)+(6,7)$	F_1'	$1,5,4,9,10,3,7,6,8,2$
	$Q_1+(1,9)+(2,6)$	F_4	$1,10,4,9,6,8,3,7,2,5$
	$Q_1+(1,7)+(2,9)$	D_3'	$9,7,8,2,\{1,6\},4,5,10,3$
	$Q_1+(1,7)+(2,6)$	C_3	$1,2,4,8,10,7,5,3,\{6,9\}$
3	$Q_1{*}7(2,10)$	F_1'	$\{1,6\},5,4,9,10,3,7,11,8,2$
	$Q_1{*}8(3,10)$	F_1'	$2,7,11,\{1,6\},9,8,4,5,10,3$
4	$Q_1{*}7(2,9)+(1,11)$	F_1'	$2,7,11,\{1,6\},9,8,4,5,10,3$
	$Q_1{*}7(2,9)+(6,11)$	F_1'	$3,4,5,\{8,10\},9,1,7,11,6,2$
	$Q_1{*}7(2,8)+(3,11)$	F_1'	$8,7,2,3,11,1,9,4,\{5,10\},6$
	$Q_1{*}8(3,7)+(2,11)$	F_1'	$\{5,10\},1,6,11,2,9,3,8,7,4$
	$Q_1{*}8(3,7)+(1,8)$	F_1'	$\{5,10\},11,6,1,8,9,3,2,7,4$
	$Q_1{*}8(3,7)+(5,8)$	F_1'	$\{1,2\},3,4,5,8,9,11,10,7,6$
	$Q_1{*}8(3,6)+(4,11)$	F_1'	$8,3,\{2,7\},11,4,1,9,10,5,6$
	$Q_1{*}8(3,6)+(9,11)$	E_{20}	$7,10,9,11,2,\{1,5,6\},4,8,3$
5	$Q_1{*}7(8,10){*}8(3,7)+(11,12)$	F_1'	$\{1,2\},6,9,11,12,\{3,4\},10,7,8,5$
	$Q_1{*}7(2,8){*}8(7,10)+(11,12)$	F_1'	$\{3,4\},9,6,12,11,\{1,2\},10,8,7,5$
	$Q_1{*}7(2,9){*}9(4,6)+(9,11)$	F_1'	$\{3,4\},8,6,9,11,\{1,2\},10,12,7,5$
	$Q_1{*}7(2,8){*}9(4,10)+(7,9)$	F_1'	$8,\{3,4\},5,10,9,\{1,2\},12,11,7,6$
	$Q_1{*}7(2,9){*}10(9,11)+(7,12)$	F_1'	$\{1,6\},2,3,\{8,11\},7,4,12,10,9,5$
	$Q_1{*}7(2,8){*}10(5,9)+(7,10)$	F_1'	$3,\{1,2\},6,8,7,9,10,12,11,\{4,5\}$
6	$Q_1{*}7(2,8)+(1,11)+(6,7)$	F_1'	$2,7,6,1,11,8,\{4,9\},5,10,3$
	$Q_1{*}8(3,7)+(4,11)+(8,9)$	F_1'	$\{1,6\},5,10,11,4,7,3,8,9,2$
	$Q_1{*}8(3,6)+(1,11)+(8,10)$	F_1'	$2,7,11,\{1,6\},9,8,4,5,10,3$
7	$Q_1{*}8(3,7){*}9(4,10)+(8,12)+(9,11)$	F_1'	$\{2,7\},10,5,\{1,6\},11,4,8,12,9,3$
9	$Q_1+(1,7-8)$	F_1'	$2,7,11,\{1,6\},9,8,4,5,10,3$

Table 2: Applying Theorem 2.1 to Q_3

Type	Enlargement	Minor	Branch sets of the minor
1	$Q_3{+}(2,4)$	F_1'	$1, 11, 3, 2, \{4, 5\}, 8, 9, 7, 10, 6$
	$Q_3{+}(2,5)$	F_1'	$1, 11, 4, 5, \{2, 3\}, 9, 8, 10, 7, 6$
2	$Q_3{+}(1,7){+}(2,6)$	D_3'	$\{3, 8, 11\}, 2, 7, 6, 1, 4, 5, 10, 9$
	$Q_3{+}(1,9){+}(2,6)$	F_1'	$9, 4, 5, 1, \{3, 11\}, 10, 2, 6, 8, 7$
	$Q_3{+}(1,9){+}(6,7)$	F_4	$5, 11, 9, 1, 10, \{3, 8\}, 6, 2, 7, 4$
	$Q_3{+}(2,9){+}(6,7)$	E_{18}	$1, \{3, 8\}, \{4, 9\}, 2, \{5, 10\}, 6, 11, 7$
	$Q_3{+}(1,7){+}(2,9)$	D_3'	$1, 2, \{3, 8\}, 7, \{6, 9\}, 5, 4, 11, 10$
	$Q_3{+}(2,8){+}(3,7)$	F_4	$11, 9, 5, \{1, 6\}, 3, 7, 10, 2, 8, 4$
	$Q_3{+}(2,10){+}(3,7)$	E_{22}	$1, 2, 3, 11, 5, 10, 7, \{4, 9\}, \{6, 8\}$
	$Q_3{+}(2,10){+}(7,8)$	F_4	$3, 7, 10, 8, 11, \{4, 9\}, 5, 6, 1, 2$
	$Q_3{+}(3,10){+}(7,8)$	F_4	$5, 11, 6, 1, 10, 3, 8, 2, 7, \{4, 9\}$
	$Q_3{+}(2,8){+}(3,10)$	F_1'	$2, 8, 6, 1, \{3, 11\}, 9, 10, 5, 4, 7$
	$Q_3{+}(3,9){+}(4,8)$	E_{22}	$\{1, 6\}, 2, 3, 11, 5, \{7, 10\}, 8, 4, 9$
	$Q_3{+}(3,9){+}(8,11)$	F_1'	$11, 4, 5, \{1, 6\}, 9, 10, 3, 2, 7, 8$
	$Q_3{+}(3,4){+}(8,11)$	D_3'	$\{4, 5\}, 11, 8, 1, \{2, 3\}, 9, 7, 10, 6$
	$Q_3{+}(6,11){+}(8,9)$	F_1'	$10, 7, 2, \{3, 8\}, 9, 1, 4, 11, 6, 5$
	$Q_3{+}(3,9){+}(6,11)$	F_1'	$10, 7, 2, \{3, 8\}, 9, 1, 4, 11, 6, 5$
	$Q_3{+}(3,4){+}(6,11)$	D_3'	$\{1, 2\}, 11, 4, 6, \{3, 8\}, 7, 10, 5, 9$
	$Q_3{+}(4,6){+}(8,9)$	F_4	$5, 11, 6, 1, 10, \{3, 8\}, 9, 2, 7, 4$
	$Q_3{+}(3,9){+}(4,6)$	F_4	$5, 11, 6, 1, 10, \{3, 8\}, 9, 2, 7, 4$
	$Q_3{+}(3,6){+}(8,11)$	E_{20}	$2, 1, 3, 11, 7, \{6, 9\}, 8, \{4, 5\}, 10$
	$Q_3{+}(1,3){+}(2,11)$	F_1'	$2, 3, 6, 1, 11, 9, \{8, 10\}, 5, 4, 7$
3	$Q_3{*}1(2,5)$	F_1'	$7, 10, 5, \{1, 2\}, \{3, 8\}, 4, 6, 12, 11, 9$
4	$Q_3{*}1(5,6){+}(10,12)$	F_1'	$12, 1, 5, 10, \{6, 8\}, 4, \{2, 3\}, 7, 9, 11$
	$Q_3{*}1(5,6){+}(8,12)$	F_1'	$9, \{1, 6\}, 5, \{4, 11\}, 12, 10, 2, 3, 8, 7$
	$Q_3{*}1(5,6){+}(1,3)$	F_1'	$10, 8, 6, \{1, 5\}, 3, \{4, 9\}, 2, 12, 11, 7$
6	$Q_3{*}1(5,6){+}(1,7){+}(9,12)$	F_1'	$8, 10, 5, \{1, 6\}, 7, 4, 2, 12, 9, \{3, 11\}$
8	$Q_3 + (2,9,11)$	F_4	$1, 12, 3, 2, \{4, 5\}, 9, \{6, 8\}, 7, 10, 11$
9	$Q_3{+}(8,1{-}11)$	F_1'	$1, 12, 11, \{2, 3\}, \{6, 8\}, 4, 10, 7, 9, 5$
	$Q_3{+}(8,1{-}2)$	F_1'	$9, 4, 5, \{1, 6\}, \{3, 11\}, 10, 2, 12, 8, 7$
	$Q_3{+}(10,1{-}2)$	F_4	$11, 5, 9, \{1, 6\}, \{3, 8\}, 10, 7, 12, 2, 4$
	$Q_3{+}(6,2{-}3)$	F_1'	$2, 12, 6, 1, \{3, 8, 11\}, 9, 10, 5, 4, 7$
	$Q_3{+}(9,2{-}3)$	F_1'	$5, 4, 11, \{1, 6\}, 9, \{3, 8\}, 7, 2, 12, 10$
	$Q_3{+}(3,1{-}6)$	F_4	$2, 11, 12, 1, 7, \{4, 9\}, \{6, 8\}, 5, 10, 3$
	$Q_3{+}(1,3{-}8)$	F_4	$2, 11, 12, 1, 7, \{4, 6, 9\}, 8, 5, 10, 3$
	$Q_3{+}(2,6{-}8)$	F_4	$3, 7, 12, \{8, 10\}, 11, \{4, 9\}, 6, 5, 1, 2$
	$Q_3{+}(7,6{-}8)$	F_4	$11, 9, 5, \{1, 6\}, \{2, 3\}, 7, 10, 12, 8, 4$
	$Q_3{+}(3,7{-}9)$	F_4	$5, 11, 9, \{1, 6\}, 10, \{3, 8\}, 12, 2, 7, 4$
	$Q_3{+}(8,7{-}9)$	F_4	$5, 11, 9, \{1, 6\}, 10, \{3, 8\}, 12, 2, 7, 4$
	$Q_3{+}(1,7{-}10)$	E_2	$2, 9, 12, 11, 5, 8, \{1, 6\}, 3, 7, 4, 10$
	$Q_3{+}(6,7{-}10)$	E_2	$2, 9, 12, 11, 5, 8, \{1, 6\}, 3, 7, 4, 10$

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