FIVE-LIST-COLORING GRAPHS ON SURFACES III. ONE LIST OF SIZE ONE AND ONE LIST OF SIZE TWO

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ABSTRACT

Let G be a plane graph with outer cycle C and let $(L(v) : v \in V(G))$ be a family of non-empty sets. By an L-coloring of G we mean a (proper) coloring ϕ of G such that $\phi(v) \in L(v)$ for every vertex v of G. Thomassen proved that if $v_1, v_2 \in V(C)$ are adjacent, $L(v_1) \neq L(v_2)$, $|L(v)| \geq 3$ for every $v \in V(C) - \{v_1, v_2\}$ and $|L(v)| \geq 5$ for every $v \in V(G) - V(C)$, then G has an L-coloring. What happens when v_1 and v_2 are not adjacent? Then an L-coloring need not exist, but in the first paper of this series we have shown that it exists if $|L(v_1)|, |L(v_2)| \geq 2$. Here we characterize when an L-coloring exists if $|L(v_1)| \geq 1$ and $|L(v_2)| \geq 2$.

This result is a lemma toward a more general theorem along the same lines, which we will use to prove that minimally non-L-colorable planar graphs with two precolored cycles of bounded length are of bounded size. The latter result has a number of applications which we pursue elsewhere.

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1 Introduction

All *graphs* in this paper are finite and simple; that is, they have no loops or parallel edges. *Paths* and *cycles* have no repeated vertices or edges. If G is a graph and $L = (L(v) : v \in V(G))$ is a family of non-empty sets, then we say that L is a *list assignment for* G. It is a *k-list-assignment*, if $|L(v)| \geq k$ for every vertex $v \in V(G)$. An L-coloring of G is a (proper) coloring ϕ of G such that $\phi(v) \in L(v)$ for every vertex v of G. We say that a graph G is *k-choosable*, also called *k-list-colorable*, if for every k-list-assignment L for G, G has an L-coloring.

One notable difference between list coloring and ordinary coloring is that the Four Color Theorem [1, 2] does not generalize to list-coloring. Indeed, Voigt [9] constructed a planar graph that is not 4-choosable. On the other hand Thomassen [7] proved that every planar graph is 5-choosable. His proof is remarkably short and beautiful. For the sake of the inductive argument he proves the following stronger statement.

Theorem 1.1 *If G is a plane graph with outer cycle C and $P = p_1p_2$ is a path of length one in C and L is a list assignment with $|L(v)| \geq 5$ for all $v \in V(G) - V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) - V(P)$, and $|L(p_1)|, |L(p_2)| \geq 1$ with $L(p_1) \neq L(p_2)$, then G is L-colorable.*

What if p_1 and p_2 are not adjacent? In that case G need not be L-colorable, but it is possible to characterize instances when it is not. In fact, we are able to extract useful information even when more vertices are pre-colored, but it will take some effort. We began this line of research in our previous paper [6], where we proved a generalization of Theorem 1.1 conjectured by Hutchinson [4], who proved the result for outerplanar graphs.

Theorem 1.2 *If G is a plane graph with outer cycle C and $p_1, p_2 \in V(C)$ and L is a list assignment with $|L(v)| \geq 5$ for all $v \in V(G) - V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) - \{p_1, p_2\}$, and $|L(p_1)|, |L(p_2)| \geq 2$, then G is L-colorable.*

The main result of this paper is to characterize when an L-coloring exists, if in Theorem 1.2 we only assume that $|L(p_1)| \geq 1$. In order to state the theorem we need to define a family of obstructions.

Let G be a connected plane graph, and let u, v, w be distinct vertices of G incident with the outer face of G, let u be adjacent to v, let the edge uv be incident with the outer face of G and let L be a list assignment for G. We say that the pair (G, L) is a *coloring harmonica from uv to w* if either

- G is a triangle with vertex set $\{u, v, w\}$, $L(u) = L(v) = L(w)$ and $|L(u)| = 2$, or
- there exists a vertex $z \in V(G)$ incident with the outer face of G such that uvz is a triangle in G, $L(u) = L(v) \subseteq L(z)$, $|L(u)| = |L(v)| = 2$, $|L(z)| = 3$, and (G', L') is a

coloring harmonica from z to w, where G' is obtained from G by deleting one or both of the vertices u, v, and L' satisfies $L'(z) = L(z) - L(u)$ and $L'(x) = L(x)$ for every $x \in V(G') - \{z\}$.

We say that the pair (G, L) is a *coloring harmonica from u to w* if

- there exist vertices $x, y \in V(G)$ incident with the outer face of G such that uxy is a triangle in G, $|L(u)| = 1$, $L(x) - L(u) = L(y) - L(u)$, $|L(x) - L(u)| = 2$, and (G', L') is a coloring harmonica from xy to w, where $G' := G \setminus u$, $L'(x) = L(x) - L(u)$, $L'(y) = L(y) - L(u)$ and $L'(z) = L(z)$ for every $z \in V(G') - \{x, y\}$.

See Figure 1. We say that the pair (G, L) is a *coloring harmonica* if it is either a coloring harmonica from uv to w or a coloring harmonica from u to w, where u, v, w are as specified earlier. We say that the pair (G, L) *contains a coloring harmonica* (G', L') if G' is a subgraph of G and $L'(x) = L(x)$ for every $x \in V(G')$.

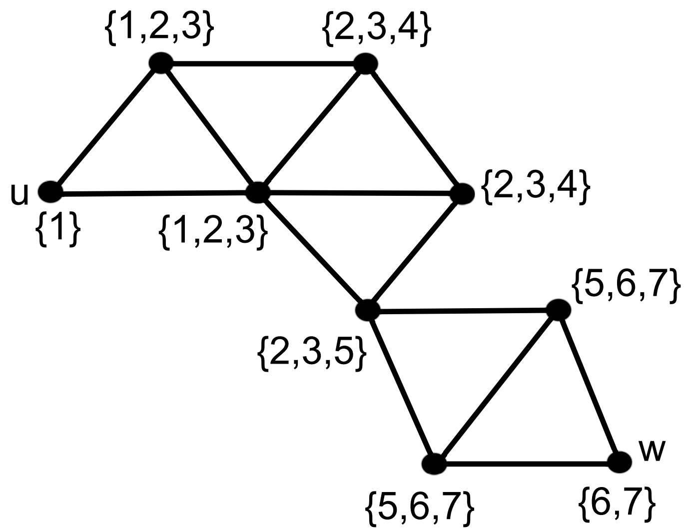

Figure 1: A coloring harmonica from u to w.

We can now state the main result of this paper. Hutchinson [4] proved it for outerplanar graphs.

Theorem 1.3 *Let G be a plane graph with outer cycle C, let $p_1, p_2 \in V(C)$, and let L be a list assignment with $|L(v)| \geq 5$ for all $v \in V(G) - V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) - \{p_1, p_2\}$, $|L(p_1)| \geq 1$ and $|L(p_2)| \geq 2$. Then G is L-colorable if and only if the pair (G, L) does not contain a coloring harmonica from p_1 to p_2.*

2 Canvases

We also recall the definition of the graphs we are working with, first introduced in our previous paper [6].

Definition 2.1 (Canvas) We say that (G, S, L) is a *canvas* if G is a plane graph, S is a subgraph of the boundary of the outer face of G, and L is a list assignment for some supergraph of G such that $|L(v)| \geq 5$ for all $v \in V(G) - V(C)$, where C is the boundary of the outer face of G, $|L(v)| \geq 3$ for all $v \in V(G) - V(S)$, and there exists a proper L-coloring of S.

We should remark that we allow L to be a list assignment of some supergraph of G merely for convenience when passing to subgraphs. Given this definition of a canvas, we can state an equivalent but slightly more general version of Theorem 1.1 as follows.

Theorem 2.2 *If (G, S, L) is a canvas and S is path of length one, then G is L-colorable.*

We can also restate Theorem 1.2 in these terms.

Theorem 2.3 (Two with List of Size Two Theorem) *If (G, S, L) is a canvas with $V(S) = \{v_1, v_2\}$ and $|L(v_1)|, |L(v_2)| \geq 2$, then G is L-colorable.*

It should be noted that Thomassen [8] characterized the canvases (G, S, L) where S is a path of length two and G is not L-colorable. For our main theorem, we do not need this full characterization. However, we do need the following lemma that can be found in [8, Lemma 1]. A *chord* of a cycle in a graph G is a subgraph of G consisting of two vertices that belong to the cycle and an edge joining them that does not belong to the cycle.

Lemma 2.4 *Let $T = (G, S, L)$ be a canvas such that S is an induced path of length two. If there does not exist a chord of the boundary of the outer face of G, then there exists at most one proper L-coloring of S that does not extend to an L-coloring of G.*

We also need a notion of containment for canvases as follows.

Definition 2.5 A canvas $T = (G, S, L)$ *contains* a canvas $T' = (G', S', L')$ if G' is a subgraph of G, $S = S'$ and the restrictions of L and L' to G' are equal.

3 Governments and Reductions

In this section, we will develop notation and definitions for characterizing how the colorings of P in Theorem 1.1 extend to colorings of other paths of length one on the boundary of the outer walk. We introduce a notion called a government to describe sets of colorings that come in two types which we call dictatorships and democracies. Our main theorem will show that a government extends to at least two governments unless a very specific structure occurs.

3.1 Coloring Extensions

Definition 3.1 Suppose $T = (G, P, L)$ is a canvas where $P = p_1p_2$ is a path of length one in the boundary C of the outer face of G. Suppose we are given a collection $\mathcal{C}$ of L-colorings of P. Let P' be an edge of G with both ends in C. We let $\Phi_T(P', \mathcal{C})$ denote the collection of proper L-colorings of P' that can be extended to a proper L-coloring ϕ of G such that ϕ restricted to P is an L-coloring in $\mathcal{C}$. We will drop the subscript T when the canvas is clear from context.

We may now restate Theorem 1.1 in these terms.

Theorem 3.2 *Let $T = (G, P, L)$ be a canvas with P a path of length one. If $\mathcal{C}$ is a nonempty collection of proper L-colorings of P, and P' is an edge of G with both ends in C, then $\Phi(P', \mathcal{C})$ is nonempty.*

Note the following easy proposition.

Proposition 3.3 *Let T, P, P' be as in Theorem 3.2. If $U = u_1u_2$ is a chord of C separating P from P', then*

$$\Phi(P', \Phi(U, \mathcal{C})) = \Phi(P', \mathcal{C}).$$

3.2 Governments

To explain the structure of extending larger sets of colorings, we focus on two special sets of colorings, defined as follows.

Definition 3.4 (Government) Let $\mathcal{C} = \{\phi_1, \phi_2, \ldots, \phi_k\}$, $k \geq 2$, be a collection of disjoint proper colorings of a path $P = p_1p_2$ of length one. For $p \in P$, let $\mathcal{C}(p)$ denote the set $\{\phi(p) | \phi \in \mathcal{C}\}$.

We say $\mathcal{C}$ is a *dictatorship* if there exists $i \in \{1, 2\}$ such that $\phi_j(p_i)$ is the same for all $1 \leq j \leq k$, in which case, we say p_i is the dictator of $\mathcal{C}$. We say $\mathcal{C}$ is a *democracy* if $k = 2$ and $\phi_1(p_1) = \phi_2(p_2)$ and $\phi_2(p_1) = \phi_1(p_2)$. We say $\mathcal{C}$ is a *government* if $\mathcal{C}$ is a dictatorship or a democracy.

We also need a generalized form of government as follows.

Definition 3.5 Let $\mathcal{C}$ be a collection of disjoint proper colorings of a path $P = p_1p_2$ of length one. We say $\mathcal{C}$ is a *confederacy* if $\mathcal{C}$ is not a government and yet $\mathcal{C}$ is the union of two governments.

3.3 Reductions

Thomassen found a useful reduction in his proof of 5-choosability. We will need a generalization of that reduction as follows.

Definition 3.6 (Democratic Reduction) Let $T = (G, S, L)$ be a canvas and L_0 be a set of two colors. Let C be the boundary of the outer face of G. Suppose that $P = p_1 \dots p_k$ is an induced path in C such that, for every vertex v in P, v is not the end of a chord of C or a cutvertex of C, $V(C) \neq V(P)$, and $L_0 \subseteq L(v)$. If $k \geq 2$, let x be the vertex of C adjacent to p_1 other than p_2 and y be the vertex of C adjacent to p_k other than p_{k-1}. If $k = 1$, let x and y be the two neighbors of p_1 on C. We assume that $L(x) - L_0 \neq \emptyset$.

We define the *democratic reduction* of P in T with respect to L_0 and centered at x, denoted as $T(P, L_0, x)$, as $(G \setminus V(P), S', L')$ where $L'(w) = L(w) - L_0$ if either $w = x$, or $w \neq y$ is a neighbor of a vertex in P, and $L'(w) = L(w)$ otherwise; and $S' = S \setminus V(P)$ if $|L'(x)| \geq 3$ and otherwise let S' be obtained from $S \setminus V(P)$ by adding x as an isolated vertex.

Proposition 3.7 *Let $T(P, L_0, x) = (G', S', L')$ be a democratic reduction of a path P in a canvas $T = (G, S, L)$ with respect to L_0 and centered at x. The following statements hold:*

1. *$T(P, L_0, x)$ is a canvas.*
2. *If ϕ is an L'-coloring of G', then ϕ can be extended to an L-coloring of G.*

Proof. Let C be the boundary of the outer face of G. If $v \in V(G')$ such that $|L'(v)| < 5$, then either $v \in C$ or v is adjacent to a vertex of w in P. In either case, v is in the boundary of the outer face of G'. Note that if $v \in V(G')$ such that $L'(v) \neq L(v)$, then either $v = x$ or $v \notin C$. In the latter case, $|L(v)| = 5$ and hence $|L'(v)| \geq 3$. Thus, if $v \in V(G')$ such that $|L'(v)| < 3$, then $v \in V(S) \cup \{x\}$. Recall that by definition, $V(S') = V(S) \cup \{x\}$ if $|L'(x)| < 3$ and $V(S') = V(S)$ otherwise. In either case, it follows that if $v \in V(G')$ such that $|L'(v)| < 3$, then $v \in V(S')$. This proves (1).

Let ϕ be an L'-coloring of G'. Let $P = p_1 \dots p_k$ where p_1 is adjacent to x. If $k = 1$, let y be the neighbor of p_1 in C other than x. If $k \geq 2$, let y be the neighbor of p_k in C other than p_{k-1}. Let $\phi(p_k) \in L_0 - \{\phi(y)\}$. For all i with $1 \leq i \leq k-1$, let $\phi(p_i) \in L_0 - \{\phi(p_{i+1})\}$. Now ϕ is an L-coloring of G. This proves (2). □

We note that Thomassen's reduction corresponds to a democratic reduction where $|V(P)| = 1$, $x \in V(S)$ and $|L(x)| = 1$.

4 Harmonicas

In this section, we rework the definition of coloring harmonica to an object involving governments which we call a harmonica. We then prove a stronger version of our main theorem which shows that harmonicas are the only obstacle to extending a government to a confederacy. We will then show that this implies that coloring harmonicas are the only obstruction to generalizing Theorem 2.3 to the case of one vertex with a list of size one and one with a list of size two. That is, we finally prove Theorem 1.3.

Definition 4.1 (Harmonica) Let $T = (G, P, L)$ be a canvas where P is path of length one. Let $\mathcal{C}$ be a government for P and let P' be another (not necessarily distinct) path of length one incident with the outer face of G. We say T is a *harmonica* from P to P' with government $\mathcal{C}$ if

- $G = P = P'$, or
- $\mathcal{C}$ is a dictatorship, $G = P \cup P'$, $V(P) \cap V(P') = \{z\}$ where z is the dictator of $\mathcal{C}$, or
- $\mathcal{C}$ is a dictatorship and there exists a triangle zu_1u_2 where $z \in V(P)$ is the dictator of $\mathcal{C}$ in color c, for $i = 1, 2$ we have $L(u_i) = L_0 \cup \{c\}$ if $u_i \notin V(P)$ and $\mathcal{C}(u_i) = L_0$ otherwise, where $|L_0| = 2$ and the canvas $(G \setminus (V(P) - V(U)), U, L)$ is a harmonica from $U = u_1u_2$ to P' with democracy $\mathcal{C}'$ where $\mathcal{C}'(u_1) = \mathcal{C}'(u_2) = L_0$, or
- $\mathcal{C}$ is a democracy and there exists $z \sim p_1, p_2$ where $P = p_1p_2$ such that $L(z) = L_0 \cup \{c\}$ where $L_0 = \mathcal{C}(p_1) = \mathcal{C}(p_2)$ and there exists $i \in \{1, 2\}$ such that the canvas $(G \setminus p_i, U, L)$ is a harmonica with dictatorship $\mathcal{C}' = \{\phi_1, \phi_2\}$, where $U = zp_{3-i}$ and $\phi_1(z) = \phi_2(z) = c$ and $\{\phi_1(p_{3-i}), \phi_2(p_{3-i})\} = L_0$.

Note that $\Phi_T(P', \mathcal{C})$ is a government. We remark that the notion of harmonica is closely related to the notion of coloring harmonica, introduced earlier. Lemma 4.16 clarifies the relation between the two.

We need the following easy lemma, whose proof we omit.

Lemma 4.2 *Let $T = (G, P, L)$ be a harmonica from P to P' with government $\mathcal{C}$, and let $v \in V(G) - V(P)$ be such that if $v \in V(P')$, then v has degree at least two. Then $|L(v)| = 3$.*

The following is our main result.

Theorem 4.3 *Let $T = (G, P, L)$ be a canvas and P, P' be paths of length one in the boundary of the outer face of G. Given a collection $\mathcal{C}$ of proper colorings of P such that $\mathcal{C}$ is a government or a confederacy, then $\Phi(P', \mathcal{C})$ contains a government, and furthermore, either*

- *$\Phi(P', \mathcal{C})$ contains a confederacy, or,*
- *$\mathcal{C}$ is a government and T contains a harmonica from P to P' with government $\mathcal{C}$.*

Proof. Suppose that $T = (G, P, L)$ is a counterexample with $|V(G)|$ minimized and, subject to that, $\mathcal{C}$ is a government if possible. Let C be the boundary of the outer face of G.

Claim 4.4 *G is connected.*

Proof. Suppose not. Let G_1 be the component of G containing P. First suppose that G_1 contains P' and let G_2 be a component of G other than G_1. Let $T' = (G \setminus V(G_2), P, L)$. By the minimality of G, $\Phi_{T'}(P', \mathcal{C})$ contains a government and hence $\Phi_T(P', \mathcal{C})$ contains a government by Theorem 1.1. Furthermore, either $\Phi_{T'}(P', \mathcal{C})$ contains a confederacy, a contradiction as then $\Phi_T(P', \mathcal{C})$ contains a confederacy by Theorem 1.1, or T' contains a harmonica from P to P' with government $\mathcal{C}$, in which case so does T, a contradiction.

So we may assume that G_1 does not contain P' and let G_2 be the component of G containing P'. It follows from Theorem 2.2 that every L-coloring of $P \cup P'$ extends to an L-coloring of G. In particular, let $P' = p_1'p_2'$ and $c_1 \in L(p_1'), c_2 \in L(p_2')$. If we let $\mathcal{C}_1 = \{\phi_1, \phi_2\}$ where $\phi_1(p_1') = \phi_2(p_1') = c_1$ and $\{\phi_1(p_2'), \phi_2(p_2')\}$ be a subset of $L(p_2') - \{c_1\}$ of size two. Similarly let $\mathcal{C}_2 = \{\psi_1, \psi_2\}$ where $\psi_1(p_2') = \psi_2(p_2') = c_2$ and $\{\psi_1(p_1'), \psi_2(p_1')\}$ be a subset of $L(p_1') - \{c_2\}$ of size two. Thus $\mathcal{C}_1$ is a dictatorship with dictator p_1' and $\mathcal{C}_2$ is a dictatorship with dictator p_2'. Hence $\mathcal{C}_1 \cup \mathcal{C}_2$ is a confederacy and $\mathcal{C}_1 \cup \mathcal{C}_2 \subseteq \Phi(\mathcal{P}', \mathcal{C})$, a contradiction. □

Claim 4.5 *There does not exist a vertex in an open disk bounded by a cycle of length at most four.*

Proof. Let C be a cycle of length at most four in G. Let Δ be the closed disk bounded by C. Let $G_1 = G \setminus (\Delta \setminus C)$ and $G_2 = G \cap \Delta$. Suppose $G \cap (\Delta \setminus C) \neq \emptyset$. Let ϕ be an L-coloring ϕ of G_1. It follows from a theorem of Bohme at al [3] that ϕ can be extended to an L-coloring of G_2 and hence to an L-coloring of G. Let $T_1 = (G_1, P, L)$. From above, $\Phi_{T_1}(P', \mathcal{C}) \subseteq \Phi_T(P', \mathcal{C})$. Since $P, P' \subseteq G_1$, it follows from the minimality of T that $\Phi_{T_1}(P', \mathcal{C})$ contains a government $\mathcal{C}'$. Furthermore, either $\Phi_{T_1}(P', \mathcal{C})$ contains a confederacy, a contradiction, or, T_1 contains a harmonica T' from P to P' with government $\mathcal{C}$, and hence so does T, a contradiction. □

Claim 4.6 *G is 2-connected.*

Proof. Suppose not. Then there exists a cutvertex v of G. So suppose v divides G into two graphs $G_1, G_2 \neq G$ such that $G_1 \cup G_2 = G$, $V(G_1) \cap V(G_2) = \{v\}$ and without loss of generality $V(P) \subseteq V(G_1)$. Consider the canvases $T_1 = (G_1, P, L)$ and $T_2 = (G_2, U', L)$ where $U' = vw$ is an edge of the outer walk of G_2 containing v. If $V(P') \subseteq V(G_1)$, then by the minimality of T, $\Phi_{T_1}(P', \mathcal{C})$ contains a government, and hence so does $\Phi_T(P', \mathcal{C})$, and either $\Phi_{T_1}(P', \mathcal{C})$ contains a confederacy, or T_1 contains a harmonica T' from P to P' with government $\mathcal{C}$. In the former case, it follows from Theorem 2.2 that every L-coloring of G_1

extends to an L-coloring of G and hence $\Phi_{T_1}(P', \mathcal{C})$ contains a confederacy, a contradiction. In the latter case, T also contains T', a contradiction. So we may assume that $V(P') \subseteq V(G_2)$.

Now suppose there exist two L-colorings ϕ_1, ϕ_2 of T_1 such that $\phi_1(v) \neq \phi_2(v)$. Then there exists a confederacy $\mathcal{C}'$ for U' (a union of two dictatorships) such that every coloring in $\mathcal{C}'$ extends back to T_1. As T is a minimum counterexample, it follows that $\Phi_{T_2}(P', \mathcal{C}')$ has a confederacy. Hence $\Phi_T(P', \mathcal{C})$ has a confederacy, contradicting that T is a counterexample.

Let U be an edge of the outer walk of G_1 containing v. Hence, by the previous paragraph we may assume that $\Phi_{T_1}(U, \mathcal{C})$ is a dictatorship with dictator v. Let c be the color of v in that dictatorship. As T is a minimum counterexample, it follows that T_1 contains a harmonica $T_1' = (G_1', P, L)$ from P to U. Let $\mathcal{C}_2 = \{\psi_1, \psi_2\}$ where $\psi_1(v) = \psi_2(v) = c$ and $\{\psi_1(w), \psi_2(w)\}$ is a subset of $L(w) - \{c\}$ of size two. Note that $\mathcal{C}_2$ is a dictatorship with dictator v. It follows from the minimality of T that $\Phi_{T_2}(P', \mathcal{C}_2)$ contains a government and hence $\Phi_T(P', \mathcal{C})$ contains a government. Furthermore, either $\Phi_{T_2}(P', \mathcal{C}_2)$ contains a confederacy, a contradiction as then $\Phi_T(P', \mathcal{C})$ contains a confederacy, or that T_2 contains a harmonica $T_2' = (G_2', U, L)$ from U' to P'. Let G' be the union of G_1' and G_2' where we delete vertices of $U \setminus V(P)$ that have degree one in G_1' and vertices of $U' \setminus V(P')$ that have degree one in G_2'. Then $T' = (G', P, L)$ is a harmonica from P to P' with government $\mathcal{C}$, a contradiction. □

Claim 4.7 *There does not exist a chord of C.*

Proof. Suppose there exists a chord U of C. Now U divides G into graphs $G_1, G_2 \neq G$ such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = U$, where we may assume without loss of generality that $P \subseteq G_1$. Consider the canvases $T_1 = (G_1, P, L)$ and $T_2 = (G_2, U, L)$. If $V(P') \subseteq V(G_1)$, then by the minimality of T, $\Phi_{T_1}(\mathcal{P}', \mathcal{C})$ contains a government, and hence so does $\Phi_T(\mathcal{P}', \mathcal{C})$, and either $\Phi_{T_1}(\mathcal{P}', \mathcal{C})$ contains a confederacy, or T_1 contains a harmonica T' from P to P' with government $\mathcal{C}$. In the former case, it follows from Theorem 2.2 that every L-coloring of G_1 extends to an L-coloring of G and hence $\Phi_T(P', \mathcal{C})$ contains a confederacy, a contradiction. In the latter case, T also contains T', a contradiction.

So we may assume that $V(P') \subseteq V(G_2)$. By the minimality of T, $\Phi_{T_1}(U, \mathcal{C})$ contains a government $\mathcal{C}'$. Furthermore, either $\Phi_{T_1}(U, \mathcal{C})$ contains a confederacy $\mathcal{C}''$, or, there exists a harmonica $T_1' = (G_1', P, L)$ from P to U with government $\mathcal{C}$. Suppose the former. But then by the minimality of T, $\Phi_{T_2}(P', \mathcal{C}'')$ contains a confederacy and hence $\Phi_T(P', \mathcal{C})$ contains a confederacy by Proposition 3.3, a contradiction.

So we may suppose the latter. By the minimality of T, $\Phi_{T_2}(P', \mathcal{C}')$ contains a government and hence $\Phi_T(P', \mathcal{C})$ contains a government. Furthermore, either $\Phi_{T_2}(P', \mathcal{C}')$ contains a confederacy or there exists a harmonica $T_2' = (G_2', U, L)$ from U to P' with government $\mathcal{C}'$. If the former holds, then $\Phi_T(P', \mathcal{C})$ contains a confederacy by Propoisition 3.3, a contradiction. So suppose the latter. Let G' be obtained from G_1' and G_2' by deleting vertices of $U \setminus (V(P) \cup$

$V(P')$) that have degree one in both G_1 and G_2. Then $T' = (G', P, L)$ is a harmonica from P to P' with government $\mathcal{C}$, a contradiction. $\square$

Claim 4.8 *$P \neq P'$.*

Proof. Suppose not. Note that every L-coloring of P extends to an L-coloring of G by Theorem 2.2 and hence $\Phi(P', C)$ contains $\mathcal{C}$. Thus if $\mathcal{C}$ is a confederacy, $\Phi_T(P', \mathcal{C})$ contains a confederacy, a contradiction. So we may assume that $\mathcal{C}$ is a government but then (P, P, L) is a harmonica from P to P' with government $\mathcal{C}$, a contradiction. $\square$

Claim 4.9 *$V(P) \cup V(P')$ does not induce a triangle.*

Proof. Suppose it does. Let $\{z\} = V(P) \cap V(P')$ and let $P = xz$ and $P' = yz$. Thus x is adjacent to y. By Claims 4.5 and 4.7, $V(G) = V(P) \cup V(P')$.

Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a government of P and let $\phi_1 \neq \phi_2 \in \mathcal{C}_0$. Note that $\Phi_T(P', \mathcal{C}_0) \subseteq \Phi_T(P', \mathcal{C})$. If $\mathcal{C}_0$ is a democracy, then for every $c \in L(y) - \{\phi_1(z), \phi_2(z)\}$, $\Phi_T(P', \mathcal{C}_0)$ contains a dictatorship with dictator y in color c. Hence if there are two such colors, $\Phi_T(P', \mathcal{C}_0)$ contains a confederacy, a contradiction. So in this case, $|\mathcal{C}_0| = 2$, $|L(y)| = 3$ and $L(y) = \{c, \phi_1(z), \phi_2(z)\}$ for some c, $\Phi_T(P', \mathcal{C}_0)$ contains a dictatorship with dictator y in color c, and hence T contains a harmonica from P to P' with government $\mathcal{C}_0$.

If $\mathcal{C}_0$ is a dictatorship with dictator z in color a, then $\Phi_T(P', \mathcal{C}_0)$ contains a dictatorship with dictator z in color a, and T contains a harmonica from P to P' with government $\mathcal{C}_0$.

Next we claim that if $\mathcal{C}_0$ is a dictatorship with dictator x in color b, then $\Phi_T(P', \mathcal{C}_0)$ contains a confederacy, a contradiction, unless $|\mathcal{C}_0| = 2$ and $L(y) = \{b, \phi_1(z), \phi_2(z)\}$, in which case $\Phi_T(P', \mathcal{C}_0)$ contains a democracy with colors $\{\phi_1(z), \phi_2(z)\}$, and T contains a harmonica from P to P' with government $\mathcal{C}_0$. To see this, note that for every color $d \in L(y) - \{b, \phi_1(z), \phi_2(z)\}$, $\Phi_T(P', \mathcal{C}_0)$ contains a dictatorship with dictator y in color d. Thus if $\Phi_T(P', \mathcal{C}_0)$ does not contain a confederacy, there exists at most one such color as otherwise it contains two dictatorships with dictator y in two different colors. But if there exists only one such color, then there exist $d' \in L(y) - \{b, d\}$ since $|L(y)| \geq 3$. But then there exists $d'' \in \{\phi_1(z), \phi_2(z)\} - \{d'\}$ and hence $\Phi_T(P', \mathcal{C})$ contains a dictatorship with dictator z in color d'', a contradiction since then $\Phi_T(P', \mathcal{C}_0)$ contains a confederacy, a contradiction. Finally note that $|\mathcal{C}_0| = 2$ as otherwise there exists $\phi_3 \in \mathcal{C}$ and the same arguments as above imply that $L(y) = \{b, \phi_1(z), \phi_3(z)\}$, a contradiction.

Thus if $\mathcal{C}$ is a government, the arguments above imply that $\Phi_T(P', \mathcal{C})$ contains a government and furthermore that $\Phi_T(P', \mathcal{C})$ contains a confederacy unless T contains a harmonica from P to P' with government $\mathcal{C}$, contradicting that T is a counterexample.

So suppose that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is a confederacy where $\mathcal{C}_1, \mathcal{C}_2$ are governments. From above, there exist governments $\mathcal{C}_1' \subseteq \Phi_T(P', \mathcal{C}_1)$ and $\mathcal{C}_2' \subseteq \Phi_T(P', \mathcal{C}_2)$. Since $C_1' \cup \mathcal{C}_2' \subseteq \Phi_T(P', \mathcal{C})$, it follows that $\mathcal{C}_1'$ and $\mathcal{C}_2'$ are either both democracies in the same colors, or they are both dictatorship with the same dictator in the same color. But then from the arguments above, it follows that the same is true of $\mathcal{C}_1$ and $\mathcal{C}_2$ and hence $\mathcal{C}$ is not a confederacy, a contradiction. $\square$

Claim 4.10 $V(P) \cap V(P') = \emptyset$.

Proof. Suppose not. By Claim 4.8 $P \neq P'$. Let $\{z\} = V(P) \cap V(P')$ and let $P = xz$ and $P' = yz$. By Claim 4.9, x is not adjacent to y. That is to say that $P \cup P'$ is an induced path of length two and yet there does not exist a chord of C by Claim 4.7. By Lemma 2.4, there exists a unique L-coloring ϕ_0 of $P \cup P'$ that does not extend to an L-coloring of G.

Suppose there do not exist $\phi_1, \phi_2 \in \mathcal{C}$ such that $\phi_1(z) \neq \phi_2(z)$. But then $\mathcal{C}$ is a dictatorship with dictator z, say in color c. Let $\psi_1(z) = \psi_2(z) = c$ and let $\{\psi_1(y), \psi_2(y)\}$ be a subset of $L(y) - \{c\}$ of size two. Thus $\mathcal{C}' = \{\psi_1, \psi_2\}$ is a dictatorship on P' with dictator z in color c. Moreover, there exists $\phi \in \mathcal{C}$ such that $\phi(x) \neq \phi_0(x)$. Thus we can extend ψ_1 and ψ_2 to L-colorings of G by letting $\psi_1(x) = \psi_2(x) = \phi(x)$. Hence $\mathcal{C}' \subseteq \Phi_T(P', \mathcal{C})$, and $(P \cup P', P, L)$ is a harmonica from P to P' with government $\mathcal{C}$, a contradiction.

So we may assume that there exist $\phi_1, \phi_2 \in \mathcal{C}$ such that $\phi_1(z) \neq \phi_2(z)$. Let $i \in \{1, 2\}$ be such that $\phi_i(z) \neq \phi_0(z)$. Hence there is a dictatorship $\mathcal{C}_1 \subseteq \Phi(P', C)$ such that $\phi(z) = \phi_i(z)$ for all $\phi \in \mathcal{C}_1$.

Suppose $L(y) - \{\phi_1(z), \phi_2(z), \phi_0(y)\} \neq \emptyset$. Let c be a color in $L(y) - \{\phi_1(z), \phi_2(z), \phi_0(y)\}$. Hence there exists a dictatorship $\mathcal{C}_2 \subseteq \Phi(P', \mathcal{C})$ such that $\phi(y) = c$ for all $\phi \in \mathcal{C}_2$. But then $\Phi(P', \mathcal{C})$ contains the confederacy $\mathcal{C}_1 \cup \mathcal{C}_2$, a contradiction.

So we may assume that $L(y) = \{\phi_0(y), \phi_1(z), \phi_2(z)\}$ as $|L(y)| \geq 3$. Hence, $\phi_0(y) \neq \phi_1(z), \phi_2(z)$. Thus the democracy $\mathcal{C}_3$ in colors $\phi_1(z), \phi_2(z)$ is in $\Phi(P', \mathcal{C})$. But then $\Phi(P', \mathcal{C})$ contains the confederacy $\mathcal{C}_1 \cup \mathcal{C}_3$, a contradiction. $\square$

Claim 4.11 *$\mathcal{C}$ is a government.*

Proof. Suppose not. Then $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is a confederacy. Note that by Claim 4.8, $P \neq P'$, and by Claim 4.9, $V(P) \cup V(P')$ does not induce a triangle. By the minimality of T, since $\Phi_T(P', \mathcal{C}_1)$ does not contain a confederacy, there exists a harmonica $T' = (G', P, L)$ from P to P' with government $\mathcal{C}_1$. Since there does not exist a chord of C by Claim 4.7 and $V(P) \cup V(P')$ does not induce a triangle, we find that $G' = P \cup P'$, contrary to Claim 4.10. $\square$

Claim 4.12 *$\mathcal{C}$ is a dictatorship.*

Proof. Suppose not. Hence $\mathcal{C}$ is a democracy. Let L_0 be the colors of $\mathcal{C}$. Let $Q = q_1 \dots q_k$ be a maximal path in C such that $E(Q) \cap E(P') = \emptyset$, $V(P) \subseteq V(Q)$, $L_0 \subseteq L(v)$ for all $v \in V(Q)$. Note that C is a cycle since G is 2-connected by Claim 4.6.

We claim that $q_1 \in V(P')$. Suppose not. Let $q_1v_1 \in E(C) - E(Q)$. Let Q_1 be the shortest subpath of Q that includes q_1 and both vertices of P. Let $T' = (G \setminus V(Q_1), v_1, L')$ be the democratic reduction $T(Q_1, L_0, v_1)$ of Q_1 with democracy L_0 centered around v_1. Note that $P' \subseteq G \setminus V(Q_1)$ since $V(P) \cap V(P') = \emptyset$ by Claim 4.10. As Q is maximal and $v_1 \notin V(Q)$, L_0 is not a subset of $L(v_1)$ as otherwise $Q + v_1$ would also be a path satisfying the above conditions, contradicting that Q is maximal. Hence $|L'(v_1)| = |L(v_1)| - |L(v_1) \cap L_0| \geq 3 - 1 = 2$. Let P'' be a path of length one of the boundary of the outer face of $G \setminus V(Q_1)$ containing v_1, and let $T'' = (G \setminus V(Q_1), P'', L')$. Now the set of L'-colorings of P'' contains a confederacy $\mathcal{C}'$. By the minimality of T, $\Phi_{T''}(P', \mathcal{C}')$ contains a confederacy. By Lemma 3.7(2), every L'-coloring of $G \setminus V(Q_1)$ extends to an L-coloring of G. Thus every coloring in $\Phi_{T''}(P', \mathcal{C}')$ is in $\Phi_T(P', \mathcal{C})$. Hence $\Phi_T(P', \mathcal{C})$ contains a confederacy, a contradiction. This proves the claim that $q_1 \in V(P')$. By symmetry, it follows that $q_k \in V(P')$.

Yet $V(P) \cap V(P') = \emptyset$ by Claim 4.10. So $q_1, q_k \notin V(P)$. Let $c_1 \in L(q_1) - L_0$ and $c_2 \in L(q_k) - L_0$. Let $\mathcal{C}_1 = \{\phi_1, \phi_1'\}$ where $\phi_1(q_1) = \phi_1'(q_1) = c_1$ and $\{\phi_1(q_k), \phi_1'(q_k)\} = L_0$. Similarly, let $\mathcal{C}_2 = \{\phi_2, \phi_2'\}$ where $\phi_2(q_k) = \phi_2'(q_k) = c_2$ and $\{\phi_2(q_1), \phi_2'(q_1)\} = L_0$. Hence $\mathcal{C}_1$ and $\mathcal{C}_2$ are distinct governments of P' and $\mathcal{C}' = \mathcal{C}_1 \cup \mathcal{C}_2$ is a confederacy.

Moreover, for all $\phi \in \mathcal{C}'$, $\phi \in \Phi_T(P', \mathcal{C})$. To see this, consider the democratic reductions $T_1 = T(Q \setminus q_1, L_0, q_1)$ and $T_2 = T(Q \setminus q_k, L_0, q_k)$. By Theorem 2.2, every L-coloring of q_1 extends to an L-coloring of T_1 and every L-coloring of q_k extends to an L-coloring of T_2. Thus if $\phi \in \mathcal{C}_1$, then ϕ extends to an L-coloring of T_1 as noted above and can then be extended to $Q \setminus q_1$ by using the colors of L_0. Hence $\phi \in \Phi_T(P', \mathcal{C})$. Similarly if $\phi \in \mathcal{C}_2$, then ϕ extends to an L-coloring of T_2 as noted above and can the be extended to $Q \setminus q_k$ by using the colors of L_0. Hence $\phi \in \Phi_T(P', \mathcal{C})$. Thus $\Phi_T(P', \mathcal{C})$ contains a confederacy, a contradiction. □

Suppose without loss of generality that p_1 is the dictator of $\mathcal{C}$ in color c. Let v_1, v_2 be the vertices of C adjacent to p_1. Suppose without loss of generality that $P = p_1v_1$. Our next objective is to consider two democratic reductions centered at p_1, one removing v_1 and the other removing v_2. However, for the one removing v_2 we need to define a new canvas so as to ensure that v_1 has a proper list. Moreover, we first have to show that $v_1, v_2 \notin V(P')$ if we are to study the colorings that extend to P' in these reductions. Let $\phi_1, \phi_2 \in \mathcal{C}$. Then $c = \phi_1(p_1)$. Let $M_1 = \{\phi_1(v_1), \phi_2(v_1)\}$. Let $L'(v_1) = M_1 \cup \{c\}$, $L'(p_1) = \{c\}$ and let $L'(w) = L(w)$ otherwise. Let $T' = (G, p_1, L')$. Let M_2 be a subset of $L(v_2) - \{c\}$ of size two.

Claim 4.13 *Neither v_1 nor v_2 is in P'.*

Proof. Suppose not. By Claim 4.10 it follows that v_2 is in P'. Let $P' = v_2z$. Let $S' = p_1v_2z$ and let $T'' = (G, S', L')$. Since there does not exist a chord of C by Claim 4.7,

S' is an induced path of length two. Thus by Lemma 2.4, there exists at most one proper L'-coloring of S', call it ϕ_0, that does not extend to an L'-coloring of G.

Let $c_1 \in L(v_2) - \{c, \phi_0(v_2)\}$. Let $\mathcal{C}'_1 = \{\psi_1, \psi'_1\}$ where $\psi_1(v_2) = \psi'_1(v_2) = c_1$ and $\{\psi_1(z), \psi'_1(z)\}$ is a subset of $L(z) - \{c_1\}$ of size two. Thus $\mathcal{C}'_1$ is a dictatorship on P' with dictator v_2. Suppose that $M_2 = \{c_2, c_3\}$. If there exists $c_4 \in L(z) - (\{\phi_0(z)\} \cup M_2)$, then let $\mathcal{C}'_2 = \{\psi_2, \psi'_2\}$ where $\psi_2(z) = \psi'_2(z) = c_4$ and $\{\psi_2(v_2), \psi'_2(v_2)\} = M_2$. Otherwise $L(z) = M_2 \cup \{\phi_0(z)\}$. In that case, let $\mathcal{C}'_2 = \{\psi_3, \psi'_3\}$ where $\psi_3(v_2) = \psi'_3(z) = c_2$ and $\psi'_3(v_2) = \psi_3(z) = c_3$. Thus in the first case, $\mathcal{C}'_2$ is a dictatorship with dictator z and in the second case $\mathcal{C}'_2$ is a democracy. Thus in either case, $\mathcal{C}' = \mathcal{C}'_1 \cup \mathcal{C}'_2$ is a confederacy. Yet in either case, $\mathcal{C}' \subseteq \Phi_{T'}(P', \mathcal{C}) \subseteq \Phi_T(P', \mathcal{C})$, since in every coloring in $\mathcal{C}'$ either v_2 or z receives a color different from the color it receives in ϕ_0, a contradiction. □

Now consider the democratic reductions $T_1 = T'(v_1, M_1, p_1), T_2 = T'(v_2, M_2, p_1)$. Suppose that $T_1 = (G_1, p_1, L_1)$ and $T_2 = (G_2, p_1, L_2)$. Let $T'_1 = (G_1, p_1v_2, L_1)$ and $T'_2 = (G_2, P, L_2)$. By the minimality of T, $\Phi_{T'_2}(P', \mathcal{C})$ contains a government $\mathcal{C}_2$. Furthermore, either $\Phi_{T'_2}(P', \mathcal{C})$ contains a confederacy, a contradiction as then so does $\Phi_T(P', \mathcal{C})$, or, T'_2 contains a harmonica $T''_2 = (G'_2, P, L_2)$ from P to P' with government $\mathcal{C}$.

Let $\mathcal{C}^* = \{\phi_1, \phi_2, \ldots \phi_k\}$ where $\phi_1(p_1) = \phi_2(p_1) = \ldots = \phi_k(p_1) = c$ and $\{\phi_1(v_2), \phi_2(v_2), \ldots, \phi_k(v_2)\} = L(v_2) - \{c\}$. By the minimality of T, where we consider the canvas T'_1, we find that $\Phi_{T'_1}(P', \mathcal{C}^*)$ contains a government $\mathcal{C}_1$. Furthermore, either $\Phi_{T'_1}(P', \mathcal{C}^*)$ contains a confederacy, a contradiction as then so does $\Phi_T(P', \mathcal{C})$, or, T'_1 contains a harmonica $T''_1 = (G'_1, p_1v_2, L_1)$ from P to P' with government $\mathcal{C}^*$.

Let $P' = p'_1p'_2$ where p'_1 is on the subpath of C from p'_2 to v_1 not containing p_1.

Claim 4.14 *There exists $v \notin C$ such that $v \sim p_1, v_1, v_2, p'_1, p'_2$*

Proof. Let W_1 be the outer walk of G'_1 and W_2 be the outer walk of G'_2. Since T''_1 is obtained by an application of the third rule by Claim 4.13, the vertex p_1 has two neighbors u_1, u_2 such that $|L_1(u_i)| = 3$ if $u_i \neq v_2$. But not both u_1, u_2 can be adjacent to v_1 by planarity and Claim 4.5, and hence one of them, say u_2 belongs to C. It follows from Claim 4.7 that $u_2 = v_2$. Thus $v_2 \in W_1$ and p_1 is not a cutvertex of G'_1. Let W'_1 be the subwalk of W_1 from p_1 to P' not containing v_2. Let W''_1 be the subwalk of W_1 from p_1 to P' containing v_2.

Note if $z \in V(G'_1) - V(C)$, then $|L_1(z)| \leq 3$ by Lemma 4.2. Hence $|L(z)| = 5$, $|L_1(z)| = 3$, $M_1 \subseteq L(z)$ and z is adjacent to v_1. However if $z \in V(W''_1) - V(W'_1)$, then z is not adjacent to v_1 since G is planar and hence $z \in V(C)$.

We claim that there exists $w_1 \in V(G'_1) - V(C)$ such that w_1 is adjacent to v_1 and p'_2. To see this, note that there exists a vertex $w_1 \in V(W'_1)$ in a triangle R in G'_1 such that either $R = w_1p'_1p'_2$ or $R = w_1p'_iu_1$ for some $i \in \{1, 2\}$ where u_1 in $V(W''_1) - V(W'_1)$. In the former case, $w_1 \notin V(C)$ by Claim 4.7 and hence w_1 is adjacent to v_1 as desired. In the latter case, it follows that that u_1 is not adjacent to v_1, and hence $u_1 \in V(C)$ by the result of the previous paragraph. But then $i = 2$ by Claim 4.7. Moreover, by Claim 4.7, $w_1 \notin V(C)$ given that w_1 is adjacent to u_1 and hence w_1 is adjacent to v_1 as desired.

By symmetry, there exists $w_2 \in V(G'_2)$ such that w_2 is adjacent to v_2 and p'_1. As G is planar, we find that $w_1 = w_2$, call it v, and hence v is adjacent to u_1, u_2, v_1, v_2. Moreover as T''_2 is a harmonica with government $\mathcal{C}$ and $\mathcal{C}$ is a dictatorship with dictator p_1, we find that v is adjacent to p_1. □

Claim 4.15 *There exist vertices $v_2 = z_1, z_2, ..., z_k = p'_2$, all adjacent to v such that for all $i = 1, 2, \ldots, k$ the canvas $(G'_1 \setminus \{p_1, z_1, ..., z_{i-1}\}, vz_i, L_1)$ is a harmonica from vz_i to P' with a democracy or a dictatorship depending on the parity of i.*

Proof. The canvas $(G'_1 \setminus p_1, vz_1, L_1)$ is a harmonica from vz_1 to P' with a democracy obtained from T''_1 by application of the third rule, $(G'_1 \setminus \{p_1, v_2\}, vz_2, L_1)$ is a harmonica from vz_2 to P' with a dictatorship obtained from $(G'_1 \setminus p_1, vz_1, L_1)$ by application of the fourth rule, and so on. We note that the vertex v cannot be the vertex that is being deleted during the construction of the next harmonica, because the next-to-last harmonica in the sequence leading up to P' involves the vertex v. This proves Claim 4.15. □

We are now ready to complete the proof of Theorem 4.3. It follows that M_2 is a subset of $L(v)$ and $L(z_i)$ for all $i, 1 \le i \le k$. Similarly there exist vertices $v_1 = w_1, ..., w_l = p'_1$ and M_1 is a subset of $L(v)$ and $L(w_i)$ for all $i, 1 \le i \le l$. Since $M_1 \cup M_2 \cup \{c\} = L(v)$ by Lemma 4.2, we see that M_1 and M_2 are disjoint. Since $|L(p'_1)| = |L(p'_2)| = 3$ by Lemma 4.2 as T''_1 and T''_2 are harmonicas, and $M_1 \subseteq L(p'_1)$ and $M_2 \subseteq L(p'_2)$, it follows that the last step in the construction of T''_1 is according to the second rule of the definition of harmonica. In other words, p'_2 is a dictator of $\mathcal{C}_1$. Similarly, p'_1 is a dictator of $\mathcal{C}_2$. Thus $\mathcal{C}_1 \cup \mathcal{C}_2$ is a confederacy, as desired. This completes the proof of Theorem 4.3. □

In the following lemma we clarify the relationship between harmonicas and coloring harmonicas before we can prove Theorem 1.3.

Lemma 4.16 *Let $T = (G, P, L)$ be a canvas and P, P' be paths of length one in the boundary of the outer face of G. Let $P = uv$, let $P' = ww'$, where u, v, w are pairwise distinct, and let T be a harmonica from P to P' with government $\mathcal{C}$. Assume that $\Phi(P', \mathcal{C})$ is a dictatorship with dictator w in color d, that if $\mathcal{C}$ is a democracy, then $|L(u)| = |L(v)| = 2$, and that if $\mathcal{C}$ is a dictatorship, then u is the dictator, $|L(u)| = 1$ and if v has degree in G of at least two, then $|L(v) - L(u)| = 2$. Let G' be obtained from G by deleting either or both of v and w' if either has degree one in G, let $L'(w) = L(w) - \{d\}$ and $L'(x) = L(x)$ for all $x \in V(G') - \{w\}$. If $\mathcal{C}$ is a dictatorship, then (G', L') is a coloring harmonica from u to w. If $\mathcal{C}$ is a democracy, then (G', L') is a coloring harmonica from uv to w.*

Proof. We proceed by induction on $|V(G)|$. If $|V(G)| = 3$, then, since w is the dictator of $\Phi(P', \mathcal{C})$, and $w \notin V(P)$, we deduce that T is obtained according to the fourth rule in the definition of harmonica. Thus $\mathcal{C}$ is a democracy, u, v, w form a triangle, $L'(u) = L'(v) = L'(w)$ and $|L(u)| = 2$. Thus (G', L') is a coloring harmonica from uv to w. We may therefore assume that $|V(G)| \geq 4$.

Assume now that $\mathcal{C}$ is a democracy. Thus T is obtained according to the fourth rule in the definition of harmonica. Consequently, there exists a vertex z adjacent to u and v such that $L(z) = L_0 \cup \{c\}$ where $L_0 = \mathcal{C}(u) = \mathcal{C}(v)$ and there exist x, y such that $\{x, y\} = \{u, v\}$, the canvas $T' = (G \setminus x, U, L)$ is a harmonica with dictatorship $\mathcal{C}' = \{\phi_1, \phi_2\}$, where $U = zy$ and $\phi_1(z) = \phi_2(z) = c$ and $\{\phi_1(y), \phi_2(y)\} = L_0$. It follows that $L(u) = L(v) = L_0$. If $z = w$, then T' is obtained according to the second rule in the definition of harmonica, and hence w' has degree one in G. It follows that (G', L') is a coloring harmonica from uv to w, because G' is obtained from G by deleting w'. We may therefore assume that $z \neq w$.

Thus T' is obtained according to the third rule in the definition of harmonica. Let u_1, u_2 be as in that rule, and let $G'_1 = G' \setminus x$ if $y \in \{u_1, u_2\}$ and $G'_1 = G' \setminus \{x, y\}$ otherwise. Let $L_1(z) = L(z) - L_0$ and $L_1(z') = L(z')$ for all $z' \in V(G) - \{z\}$, and let $L'_1(x') = L'(x')$ for all $x' \in V(G'_1)$. It follows by induction applied to the canvas $(G \setminus x, U, L_1)$ that (G'_1, L'_1) is a coloring harmonica from z to w, and hence (G', L') is a coloring harmonica from uv to w, as desired.

We may therefore assume that $\mathcal{C}$ is a dictatorship. Since $w \notin V(P)$, it follows that T is obtained according to the third rule in the definition of harmonica. Thus there exists a triangle uu_1u_2 and letting c denote the color in which u is the dictator, for $i = 1, 2$ we have $L(u_i) = L_0 \cup \{c\}$ if $u_i \notin V(P)$ and $\mathcal{C}(u_i) = L_0$ otherwise, where $|L_0| = 2$ and the canvas $(G \setminus (V(P) - V(U)), U, L)$ is a harmonica from $U = u_1u_2$ to P' with democracy $\mathcal{C}'$, where $\mathcal{C}'(u_1) = \mathcal{C}'(u_2) = L_0$. Let us note that $|L(u)| = 1$ and if $u_i \in V(P)$, then $u_i = v$, and hence $|L(u_i) - L(u)| = 2$. Let $G'_1 = G' \setminus u$, let $L_1(u_1) = L'_1(u_1) = L(u_1) - L(u)$, $L_1(u_2) = L'_1(u_2) = L(u_2) - L(u)$, and let $L_1(u') = L(u')$ and $L'_1(u') = L'(u')$ for all $u' \in V(G'_1) - \{u_1, u_2\}$. It follows by induction applied to the canvas $(G \setminus (V(P) - V(U)), U, L_1)$ that (G'_1, L'_1) is a coloring harmonica from u_1u_2 to w, and hence (G', L') is a coloring harmonica from u to w, as desired. □

Proof of Theorem 1.3. We prove only the backward direction as the forward direction is trivial. So let G, p_1, p_2, L be as in the statement of the theorem and suppose for the sake of contradiction that G is not L-colorable and yet (G, L) does not contain a coloring harmonica from p_1 to p_2.

Let $p_1v_1, p_2v_2 \in E(C)$ and let $P = p_1v_1$ and $P' = p_2v_2$. Let $c \in L(p_1)$. Let $\mathcal{C}_c = \{\phi_1, \phi_2, \ldots \phi_k\}$ where $\phi_i(p_1) = c$ for all $i, 1 \leq i \leq k$, and $\{\phi_1(v_1), \phi_2(v_2), \ldots, \phi_k(v_1)\} = L(v_1) - \{c\}$. Note that $\mathcal{C}_c$ is a dictatorship on P with dictator p_1 in color c. If there exists

$c' \in L(p_1) - \{c\}$, then let $\mathcal{C} = \mathcal{C}_c \cup \mathcal{C}_{c'}$, in which case $\mathcal{C}$ is a confederacy; otherwise let $\mathcal{C} = \mathcal{C}_c$, in which case $\mathcal{C}$ is a dictatorship on P with dictator p_1 in color c. Let c_0 be a new color and let $L'(p_2) = L(p_2) \cup \{c_0\}$, and let $L'(x) = L(x)$ for all $x \in V(G) - \{p_2\}$. Let $T = (G, P, L')$. By Theorem 4.3, $\Phi_T(P', \mathcal{C})$ contains a government $\mathcal{C}'$. Furthermore, either $\Phi_T(P', \mathcal{C})$ contains a confederacy $\mathcal{C}''$, or, $\mathcal{C}$ is a government and there exists a harmonica $T' = (G', P, L')$ from P to P' with government $\mathcal{C}$.

In the former case, there exists a coloring $\phi \in \mathcal{C}''$ such that $\phi(p_2) \neq c_0$. But then ϕ extends to an L'-coloring of G as $\phi \in \Phi_T(P', \mathcal{C})$ and yet ϕ is an L-coloring of G since $\phi(p_2) \neq c_0$, a contradiction. So we may assume the latter case. Thus $|L(p_1)| = 1$, because $\mathcal{C}$ is a government; and if v_1 has degree at least two in G, then $k = 2$, because T' is a harmonica obtained according to the third rule. As above, there does not exist a coloring $\phi \in \mathcal{C}'$ such that $\phi(p_2) \neq c_0$. Hence $\mathcal{C}'$ is a dictatorship with dictator p_2 in color c_0. Let G'' be obtained from G' by deleting either or both of v_1 and v_2 if either has degree one in G'. Now (G'', L) is a coloring harmonica from p_1 to p_2 by Lemma 4.16, a contradiction. □

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References

[1] K. Appel and W. Haken, Every planar map is four colorable, Part I: discharging, Illinois J. of Math. 21 (1977), 429–490.

[2] K. Appel, W. Haken, J. Koch, Every planar map is four colorable, Part II: reducibility, Illinois J. of Math. 21 (1977), 491–567.

[3] T. Bohme, B. Mohar and M. Stiebitz, Dirac's map-color theorem for choosability, J. Graph Theory 32 (1999), 327–339.

[4] J. Hutchinson, On list-coloring extendable outerplanar graphs, Ars Mathematica Contemporanea 5 (2012) 171–184.

[5] L. Postle, 5-list-coloring graphs on surfaces, Ph.D. Dissertation, Georgia Institute of Technology, 2012.

[6] L. Postle and R. Thomas, Five-List-Coloring Graphs on Surfaces 1. Two Lists of Size Two in Planar Graphs. Journal of Combinatorial Theory Ser. B 111 (2015), pp. 234–241.

[7] C. Thomassen, Every planar graph is 5-choosable, J. Combin. Theory Ser. B 62 (1994), 180–181.

[8] C. Thomassen, Exponentially many 5-list-colorings of planar graphs, J. Combin. Theory Ser. B 97 (2007), 571–583.

[9] M. Voigt, List colourings of planar graphs, Discrete Mathematics 120 (1993) 215–219.

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