Coloring planar graphs with triangles far apart*

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Abstract

We settle a problem of Havel by showing that there exists an absolute constant d such that if G is a planar graph in which every two distinct triangles are at distance at least d, then G is 3-colorable.

1 Introduction

In this paper we are concerned with 3-coloring planar graphs. All *graphs* in this paper are finite and simple; that is, have no loops or multiple edges. The following is a classical theorem of Grötzsch [3].

Theorem 1. *Every triangle-free planar graph is* 3*-colorable.*

There is a long history of generalizations that extend the theorem to classes of graphs that include triangles. We will survey them in a future version of this paper. Let G be a graph, and let $X, Y \subseteq V(G)$. We say that the sets X, Y are *at distance* d in G if d is the maximum integer such that every path with one end in X and the other end in Y has length at least d. We say that two subgraphs are at distance d if their vertex-sets are at distance d. The purpose of this paper is to describe a solution of a problem of Havel [4, 5]:

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Theorem 2. *There exists an absolute constant d such that if G is a planar graph and every two distinct triangles in G are at distance at least d, then G is 3-colorable.*

Our proof relies heavily on the following theorem, which we will prove in [2]. If f is a face of a planar graph, then we denote by $|f|$ the sum of the lengths of the boundary walks of f.

Theorem 3. *There exists an absolute constant K with the following property. Let G be a planar graph with no separating cycles of length at most four, let C be a subgraph of G such that C is either the null graph or an induced facial cycle of G of length at most five, and assume that there exists a 3-coloring of C that does not extend to a 3-coloring of G, but extends to every proper subgraph of G that includes C. Then $\sum |f| \leq Kt$, where the summation is over all faces f of G of length at least five and t is the number of triangles in G.*

While the idea behind our proof of Theorem 3 is fairly simple, the details are quite laborious.

2 Extending a coloring to a cylindrical grid

In this section we prove a lemma about extending a precoloring to a "cylindrical grid".

Let G be a graph drawn (without crossings) in an orientable surface Σ, and assume that we have chosen an orientation of Σ, which we shall refer to as the *clockwise orientation*. Now let C be a cycle bounding a face f in G, let $v_1, v_2, \ldots, v_k$ be the vertices of C listed in the clockwise order of their appearance on C, and let $\phi : V(C) \to \{1, 2, 3\}$ be a 3-coloring of C. We can view ϕ as a mapping of $V(C)$ to the vertices of a triangle, and speak of the *winding number of ϕ on C*, defined as the number of indices $i \in \{1, 2, \ldots, k\}$ such that $\phi(v_i) = 1$ and $\phi(v_{i+1}) = 2$ minus the number of indices i such that $\phi(v_i) = 2$ and $\phi(v_{i+1}) = 1$, where v_{k+1} means v_1. If the graph G is understood from the context and it is not a cycle, then we denote the winding number of ϕ on C by $w_\phi(C)$. Let us emphasize that the orientation of C is determined by the face it bounds. Thus if $G = C$, then $w_\phi(C)$ is ambiguous, because it does not specify the face that determines the orientation of G. In that case there are two faces bounded by C. They give rise to opposite orientations of C, and hence the corresponding winding numbers sum up to zero.

The following two propositions are easy to prove.

Proposition 4. *Let G be a graph drawn in an orientable surface in such a way that every face is bounded by a cycle, and let $\phi : V(G) \to \{1,2,3\}$ be a 3-coloring. Then the sum of the winding numbers of all the face boundaries of G is zero.*

Proposition 5. *The winding number of every 3-coloring on a cycle of length four is zero.*

Let $r \geq 3$ and $s \geq 1$ be integers. By the $r \times s$ *cylindrical grid* we mean the Cartesian product of a cycle of length r and a path on s vertices. More precisely, the $r \times s$ cylindrical grid H is obtained from a union of disjoint cycles $D_1, D_2, \ldots, D_s$ of length r by adding edges so that the i^{th} vertex of D_j is adjacent to the i^{th} vertex of D_{j+1} for all $i = 1,2,\ldots,r$ and $j = 1,2,\ldots,s-1$. The cycles $D_1, D_2, \ldots, D_s$ will be called the *hoops* of H, and the cycles D_1 and D_s will be called the *cuffs* of H. We will regard H as drawn in the sphere with a specified orientation so that the notions of the paragraph prior to Proposition 4 can be applied. We will also need to apply the notion of a winding number to the cycles D_i for $i = 2,3,\ldots,s-1$. In that case the winding number will be interpreted in a subgraph of H specified below, in which D_i will be a face boundary.

Lemma 6. *Let $r \geq 3$ be an integer, let $s = \lceil (r+3)/2 \rceil$, let G be the $r \times s$ cylindrical grid, let C_1 and C_2 be its cuffs, let $v_0 \in V(C_2)$, and let ϕ be a 3-coloring of C_1 satisfying $|w_\phi(C_1)| \leq 1$. Then ϕ can be extended to a 3-coloring ψ of G such that the restriction of ψ to $V(C_2) - \{v_0\}$ uses only two colors.*

Proof. Let $D_1, D_2, \ldots, D_s$ be the hoops of G so that $C_1 = D_1$ and $C_2 = D_s$. For $p = 1,2,\ldots,s$ let G_p be the subgraph of G induced by $V(D_1 \cup D_2 \cup \cdots \cup D_p)$, and let ψ be an extension of ϕ to a 3-coloring of G_p. Let P be a subpath of D_p of even length with ends u, u' such that the restriction of ψ to $V(P)$ uses at most two colors $\alpha, \beta \in \{1,2,3\}$ such that $\psi(u) = \alpha \equiv \beta - 1 (\mathrm{mod}\ 3)$. It follows that $\psi(u) = \psi(u')$. In other words, if the ends of P are colored 1, say, then the other color that ψ uses on P is 2. In those circumstances we say that $V(P)$ is a *segment* of D_p (with respect to ψ), and we say that α is its *flag*. By a *segmentation* of D_p we mean a partition of $V(D_p)$ into disjoint segments. We say that the integer p is *progressive* if D_p has a segmentation

$(X_1, X_2, \ldots, X_k)$ with $k \leq r - 2p + 2$. Since the partition of $V(D_p)$ into singletons is a segmentation, we see that the integer 1 is progressive. Thus we may assume that p is the maximum progressive integer in $\{1, 2, \ldots, s-2\}$, and let ψ be the corresponding extension of ϕ.

For $i = 1, 2, \ldots, s-1$ and $v \in V(D_i)$ we define $f(v)$ to be the unique neighbor of v in D_{i+1}. We claim that $p = s-2$. To prove this claim suppose for a contradiction that $p < s-2$. Let $(X_1, X_2, \ldots, X_k)$ be a segmentation of D_p with $k \leq r-2p+2$. It follows that $k = r-2p+2$, for otherwise $k \leq r-2p$, because k and r have the same parity (since each $|X_i|$ is odd), and on giving the vertex $f(v)$ color $\psi(v) + 1 \pmod 3$ we find that $p+1$ is progressive, contrary to the maximality of p. Thus $k = r-2p+2 \geq r-2\lceil (r+3)/2 \rceil + 8 \geq 4$. Let C_k denote the cycle with vertex-set $\{1, 2, \ldots, k\}$, in order, and let λ be the 3-coloring of C_k defined by saying that $\lambda(i)$ is the flag of X_i. This is clearly a proper 3-coloring of C_k and it has the same winding number as ψ on D_p for an appropriately chosen direction of C_k, when D_p is regarded as a face of G_p. But $w_\psi(D_p) + w_\phi(C_1) = 0$ by Propositions 4 and 5 applied to the graph G_p. Since $k \geq 4$ and $|w_\phi(C_1)| \leq 1$ it follows that there exist consecutive segments, say X_1, X_2, X_3, such that the flags of X_1 and X_3 are equal. From the symmetry we may assume that X_1 and X_3 have flag 1. Let us assume first that X_2 has flag 2. Let $\psi(f(v)) \equiv \psi(v) + 1 \pmod 3$ for $v \in V(D_p)$, except for $v \in X_2$ with $\psi(v) = 3$; for those vertices we define $\psi(f(v)) = 2$. This extends ψ to G_{p+1}. Then $f(X_1 \cup X_2 \cup X_3)$ is a segment of D_{p+1} with respect to ψ, and hence $(f(X_1 \cup X_2 \cup X_3), f(X_4), f(X_5), \ldots, f(X_k))$ is a segmentation of D_{p+1} with at most $k-2 = r-2(p+1)+2$ blocks. Consequently $p+1$ is progressive, contrary to the choice of p. If X_2 has flag 3, then we proceed analogously, using $\psi(f(v)) = 1$ for $v \in X_2$ with $\psi(v) = 3$ and $\psi(f(v)) \equiv \psi(v) - 1 \pmod 3$ for all other $v \in V(D_p)$. This proves our claim that $p = s-2$.

Thus D_{s-2} has a segmentation into at most three blocks (at most two is r is even). If r is even, then the result follows easily, and so we assume that r is odd. We describe how to extend ψ to D_{s-1} and D_s. Let v_1 be the unique vertex of D_{s-2} with $f(f(v_1)) = v_0$. We may assume without loss of generality that X_i has flag i for $i = 1, 2, 3$, and that $v_1 \in X_3$. Let X_3' and X_3'' be the vertex-sets of the two paths of the graph induced in D_{s-2} by $X_3 - \{v_1\}$, numbered so that X_3' has a neighbor in X_2 and X_3'' has a neighbor in X_1. One of the sets X_3', X_3'' may be empty. Let (A, B) be a 2-coloring of $D_{s-2} \backslash v_1$ such that the ends of X_1 belong to A. Let $a = 1$ and $b = 3$ if $|X_3'|, |X_3''|$ are both even, and let let $a = 3$ and $b = 1$ if $|X_3'|, |X_3''|$ are both odd. We define $\psi(f(v)) = 2$ for all $v \in A$, $\psi(f(v)) = 3$ for all $v \in B - X_3''$, $\psi(f(v)) = 1$ for all $v \in B \cap X_3''$, $\psi(f(v_1)) = a$, $\psi(f(f(v_1))) = b$, $\psi(f(f(v))) = a$ for all $v \in A$, and $\psi(f(f(v))) = 2$ for all $v \in B$. Then ψ is a desired coloring of G. $\square$

Lemma 7. *Let $r \geq 3$ be an integer, let G be the $r \times (r+5)$ cylindrical grid, let C_1 and C_2 be the two cuffs of G, and let ϕ be a 3-coloring of $C_1 \cup C_2$ such that $|w_\phi(C_1)| \leq 1$ and $w_\phi(C_1) + w_\phi(C_2) = 0$. Then there exists a 3-coloring ψ of G such that $\psi(v) = \phi(v)$ for every $v \in V(C_1 \cup C_2)$.*

Proof. Let $D_1, D_2, \ldots, D_{r+5}$ be the hoops of G. Let $s = \lceil (r+3)/2 \rceil$, let $u \in V(D_s)$ be arbitrary, and let $v \in V(D_{s+2})$ be the nearest vertex to u. By two applications of Lemma 6 we deduce that ψ can be extended to a 3-coloring ψ of $G \backslash V(D_{s+1})$ such that the restriction of ψ to $D_s \backslash u$ uses only two colors, and likewise the restriction of ψ to $D_{s+2} \backslash v$ uses only two colors. We regard D_s as a face of the subgraph of G induced by $V(D_1 \cup D_2 \cup \cdots \cup D_s)$, and we regard D_{s+2} as a face of the subgraph of G induced by $V(D_{s+2} \cup D_{s+3} \cup \cdots \cup D_{r+5})$. The condition $w_\phi(C_1) + w_\phi(C_2) = 0$ and Propositions 4 and 5 imply that the winding number of ψ on D_s and the winding number of ψ on D_{s+2} add up to zero. It follows that the coloring ψ can be extended to a 3-coloring of all of G, as desired. $\square$

3 Proof of Theorem 2

Let C be a cycle in a graph G, and let $S \subseteq V(G)$. We say that the cycle C is *S-tight* if C has length four and the vertices of C can be numbered v_1, v_2, v_3, v_4 in order such that for some integer $t \geq 0$ the vertices v_1, v_2 are at distance exactly t from S, and the vertices v_3, v_4 are at distance exactly $t+1$ from S.

Lemma 8. *Let $d \geq 1$ be an integer, let G be a graph, and let $\mathcal{S}$ be a family of distinct subsets of $V(G)$ such that every two distinct sets of $\mathcal{S}$ are at distance at least $2d$. Let C be a cycle in G of length four that is at distance at most $d-1$ from $S_0 \in \mathcal{S}$, and assume that for each pair u, v of diagonally opposite vertices of C, some pair of distinct sets in $\mathcal{S}$ are at distance at most $2d-1$ in the graph obtained from G by identifying u and v. Then C is S_0-tight.*

Proof. Let the vertices of C be v_1, v_2, v_3, v_4 in order. By hypothesis there exist sets $S_1, S_2, S_3, S_4 \in \mathcal{S}$, where S_i is at distance d_i from v_i, such that $S_1 \neq S_3$, $S_2 \neq S_4$, $d_1 + d_3 \leq 2d-1$, and $d_2 + d_4 \leq 2d-1$. From the symmetry we may assume that $d_1 \leq d-1$ and $d_2 \leq d-1$. That implies that S_1, S_2 and at least one of the pairs S_0, S_1 and S_0, S_2 are at distance at most $2d-1$, and hence $S_0 = S_1 = S_2$. But $S_4 \neq S_2 = S_1$, and hence $d_1 + d_4 + 1 \geq 2d$, because S_1 and S_4 are at distance at least $2d$. This and the inequality $d_2+d_4 \leq 2d-1$ imply that $d_1 \geq d_2$. But there is symmetry between d_1 and d_2, and hence an analogous argument shows that $d_1 \leq d_2$. Thus for $t := d_1 = d_2$ the vertices v_1, v_2 are both at distance t from $S_0 = S_1 = S_2$. If v_4 was at distance t or less from S_0, then S_0 and S_4 would be at distance at most $t+d_4 = d_2+d_4 \leq 2d-1$, a contradiction. The same holds for v_3 by symmetry, and hence v_3 and v_4 are at distance $t+1$ from S_0, as desired. □

Let G be a graph, let $S \subseteq V(G)$ and let C be a cycle or a path in G. We say that C is *equidistant* from S if for some integer $t \geq 0$ every vertex of C is at distance exactly t from S. We will also say that C is equidistant from S at distance t.

Lemma 9. *Let G be a plane graph, let $s, i_0 \geq 1$ be integers, and let $S \subseteq V(G)$ induce a connected subgraph of G. Assume that for every integer i satisfying $i_0 \leq i \leq i_0 + 2s + 7\log_2 s + 7$ every face of G at distance exactly i from S is bounded by an S-tight cycle. Assume also that there exists an equidistant cycle C_0 at distance i_0 from S of length at most s. Then G has a subgraph isomorphic to an $r \times (r+5)$ cylindrical grid for some integer r satisfying $3 \leq r \leq s$.*

Proof. Let C_0 be as stated. We may assume, by replacing C_0 by a shorter cycle, that C_0 is induced. We may choose the maximum integer $j \geq 0$ such that there exists an induced equidistant cycle C at distance t from S of length r, where $i_0 \leq t \leq i_0 + s(1 + 1/2 + \cdots + 1/2^{j-1}) + 7j$ (or $t = i_0$ if $j = 0$), and $r \leq s/2^j + 1 + 1/2 + \cdots + 1/2^{j-1}$. Let p be the maximum integer such that G has a subgraph H isomorphic to the $r \times p$ cylinder with one cuff C and such that each D_i is equidistant from S at distance $t + i - 1$ for all $i = 1, 2, \ldots, p$, where $D_1 = C, D_2, \ldots, D_p$ are the hoops of H. Such integers j and p exist, because the $r \times 1$ cylindrical grid C_0 satisfies the requirements.

We claim that H satisfies the conclusion of the theorem. To prove that it suffices to show that $p \geq s/2^j + 1 + 1/2 + \cdots + 1/2^{j-1} + 5$, and so we may assume for a contradiction that $p \leq s/2^j + 7$. If D_p is not induced, then $V(D_p)$ includes the vertex-set of an induced equidistant cycle of length at most $|V(D_p)|/2 + 1 \leq r/2 + 1 \leq s/2^{j+1} + 1 + 1/2 + \cdots + 2^j$ at distance $t + p - 1 \leq i_0 + s(1 + 1/2 + \cdots + 1/2^j) + 7(j+1)$ from S, contrary to the maximality of j. Thus D_p is induced.

Let Δ be the open disk bounded by D_p; using the fact that S induces a connected subgraph of G and the symmetry between Δ and the other component of $\mathbf{R}^2 - D_p$ we may assume that Δ includes the set S. Thus the closure of Δ includes H. Now let uv be an edge of D_p, and let f be the face of G incident with uv that is not a subset of Δ. Then f is at distance $t+p-1$ from S. From the upper bound on r and the fact that $r \geq 3$ we deduce that $j \leq \log_2 s$. It follows that $t+p-1 \leq i_0+2s+7\log_2 s+7$, and hence the face f is bounded by an S-tight cycle. Thus the vertices on the boundary on f may be denoted by u, v, v', u' in order. Since f is not a subset of Δ it follows that u', v' are at distance $t+p$ from S. Now let w be the other neighbor of v on C, and let us repeat the same argument to the edge vw, obtaining a face boundary v, w, w', v''.

We claim that $v' = v''$. Indeed, if not, then there exists a face f' incident with v, not incident with either of the edges uv, vw, and not contained in Δ. Since the face f' is S-tight, a neighbor of v in the face boundary of f' is at distance $t+p-1$ from S, and hence belongs to D_p, contrary to the fact that D_p is induced. This proves our claim that $v' = v''$.

Thus for every $v \in V(D_p)$ there exists a unique vertex v' as above. Let D_{p+1} be the subgraph of G consisting of all vertices v' for $v \in V(D_p)$ and all edges $u'v'$ for all edges $uv \in E(D_p)$. We claim that $v_1' \neq v_2'$ for distinct vertices $v_1, v_2 \in V(D_p)$. Indeed, if $v_1' = v_2'$ for distinct $v_1, v_2 \in V(D_p)$, then we may select v_1, v_2 and a subpath P of D_p joining them such that P is as short as possible. Then $|E(P)| \leq |E(D_p)|/2$, and the vertices v' for $v \in V(P)$ form an equidistant cycle of length $|E(P)|$ at distance $t+p$ from S, leading to a contradiction in the same way as the earlier proof that D_p is induced. This proves our claim that $v_1' \neq v_2'$ for distinct vertices $v_1, v_2 \in V(D_p)$. It follows that D_{p+1} is a cycle.

Now adding D_{p+1} to H produces an $r \times (p+1)$ cylindrical grid, contrary to the maximality of p. This proves that $p \geq s/2^j + 1 + 1/2 + \cdots + 1/2^{j-1} + 5$, and hence H satisfies the conclusion of the theorem. $\square$

We will need the following lemma of Aksionov [1].

Lemma 10. *Let G be a planar graph with at most one triangle, and let C be either the null graph or a facial cycle of G of length at most five. Assume that if C has length five and G has a triangle T, then C and T are edge-disjoint. Then every 3-coloring of C extends to a 3-coloring of G.*

In order to prove Theorem 2 we prove the following more general theorem.

Theorem 2 follows by letting C_0 be the null graph.

Theorem 11. *There exists an absolute constant d with the following property. Let G be a plane graph, and let C_0 be either the null graph or an induced facial cycle of G of length at most five such that every two distinct triangles in G are at distance at least $2d$, and if C_0 has length exactly five, then it is edge-disjoint from every triangle of G. Then every 3-coloring of C_0 extends to a 3-coloring of G.*

Proof. Let K be an integer such that the integer $K-4$ satisfies the conclusion of Theorem 3, and let $d = (2K + 7\lfloor \log_2(K+8) \rfloor + 28)(K+1) + 1$. We will prove by induction on $|V(G)|$ that d satisfies the conclusion of the theorem. Let G be as stated, let ϕ_0 be a 3-coloring of C_0, and assume for a contradiction that ϕ_0 does not extend to a 3-coloring of G. We may assume, by taking a subgraph of G, that ϕ_0 extends to every proper subgraph of G that includes C_0. If G has at most one triangle, then the theorem follows from Lemma 10. In particular, the theorem holds if G has fewer than $2d$ vertices. We may therefore assume that G has at least two triangles, and that the theorem holds for all graphs with strictly fewer than $|V(G)|$ vertices. We claim that

(1) *if G has a separating cycle C of length at most five, then C has length exactly five and $E(C)$ includes an edge of a triangle of G.*

To prove (1) let C be a separating cycle in G of length at most five, and let Δ be the open disk bounded by C. From the symmetry we may assume that Δ is disjoint from C_0. Let G' be the subgraph of G consisting of C and all vertices and edges drawn in Δ. We wish to apply the induction hypothesis to G' and C. By the minimality of G the coloring ϕ_0 extends to a 3-coloring ϕ of $G \backslash (V(G') - V(C))$. It follows that the restriction of ϕ to $V(C)$ does not extend to a 3-coloring of G'. Clearly every two triangles in G' are at distance at least $2d$. By the induction hypothesis applied to G' and C we deduce C has length exactly five, and that it shares an edge with a triangle in G, as desired. This proves (1).

Let $\mathcal{S}$ denote the set of vertex-sets of all triangles in G. Then $\mathcal{S} \neq \emptyset$, because G has at least two triangles.

(2) *If C is a cycle in G of length four at distance at most $d-1$ from a set $S \in \mathcal{S}$ with $|V(C) \cap V(C_0)| \leq 1$ and C shares no edge with a triangle of*

G, then either C is S-tight, or $V(C) \cap S = \emptyset$ and at least two vertices of C are adjacent to a vertex in S.

To prove (2) let the vertices of C be numbered u_1, u_2, u_3, u_4 in order. By (1) the cycle C is facial. Let G_{13} be the graph obtained from G by identifying u_1 and u_3 and deleting all resulting loops and parallel edges, and let G_{24} be defined analogously. The graph G_{13} may have a new triangle that does not appear in G, one that resulted from the identification of u_1 and u_3. If that happens we say that G_{13} is *triangular*, and we apply the same terminology to G_{24}.

We claim that if G_{24} is not triangular, then some pair of distinct sets in $\mathcal{S}$ are at distance at most $2d-1$ in G_{24}. Since $|V(C) \cap V(C_0)| \leq 1$ the cycle C_0 is a cycle in G_{24}, and ϕ_0 does not extend to a 3-coloring of G_{24}. Furthermore, since C shares no edge with a triangle of G, it follows from (1) that if C_0 has length five, then it shares no edge with a triangle of G_{24}. It follows by induction applied to G_{24} and C_0 that in G_{24} some pair of distinct sets of $\mathcal{S}$ are at distance at most $2d-1$, as claimed. Since the claim applies to G_{13} by symmetry, it follows from Lemma 8 that if neither G_{13} nor G_{24} is triangular, then the cycle C is S-tight, as desired.

We may therefore assume from the symmetry that G_{13} is triangular. Thus G has a separating cycle C' of length five that shares two edges with C. Let the vertices of C' be u_1, u_5, u_6, u_3, u_2 in order. By (1) the cycle C' includes an edge of a triangle T of G. It follows that $V(T) = S$, and hence C is at distance zero or one from S. Since C shares no edge with a triangle of G, we deduce that G_{24} is not triangular. Thus by the claim of the previous paragraph some pair of distinct sets $S', S'' \in \mathcal{S}$ are at distance at most $2d-1$ in G_{24}. It follows that one of S', S'' is at distance at most $2d-1$ from S in G, and hence is equal to S. From the symmetry we may assume that $S'' = S$. Since S and S' are at distance at most $2d-1$ in G_{24} but not in G, we conclude that $u_1, u_3 \notin S$, and hence the edge T and C share is the edge u_5u_6. Thus u_1 and u_3 are adjacent to vertices in S, as desired. This proves (2).

By (1) we may apply Theorem 3 to G and C_0. The choice of K implies that

(3) $\sum |f| \leq K|\mathcal{S}|$, *where the summation is over all faces f of G of length at least five and the face bounded by C_0.*

By an *angle* in G we mean a pair (v, f), where $v \in V(G)$ and f is a face of G incident with v. Let $S \in \mathcal{S}$. We say that an angle (v, f) is *S-contaminated* if v is at distance at most $d-1$ from S and f either has length at least five, or is bounded by C_0. Since every S-contaminated angle contributes at least one toward the sum in (3), we deduce that there exists $S \in \mathcal{S}$ such that

(4) *there are at most K angles that are S-contaminated.*

From now on we fix this set S. We say that an integer $i \in \{0, 1, \ldots, d-1\}$ is *S-contaminated* if some angle (v, f) is S-contaminated, where v is at distance exactly i from S. It follows from (4) and the choice of d that there exists an integer i_0 such that

(5) $2 \le i_0 \le i_0 + 2K + 7\lfloor \log_2(K+8) \rfloor + 26 \le d-1$ *and no integer in* $\{i_0 - 1, i_0, \ldots, i_0 + 2K + 7\lfloor \log_2(K+8) \rfloor + 26\}$ *is S-contaminated.*

(6) *If $v \in V(G)$ is at distance i from S, where $i_0 \le i \le i_0 + 2K + 7\log_2(K+8) + 25$, then every face incident with v is S-tight and is not bounded by C_0.*

To prove (6) let v be as stated, and let f be a face incident with v. By (5) the face f is bounded by a cycle C of length four, and $|V(C) \cap V(C_0)| \le 1$, because otherwise some vertex of $V(C) \cap V(C_0)$ gives a contradiction to (5). Furthermore, C is incident with no edge of a triangle, because every triangle either has vertex-set S or is at distance at least $2d$ from S. Since $i \ge 2$ it follows from (2) that C is S-tight, as desired. This proves (6).

We now claim that

(7) *there exists an equidistant cycle at distance i_0 from S.*

We prove (7) by showing that the subgraph J of G induced by vertices at distance exactly i_0 from S has minimum degree at least two. To this end, let $v \in V(J)$. By (6) the vertex v is incident with an S-tight face f bounded by a cycle C of length four, and C includes a neighbor u of v that is also in J. Thus J has minimum degree at least one. Now let v' be the neighbor of v in $C \backslash u$, and let f' be the other face incident with the edge vv'. Since v' is not at distance i_0 from S by the definition of S-tight, the other neighbor of

v in the boundary of f', say w, is at distance exactly i_0 from S, because f' is S-tight. Thus v has degree at least two in J, as desired. This proves (7).

Our next claim bounds the length of equidistant cycles.

(8) *For every $i = 1, 2, \ldots, d-1$, every equidistant cycle at distance i from S has length at most $K + 8$.*

To prove (8) let C be an equidistant cycle at distance i from S. For every edge $e \in E(C)$ we will construct an open set $\Delta_e \subseteq \mathbf{R}^2$ such that

(a) $\Delta_e \cap \Delta_{e'} = \emptyset$ for distinct $e, e' \in E(C)$, and

(b) each Δ_e either includes a face of G of length at least five at distance at most $d-1$ from S, or it includes a face incident with an edge of C_0, or the boundary of Δ_e includes an edge joining two elements of S.

The construction is as follows. Let Δ_0 be the component of $\mathbf{R}^2 - C$ containing S, and let $e = u_1u_2$ be an edge of C. For $j = 1, 2$ there is a path P_j from u_j to S of length i, and none of length less than i. We may assume that P_1 and P_2 are coterminal; that is, if they intersect, then their intersection is a path one end of which is a common end of P_1 and P_2. If P_1 is disjoint from P_2, then let f be the edge of G joining the two ends of P_1, P_2 that belong to S; if P_1, P_2 intersect, then let $f = \emptyset$. Let Δ be the component of $\mathbf{R}^2 - P_1 - P_2 - f$ that is contained in Δ_0. We may assume that P_1 and P_2 are chosen so that Δ is minimal, and we define $\Delta_e := \Delta$.

We now prove that the sets Δ_e satisfy (a). To that end let e, P_1, P_2 be as in the previous paragraph, let $e' \in E(C) - \{e\}$, and let P_1', P_2' be the corresponding paths for e'. If $\Delta_e \cap \Delta_{e'} \neq \emptyset$, then Δ_e includes an edge of $P_1' \cup P_2'$. From the symmetry we may assume that a subpath Q of P_1' joins a vertex $x \in V(P_1)$ to a vertex $y \in V(P_1 \cup P_2)$, and otherwise lies in Δ_e. We may also assume that y is closer to S than x. If $y \in V(P_1)$, then we replace the subpath of P_1 from x to y by Q, and if $y \in V(P_2)$, then we replace the subpath of P_1 from x to S by the union of Q and the subpath of P_2 from y to S. In either case we obtain contradiction to the minimality of Δ_e. This proves that the sets Δ_e satisfy (a).

To prove that the sets Δ_e satisfy (b) let e, P_1, P_2 as in the paragraph before last. If P_1 and P_2 are disjoint, then the closure of Δ_e includes an edge of the triangle with vertex-set S, as desired. Thus we may assume that P_1 and P_2 intersect. Let $z \in V(P_1 \cap P_2)$ be the vertex farthest away from S. Then there exists a face f of G incident with z and contained in Δ_e. The face f has length at least four, because its boundary includes at most one vertex of S and it is at distance at most $d-1$ from S. If f has length at least five, then (b) holds, and so we may assume that f has length four. But z is the only vertex incident with f at the same or smaller distance from S than z, and hence (2) implies that at least two vertices of C_0 are incident with f. Since f has length four and C_0 has length at most five, it follows from (1) that f is incident with an edge of C_0. Thus the sets Δ_e satisfy (b).

It follows from (b) that to each edge of C we can assign either an edge of the triangle with vertex-set S, or a face of G of length at least five at distance at most $d-1$ from S, or a face of G incident with an edge of C_0, and this assignment is injective by (a). Since C_0 has length at most five, by (4) at most $K+5$ faces can be assigned as above, and obviously at most three edges of the triangle can be assigned. This complets the proof of (8).

By (6), (7), (8) and Lemma 9 the graph G has a subgraph H isomorphic to an $r \times (r+5)$ cylindrical grid for some $r \leq K+8$ such that H and C_0 are disjoint. Let $D_1, D_2, \ldots, D_{r+5}$ be the hoops of H. Let Δ_1 be the component of $\mathbf{R}^2 - D_2$ containing D_1, and let Δ_{r+5} be the component of $\mathbf{R}^2 - D_{r+4}$ containing D_{r+5}. Let G_1 be obtained from G by deleting all vertices and edges drawn in the interior of Δ_{r+5}, and then adding edges into the face bounded by D_{r+4} in such a way that all faces contained in Δ_{r+5} are bounded by cycles of length four, except possibly one, and if there is an exceptional face, then it is bounded by a cycle of length five. We shall refer to this as "the near-quadrangulation property". Let G_{r+5} be defined analogously. Since G has no separating cycles of length four, it follows that if C_0 is not null, then it is a subgraph of exactly one of G_1, G_{r+5}. From the symmetry we may assume that C_0 is a subgraph of G_1.

By induction ϕ_0 extends to a 3-coloring ψ_1 of G_1. Similarly, the graph G_{r+5} has a 3-coloring ψ_2. It follows from the near quadrangulation property that $|w_{\psi_1}(D_1)| \leq 1$ and $|w_{\psi_2}(D_{r+5})| \leq 1$, where the winding numbers refer to the drawing of H. Furthermore, if r is even, then $w_{\psi_1}(D_1) = w_{\psi_2}(D_{r+5}) = 0$, because there is no exceptional face bounded by a pentagon, and if r is odd, then $|w_{\psi_1}(D_1)| = |w_{\psi_2}(D_{r+5})| = 1$. In the latter case, we may assume, by permuting two colors in ψ_2 if necessary, that $w_{\psi_1}(D_1) + w_{\psi_2}(D_{r+5}) = 0$. By applying Lemma 7 to H and the coloring of $D_1 \cup D_{r+5}$ obtained by restricting ψ_1 to D_1 and restricting ψ_2 to D_{r+5} we obtain a 3-coloring of G that extends ϕ_0, a contradiction. $\square$

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