GIRTH SIX CUBIC GRAPHS
HAVE PETERSEN MINORS

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ABSTRACT

We prove that every 3-regular graph with no circuit of length less than six has a subgraph isomorphic to a subdivision of the Petersen graph.

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1. INTRODUCTION

All *graphs* in this paper are finite, and may have loops and parallel edges. A graph is *cubic* if the degree of every vertex (counting loops twice) is three. The *girth* of a graph is the length of its shortest circuit, or infinity if the graph has no circuits. (*Paths* and *circuits* have no "repeated" vertices.) The *Petersen graph* is the unique cubic graph of girth five on ten vertices. The Petersen graph is an obstruction to many properties in graph theory, and often is, or is conjectured to be, the *only* obstruction. Such is the case for instance in the following result of Alspach, Goddyn and Zhang [1]. Let G be a graph, and let $p : E(G) \to \mathbf{Z}$ be a mapping. We say that p is *admissible* if $p(e) \geq 0$ for every edge e of G, and for every edge-cut C, $\sum_{e \in C} p(e)$ is even and at least twice $p(f)$ for every edge $f \in C$. We say that a graph G is a *subdivision* of a graph H if G can be obtained from H by replacing the edges of H by internally disjoint paths with the same ends and at least one edge. We say that a graph G *contains* a graph H if G has a subgraph isomorphic to a subdivision of H.

(1.1) *For a graph G, the following two conditions are equivalent.*

(i) *For every admissible mapping $p : E(G) \to \mathbf{Z}$ there exists a list of circuits of G such that every edge e of G belongs to precisely $p(e)$ of these circuits.*

(ii) *The graph G does not contain the Petersen graph.*

Thus it appears useful to have a structural characterization of graphs that do not contain the Petersen graph, but that is undoubtedly a hard problem. In [2] we managed to find such a characterization for cubic graphs under an additional connectivity assumption. We need a few definitions before we can state the result. If G is a graph and $X \subseteq V(G)$, we denote by $\delta_G(X)$ or $\delta(X)$ the set of edges of G with one end in X and the other in $V(G) - X$. We say that a cubic graph is *theta-connected* if G has girth at least five, and $|\delta_G(X)| \geq 6$ for all $X \subseteq V(G)$ such that $|X|, |V(G) - X| \geq 7$. We say that a graph G is *apex* if $G \backslash v$ is planar for some vertex v of G ($\backslash$ denotes deletion). We say that a graph G is *doublecross* if it has four edges e_1, e_2, e_3, e_4 such that the graph $G \backslash \{e_1, e_2, e_3, e_4\}$ can be drawn in the plane with the unbounded face bounded by a circuit C, in which $u_1, u_2, v_1, v_2, u_3, u_4, v_3, v_4$ are pairwise distinct and occur on C in the order listed, where the edge e_i has ends u_i and v_i for $i = 1, 2, 3, 4$. The graph *Starfish* is shown in Figure 1. Now we can state the result of [2].

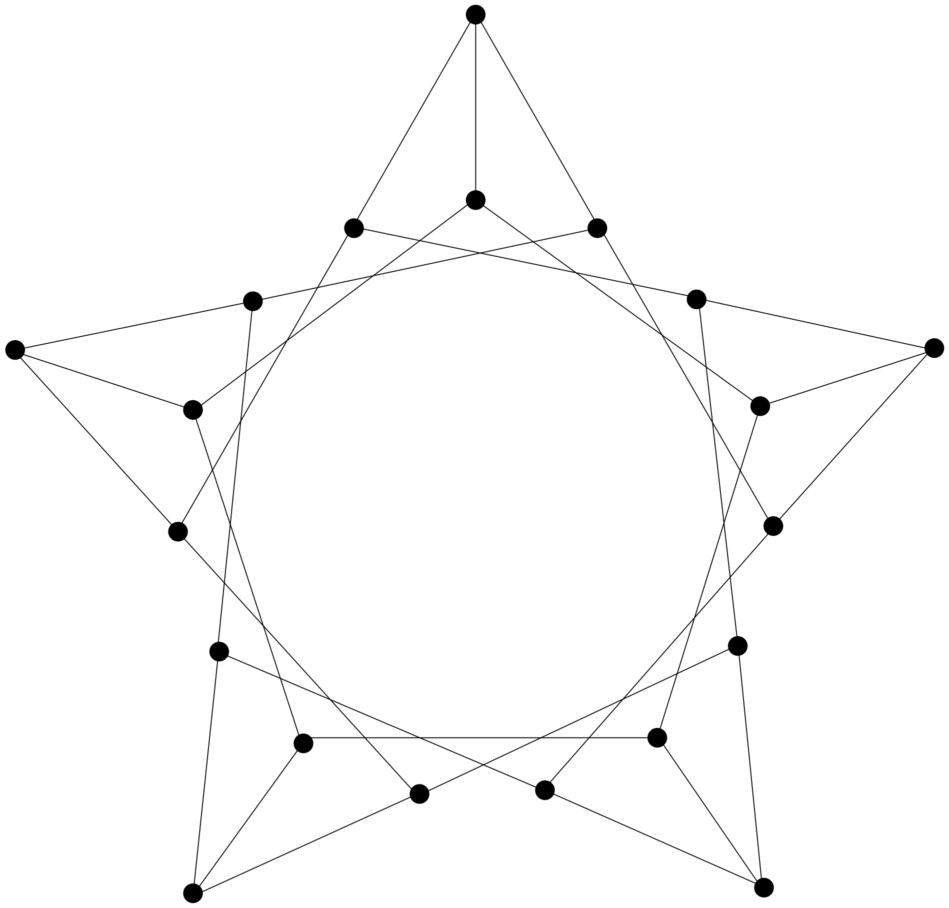

Figure 1: Starfish

(1.2) *Let G be a cubic theta-connected graph. Then G does not contain the Petersen graph if and only if either G is apex, or G is doublecross, or G is isomorphic to Starfish.*

In the present paper we use (1.2) to prove the result stated in the title and in the abstract, formally the following.

(1.3) *Every cubic graph of girth at least six contains the Petersen graph.*

Theorem (1.3) does not extend to graphs of minimum degree three. For instance, let H be (the 1-skeleton of) the Dodecahedron. The graph H has an induced matching M of size six. Let G be obtained from H by subdividing every edge of M, adding a new vertex

v and joining v to all the vertices that resulted from subdividing the edges of M. Then G has girth six, but it is apex, and hence does not contain the Petersen graph.

We prove (1.3) by induction, but in order for the inductive argument to work we need to prove a stronger statement which we now introduce. We say that two circuits of a graph *meet* if they have at least one vertex in common. Thus if two circuits of a cubic graph meet, then they have at least one edge in common. We say that a circuit of a graph is *short* if it has at most five edges. A short circuit of a graph G which meets every short circuit of G is called a *breaker*. We say that a graph G is *interesting* if it is cubic, it has at least ten vertices, and either it has girth at least six or it has a breaker. We shall see later that every interesting graph has at least fourteen vertices. In fact, it can be shown that there is exactly one interesting graph on fourteen vertices; this graph has girth six, and is usually called the *Heawood graph*. The result we prove is the following.

(1.4) *Every interesting cubic graph contains the Petersen graph.*

Since the Petersen graph is not interesting, one might ask if it is perhaps true that every interesting graph contains the Heawood graph. Unfortunately, it is not. If a graph contains another graph, and the first admits an embedding in the Klein Bottle (in fact, any fixed surface), then so does the second. However, the Heawood graph does not admit an embedding in the Klein bottle, and yet there are cubic graphs of girth six that do.

To prove (1.4) we first show in Section 2, using (1.2), that (1.4) holds for theta-connected interesting graphs, and then prove (1.4) for all interesting graphs in Section 3.

Andreas Huck (private communication) informed us that he can use (1.3) to deduce the following result. A graph is *Eulerian* if every vertex has even degree.

(1.5) *Let G be a cubic 2-edge-connected graph not containing the Petersen graph. Then there exist five Eulerian subgraphs of G such that every edge of G belongs to exactly two of these graphs.*

2. APEX AND DOUBLECROSS GRAPHS

The objective of this section is to prove (2.6) below, our main theorem for theta-connected graphs. We begin with the following.

(2.1) *Every interesting cubic graph has at least fourteen vertices.*

Proof. Let G be an interesting graph. It is easy to see that every cubic graph of girth at least six has at least fourteen vertices. Thus we may assume that G has a breaker C. Let H be the graph obtained from G by deleting the edges of C. Then H has at most five vertices of degree one, and hence at least five vertices of degree three, because G has at least ten vertices. It follows that H has a circuit. Let C' be the shortest circuit of H. The circuit C' has length at least six, because it is disjoint from C. Let Z be the set of all vertices in $V(G) - V(C')$ that are adjacent to a vertex of C'. Then $|Z| = |V(C')|$ by the choice of C'. If C' has length at least seven, then $|V(G)| \geq |V(C')| + |Z| \geq 14$, as desired, and so we may assume that C' has length six. The above argument shows that G has at least twelve vertices, and so we assume for a contradiction that G has exactly twelve vertices. Thus $V(G) = V(C') \cup Z$. Hence $V(C) \subseteq Z$, and the inclusion is proper, because C is short. The subgraph of G induced by $Z - V(C)$ is 2-regular, and hence has a circuit. But this circuit is short and disjoint from C, a contradiction. Thus G has at least fourteen vertices, as desired. ☐

A *pentagon* is a circuit of length five.

(2.2) *Every two distinct pentagons in an interesting theta-connected cubic graph have at most one edge in common.*

Proof. Let G be an interesting theta-connected graph. Suppose for a contradiction that G has two distinct pentagons C and C' with more than one edge in common. Then $|\delta_G(V(C) \cup V(C'))| = 5$ and $|V(C) \cup V(C')| = 7$, because G has girth at least five, and hence $|V(G) - (V(C) \cup V(C'))| \leq 6$ by the theta-connectivity of G, contrary to (2.1). ☐

If G is a graph and $X \subseteq V(G)$, we denote by $G|X$ the graph $G \backslash (V(G) - X)$.

(2.3) *Every interesting theta-connected cubic graph has at most five pentagons.*

Proof. Let G be an interesting theta-connected graph. Since G is theta-connected, every short circuit in G is a pentagon. Suppose for a contradiction that G has at least six pentagons. Let C_0 be a breaker in G, and let C_1, C_2, C_3, C_4 and C_5 be five other pentagons of G. The sets $E(C_0) \cap E(C_i)$ ($i = 1, 2, \ldots, 5$) are nonempty, and, by (2.2), they are pairwise disjoint and each has cardinality one. Thus G has no other short circuit. For $i = 1, 2, \ldots, 5$ let $E(C_0) \cap E(C_i) = \{e_i\}$. We may assume that $e_1, e_2, \ldots, e_5$ occur on C_0 in the order listed. By (2.2) consecutive circuits in the sequence $C_1, C_2, \ldots, C_5, C_1$ have precisely one edge and its ends in common, and non-consecutive circuits are vertex-disjoint, as otherwise G has a short circuit distinct from $C_0, C_1, \ldots, C_5$. We conclude that $\left|\bigcup_{i=0}^{5} V(C_i)\right| = 15$. Let $X = V(G) - \bigcup_{i=0}^{5} V(C_i)$; then $|\delta_G(X)| \leq 5$, and hence $|X| \leq 6$ by the theta-connectivity of G. Moreover, X has an odd number of elements, and hence is not empty. Since X has at most five vertices and is disjoint from $V(C_0)$, we deduce that $G|X$ has no circuit, and that $G|X$ is a path on at most three vertices. Hence every vertex of X is incident with an edge in $\delta(X)$, and there exists a vertex $v \in X$ adjacent to every vertex of $X - \{v\}$. We may assume v has a neighbor $c_1 \in V(C_1)$, and some vertex $c_2 \in V(C_2)$ has a neighbor in X. Thus c_1, c_2 are joined by a two-edge path with interior in $(C_1 \cup C_2) \backslash V(C_0)$, and by a path of length at most three with interior in X, and their union is a short circuit disjoint from C_0, a contradiction. $\square$

(2.4) *Every cubic doublecross graph of girth at least five has at least six pentagons.*

Proof. Let G be a doublecross graph of girth at least five, and let $e_1, u_1, v_1, \ldots, e_4, u_4, v_4$ and C be as in the definition of doublecross. Let P_1 be the subpath of C with ends u_1 and u_2 not containing v_1, let P_2 be the subpath of C with ends u_2 and v_1 not containing v_2, and let $P_3, P_4, \ldots, P_8$ be defined similarly. Thus $C = P_1 \cup P_2 \cup \ldots \cup P_8$, and the paths $P_1, P_2, \ldots, P_8$ appear on C in the order listed. Let $G' := G \backslash \{e_1, e_2, e_3, e_4\}$. We will regard G' as a plane graph with outer cycle C. Let f be the number of bounded faces of G', and let p be the number of those that are bounded by a pentagon. By Euler's formula $|V(G)| + f + 1 = |E(G)| - 4 + 2$, and since G is cubic, $2|E(G)| = 3|V(G)|$. We deduce that $|E(G)| = 3f + 9$. For $i = 1, 2, \ldots, 8$ let $d_i = |E(P_i)|$. Since every edge of $G'\backslash E(C)$ is incident with two bounded faces, and every edge of C is incident with one we obtain $2|E(G)| \geq 6f - p + 8 + \sum_{i=1}^{8} d_i$, and hence $\sum_{i=1}^{8} d_i \leq 10 + p$.

Since $d_4, d_8 \geq 1$, we get $d_1 + d_2 + d_3 + d_5 + d_6 + d_7 \leq 8 + p$. Let q be the number of pentagons in the subgraph formed by P_1, P_2, P_3 and the edges u_1v_1, u_2v_2. Then $d_1 + d_2 + 1$ is at least five, and at least six unless the cycle $P_1 \cup P_2 + u_1v_1$ is a pentagon, and similarly for $d_2 + d_3 + 1$. Furthermore, $d_1 + d_3 + 2$ is at least six unless $P_1 \cup P_3 + u_1v_1 + u_2v_2$ is a pentagon. By adding,

$$(d_1 + d_2 + 1) + (d_2 + d_3 + 1) + (d_1 + d_3 + 2) \geq 6 + 6 + 6 - q;$$

that is, $d_1 + d_2 + d_3 \geq 7 - q/2$. Similarly if there are r pentagons in the opposite crossing, then $d_5 + d_6 + d_7 \geq 7 - r/2$.

So, adding,

$$8 + p \geq d_1 + d_2 + d_3 + d_5 + d_6 + d_7 \geq 14 - (q + r)/2.$$

So $p + (q + r)/2 \geq 6$, and hence $p + q + r \geq 6$ as required. □

(2.5) *Every cubic apex graph of girth at least five has at least six pentagons.*

Proof. Let G be a cubic apex graph of girth at least five, and let v be a vertex of G such that $G\backslash v$ is planar. Let f be the number of faces in some planar embedding of $G\backslash v$, and let p be the number of them that are bounded by a pentagon. Then $2|E(G)| = 3|V(G)|$ and $|E(G)| = |E(G\backslash v)| + 3$ because G is cubic, $|V(G\backslash v)| + f = |E(G\backslash v)| + 2$ by Euler's formula, and $2|E(G\backslash v)| \geq 6f - p$, since G has girth at least five. We deduce that $p \geq 6$, as desired. □

In view of (2.4) and (2.5) it is natural to ask whether every cubic graph of girth at least five not containing the Petersen graph has at least six pentagons. That is not true, because Starfish is a counterexample.

(2.6) *Every interesting theta-connected graph contains the Petersen graph.*

Proof. Let G be an interesting theta-connected graph. By (2.3) G has at most five pentagons. Thus by (2.4) G is not doublecross, by (2.5) G is not apex, and G is not isomorphic to Starfish, because Starfish is not interesting. Thus G contains the Petersen graph by (1.2). □

3. INTERESTING GRAPHS

In this section we prove (1.4), which we restate below as (3.5). Let G be an interesting graph. We say that G is *minimal* if G contains no interesting graph on fewer vertices.

(3.1) *Every minimal interesting graph has girth at least four.*

Proof. Let G be a minimal interesting graph; then G is clearly connected. Suppose for a contradiction that C is a circuit in G of length at most three. Let C' be a breaker in G, and let $e \in E(C) \cap E(C')$. Let H be obtained from G by deleting e, deleting any resulting vertex of degree one, and then suppressing all resulting vertices of degree two. Then H has at least ten vertices by (2.1). Also, it follows that either H has girth at least six or H has a breaker (if $C \neq C'$ then the latter can be seen by considering the circuit of H that corresponds to the circuit of $C \cup C' \backslash e$). Thus H is interesting, contrary to the minimality of G. □

We say that X is a *shore* in a graph G if X is a set of vertices of G such that $|\delta(X)| \leq 5$ and both $G|X$ and $G\backslash X$ have at least two circuits. The following is easy to see.

(3.2) *A cubic graph of girth at least five is theta-connected if and only if it has no shore.*

(3.3) *Let G be an interesting graph, let X be a shore in G, and let C be a breaker in G. Then G has a shore Y such that $|\delta(Y)| \leq |\delta(X)|$ and $V(C) \cap Y = \emptyset$.*

Proof. Let Y be a shore in G chosen so that $|\delta(Y)|$ is minimum, and subject to that, $|Y \cap V(C)|$ is minimum. We claim that Y is as desired. From the minimality of $|\delta(Y)|$ we deduce that $|\delta(Y)| \leq |\delta(X)|$ and that $\delta(Y)$ is a matching, and from the minimality of $|Y \cap V(C)|$ we deduce (by considering $V(G) - Y$) that $|Y \cap V(C)| \leq 2$. Suppose for a contradiction that $Y \cap V(C) \neq \emptyset$; then $Y \cap V(C)$ consists of two vertices, say u and v, that are adjacent in C. Let $Y' = Y - V(C)$. We deduce that $|\delta(Y')| \leq |\delta(Y)|$, and so it follows from the choice of Y that $G|Y'$ has at most one circuit. On the other hand since u and v are adjacent and have degree two in $G|Y$ we see that $G|Y'$ has a circuit, and since $|\delta(Y')| \leq 5$ this is a short circuit disjoint from $V(C)$, a contradiction. □

(3.4) *No minimal interesting graph has a shore.*

Proof. Suppose for contradiction that G is a minimal interesting graph, and that X is a shore in G with $|\delta(X)|$ minimum. Let $k = |\delta(X)|$; then $k \leq 5$. If G has a short circuit let C be a breaker in G; otherwise let C be the null graph. By (3.3) we may assume that $V(C) \cap X = \emptyset$. By (3.1) and the minimality of k we may choose a circuit C' of $G \backslash X$ with $|V(C')| \geq k$. By the minimality of k there exist k disjoint paths between $V(C')$ and Z, where Z is the set of all vertices of X that are incident with an edge in $\delta(X)$. Let the paths be $P_1, P_2, \ldots, P_k$, and for $i = 1, 2, \ldots, k$ let the ends of P_i be $u_i \in Z$ and $v_i \in V(C')$ numbered so that $v_1, v_2, \ldots, v_k$ occur on C' in this order. Let H be obtained from $G|X$ by adding a circuit C'' with vertex-set $\{w_1, w_2, \ldots, w_k\}$ in order and one edge with ends u_i and w_i for $i = 1, 2, \ldots, k$. Then C'' is a breaker in H. Since $G|X$ has a circuit, and that circuit, being disjoint from C, has length at least six, we deduce that H has at least ten vertices, and so is interesting. Moreover, G contains H, and is not isomorphic to H, because $G \backslash X$ is not a circuit, contradicting the minimality of G. □

We are now ready to prove (1.4), which we restate.

(3.5) *Every interesting graph contains the Petersen graph.*

Proof. It suffices to show that every minimal interesting graph contains the Petersen graph. To this end let G be a minimal interesting graph. If G has girth at least five, then G is theta-connected by (3.2) and (3.4), and hence contains the Petersen graph by (2.6). Thus we may assume that G has a circuit of length less than five, say C. Since G has girth at least four by (3.1), we deduce that C has length four.

We claim that C is the only short circuit in G. To prove this claim suppose for a contradiction that C' is a short circuit in G other than C. Since G is interesting, we may assume that the pair C, C' is chosen in such a way that C or C' is a breaker in G. Then $|\delta(V(C) \cup V(C'))| \leq 5$. Let $X = V(G) - (V(C) \cup V(C'))$. By (3.4) X is not a shore, and so $G|X$ has at most one circuit, because $|\delta_G(X)| \leq 5$. Thus $|X| \leq 5$. It follows that G has at most twelve vertices, contrary to (2.1). Thus C is the only short circuit in G, as claimed.

Let the vertices of C be u_1, u_2, u_3, u_4 (in order), for $i = 1, 2, 3, 4$ let e_i be the edge of C with ends u_i and u_{i+1} (where u_5 means u_1), and let f_i be the unique edge of $E(G) - E(C)$ incident with u_i. Let H be the graph obtained from $G \backslash e_1$ by contracting the edges e_2 and e_4. Then H is a cubic graph with girth at least five, and hence $|V(H)| \geq 10$, as is easily seen. Moreover, every pentagon in H contains one end of e_3.

Let us assume first that H is theta-connected. By the minimality of G, the graph H is not interesting; in particular, the edge e_3 belongs to no pentagon of H. Since no two pentagons in a cubic graph of girth at least five share more than two edges, we deduce that the ends of e_3 belong to at most two pentagons each. Thus H is not doublecross by (2.4), it is not apex by (2.5), and it is not isomorphic to Starfish, because Starfish has three pairwise vertex-disjoint pentagons. Thus H contains the Petersen graph by (1.2), and hence so does G, as desired.

We may therefore assume that H is not theta-connected. By (3.2) H has a shore. By (3.4) applied to G there exists a set $X_1 \subseteq V(G)$ such that $|X_1|, |V(G) - X_1| \geq 7$, $|\delta_G(X_1)| = 6$, $u_1, u_4 \in X_1$, $u_2, u_3 \notin X_1$, and that $\delta_G(X_1)$ is a matching. Thus $|X_1|, |V(G) - X_1| \geq 8$. By arguing similarly for the graph $G \backslash e_2$ we deduce that either G contains the Petersen graph, or there exists a set $X_2 \subseteq V(G)$ such that $|X_2|, |V(G) - X_2| \geq 8$, $|\delta_G(X_2)| = 6$, $u_1, u_2 \in X_2$, and $u_3, u_4 \in V(G) - X_2$. We may assume the latter. Since $|\delta_G(X_1 \cap X_2)| + |\delta_G(X_1 \cup X_2)| \leq |\delta_G(X_1)| + |\delta_G(X_2)| = 12$, we deduce that $\delta_G(X_1 \cap X_2)$ or $\delta_G(X_1 \cup X_2)$ has at most six elements. From the symmetry we may assume that it is the former. Since $e_1, e_4 \in \delta_G(X_1 \cap X_2)$, it follows that $|\delta_G(Y)| \leq 5$, where $Y = X_1 \cap X_2 - \{u_1\}$. By (3.4) Y is not a shore, and hence the graph $G|Y$ has at most one circuit. However, if $G|Y$ has a circuit, then that circuit does not meet C, and yet it has length at most five (because $|\delta(Y)| \leq 5$), which is impossible. Thus $G|Y$ has no circuit, and hence $|Y| \leq 3$. Similarly, either $|X_1 - X_2 - \{u_4\}| \leq 3$ or $|X_2 - X_1 - \{u_2\}| \leq 3$, and from the symmetry we may assume the former. Thus $|X_1| \leq 8$. Since $|X_1| \geq 8$ as we have seen earlier, the above inequalities are satisfied with equality. In particular, $|\delta_G(X_1 \cup X_2)| = 6$, and hence $|\delta_G(X_1 \cup X_2 \cup \{u_3\})| \leq 5$, and likewise $|\delta_G(X_2 - X_1 - \{u_2\})| \leq 5$. As above we deduce that $|V(G) - X_1| = 8$. Thus $G|X_1$ and $G \backslash X_1$ both have eight vertices, six vertices of degree two, two vertices of degree three, and girth at least six. It follows that $G|X_1$ and $G \backslash X_1$ are both isomorphic to the graph that is the union of three paths on four vertices each, with the same ends and otherwise vertex-disjoint.

We now show that H is isomorphic to the graph shown in Figure 2. Let $G|X_1$ consist of three paths ab_ic_id for $i = 1, 2, 3$, and let $G \backslash X_1$ have three paths pq_ir_is similarly. So there is a six-edge matching M in G between $\{b_1, b_2, b_3, c_1, c_2, c_3\}$ and $\{q_1, q_2, q_3, r_1, r_2, r_3\}$. Since C exists we can assume that b_3 is matched by M to q_3 and c_3 to r_3. Thus $V(C) = \{b_3, c_3, r_3, q_3\}$.

Now X_2 exists and contains a, p and not d, s; and each of the six paths of the previous paragraph includes an edge of $\delta(X_2)$. Thus no edge of M belongs to $\delta(X_2)$. If X_2 contains both b_1 and c_1, then b_1, c_1 are matched by M to vertices with distance at most two in X_2, and hence G has a circuit of length at most five disjoint from C, a contradiction. Thus X_2 contains at most one of b_1, c_1, and similarly for the pairs (b_2, c_2), (q_1, r_1) and (q_2, r_2). We may therefore assume that $X_2 = \{a, b_1, b_2, b_3, p, q_1, q_2, q_3\}$. So b_1, b_2 are matched to q_1, q_2 and c_1, c_2 to r_1, r_2 in some order. Because G is interesting, we may assume the pairs are (b_1, q_1), (b_2, q_2), (c_1, r_2), (c_2, r_1). Thus H is isomorphic to the graph shown in Figure 2. That graph, however, contains the Petersen graph, as desired. $\square$

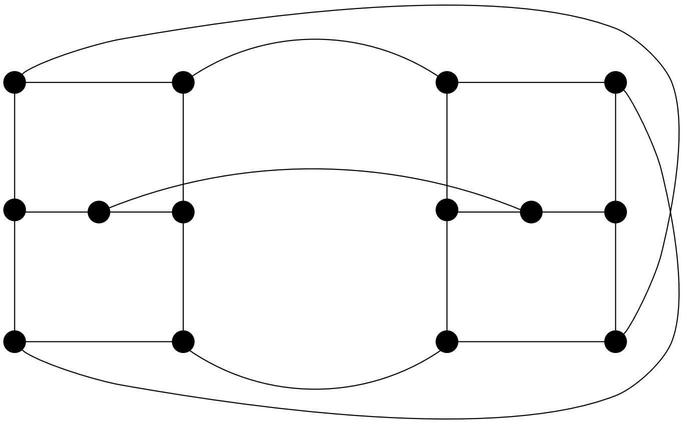

Figure 2

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