# $K_6$ MINORS IN 6-CONNECTED GRAPHS OF BOUNDED TREE-WIDTH

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## ABSTRACT

We prove that every sufficiently big 6-connected graph of bounded tree-width either has a $K_6$ minor, or has a vertex whose deletion makes the graph planar. This is a step toward proving that the same conclusion holds for all sufficiently big 6-connected graphs. Jørgensen conjectured that it holds for all 6-connected graphs.

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# 1 Introduction

Graphs in this paper are allowed to have loops and multiple edges. A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. An $H$ *minor* is a minor isomorphic to $H$. A graph $G$ is *apex* if it has a vertex $v$ such that $G\backslash v$ is planar. (We use \ for deletion.) Jørgensen [7] made the following beautiful conjecture.

**Conjecture 1.1** *Every* 6*-connected graph with no* $K_6$ *minor is apex.*

In a companion paper [8] we prove that Conjecture 1.1 holds for all sufficiently big 6-connected graphs. Here we establish the first step toward that goal, the following.

**Theorem 1.2** *For every integer* $w \geq 1$ *there exists an integer* $N$ *such that every* 6*-connected graph on at least* $N$ *vertices and tree-width at most* $w$ *with no* $K_6$ *minor is apex.*

We define tree-width later in this section, but let us discuss Jørgensen's conjecture first. It is related to Hadwiger's conjecture [5], the following.

**Conjecture 1.3** *For every integer* $t \geq 1$*, if a loopless graph has no* $K_t$ *minor, then it is* $(t-1)$*-colorable.*

Hadwiger's conjecture is known for $t \leq 6$. It is trivial for $t \leq 3$, and is still fairly easy for $t = 4$, as shown by Dirac [4]. However, for $t \geq 5$ Hadwiger's conjecture implies the Four-Color Theorem. Wagner [22] gave a structural characterization of graphs with no $K_5$ minor, which implies that Hadwiger's conjecture for $t = 5$ is actually equivalent to the Four-Color Theorem. The same conclusion has been obtained for $t = 6$ in [16] by showing that a minimal counterexample to Hadwiger's conjecture for $t = 6$ is apex. The proof uses an earlier result of Mader [9] that every minimal counterexample to Conjecture 1.3 is 6-connected. Thus Conjecture 1.1, if true, would give more structural information. Furthermore, the structure of all graphs with no $K_6$ minor is not known, and appears complicated and difficult. Thus obtaining a structural characterization of graphs with no $K_6$ minor, an analogue of Wagner's theorem mentioned above, appears beyond reach at the moment. On the other hand, Conjecture 1.1 provides a nice necessary and sufficient condition for 6-connected graphs. Unfortunately, it, too, appears to be a difficult problem.

Let us turn to tree-width and our proof method. Tree-width of a graph was first defined by Halin [6], and was later rediscovered in [12], and, independently, in [1]. The definition is as follows. A *tree-decomposition* of a graph $G$ is a pair $(T, Y)$, where $T$ is a tree and $Y$ is a family $\{Y_t \mid t \in V(T)\}$ of vertex sets $Y_t \subseteq V(G)$, such that the following two properties hold:

(W1) $\bigcup_{t \in V(T)} Y_t = V(G)$, and every edge of $G$ has both ends in some $Y_t$.

(W2) If $t, t', t'' \in V(T)$ and $t'$ lies on the path in $T$ between $t$ and $t''$, then $Y_t \cap Y_{t''} \subseteq Y_{t'}$.

The *width* of a tree-decomposition $(T, Y)$ is $\max_{t \in V(T)}(|Y_t| - 1)$, and the *tree-width* of a graph $G$ is the minimum width of a tree-decomposition of $G$.

Our proof of Theorem 1.2 proceeds as follows. We choose a tree-decomposition $(T, W)$ of $G$ of width $w$ with no "redundancies". It follows easily that if $T$ has a vertex of large degree, then $G$ has a $K_6$ minor, and so we may assume that $T$ has a long path. For the rest of the proof we concentrate our effort on this long path. Since other branches of $T$ are inconsequential, we convert $(T, W)$ to a "linear decomposition", which is really just a tree-decomposition, where the underlying tree is a path, but we find it more convenient to number the sets of vertices $W_0, W_1, \ldots, W_l$, rather than index them by the vertices of a path. At this point we no longer require that the width be bounded; all we need is that the intersections $W_{i-1} \cap W_i$ are bounded and that $l$ is sufficiently large. Thus we may assume (by trimming our linear decomposition) that all the sets $W_{i-1} \cap W_i$ have the same size, say $q$. Furthermore, it can be arranged (by invoking the result from [20] or by a direct argument) that there exist $q$ disjoint paths $P_1, P_2, \ldots, P_q$ from $W_0 \cap W_1$ to $W_{l-1} \cap W_l$. We apply the pigeon hole principle many times, each time trimming the linear decomposition, but still keeping it sufficiently long, to make sure that if the subgraph $G[W_i]$ has some useful property for some $i \in \{1, 2, \ldots, l-1\}$, then all the graphs $G[W_i]$ have that property for all $i \in \{1, 2, \ldots, l-1\}$.

A prime example of a useful property is the existence of two disjoint paths $Q_1, Q_2$ in $G[W_i]$, internally disjoint from $P_1, P_2, \ldots, P_q$, with ends $u_1, v_1$ and $u_2, v_2$, respectively, such that $u_1, v_2 \in V(P_1)$, $u_2, v_1 \in V(P_2)$ and they appear on those paths in the order listed as $P_1$ and $P_2$ are traversed from $W_0 \cap W_1$ to $W_{l-1} \cap W_l$. In those circumstances we say that $P_1$ and $P_2$ *twist* in $W_i$. Thus, in particular, we can arrange that if two paths $P_j$ and $P_{j'}$ twist in $W_i$ for some $i \in \{1, 2, \ldots, l-1\}$, then they twist in $W_i$ for all $i \in \{1, 2, \ldots, l-1\}$. On the other hand, if two paths $P_j$ and $P_{j'}$ twist in $W_i$ for all $i \in \{1, 2, \ldots, l-1\}$ and $l$ is not too small, then $G$ has a $K_6$ minor. This is the sort of argument we will be using, but the details are too numerous to be described in their entirety here.

In [8] we use Theorem 1.2 to prove Jørgensen's conjecture for sufficiently big graphs, formally the following:

**Theorem 1.4** *There exists an integer $N$ such that every 6-connected graph on at least $N$ vertices and tree-width at most $w$ with no $K_6$ minor is apex.*

How does Theorem 1.2 help in the proof of Theorem 1.4? By the excluded grid theorem of Robertson and Seymour [13] (see also [3, 11, 17]) it suffices to prove Theorem 1.4 for graphs that have a sufficiently large grid minor. We then analyze how the remainder of the graph attaches to the grid. We refer to [8] for details.

The paper is organized as follows. In Section 2 we state a few lemmas, mostly from other papers. In Section 3 we convert a tree-decomposition into a linear decomposition, as described above, and we prove that the linear decomposition can be chosen with some additional useful properties. In Section 4 we introduce the auxiliary graph—its vertices are the paths $P_1, P_2, \ldots, P_q$, and two of them are adjacent if they are joined by a path avoiding all the other paths $P_1, P_2, \ldots, P_q$. By joined we mean in some or every $W_i$; by now the two are equivalent. We use the auxiliary graph to further refine the linear decomposition. A *core* is a component of the subgraph of the auxiliary graph induced by those of the paths $P_1, P_2, \ldots, P_q$ that have at least one edge. We show, among other things, that every core is a path or a cycle. In Section 5 we use the theory of "non-planar extensions" of planar graphs from [18] to get under control adjacencies in the auxiliary graph of those paths $P_i$ that are trivial. In Section 6 we further refine our linear decomposition to arrange that the part of $G$ that corresponds to a core can be drawn either in a disk or in a cylinder, depending on whether the core is a path or a cycle. In Section 7 we digress and prove a slight extension of a result of DeVos and Seymour [2]. Finally, in Section 8 we essentially complete the proof of Theorem 1.2 in the case when some core of the auxiliary graph is a cycle, and in Section 9 we do the same when some core is a path.

# 2 Rerouting and rural societies

Let $S$ be a subgraph of a graph $G$. An *$S$-bridge* in $G$ is a connected subgraph $B$ of $G$ such that $E(B) \cap E(S) = \emptyset$ and either $E(B)$ consists of a unique edge with both ends in $S$, or for some component $C$ of $G \backslash V(S)$ the set $E(B)$ consists of all edges of $G$ with at least one end in $V(C)$. The vertices in $V(B) \cap V(S)$ are called the *attachments* of $B$. We say that an $S$-bridge $B$ *attaches to* a subgraph $H$ of $S$ if $V(H) \cap V(B) \neq \emptyset$.

Now let $S$ be such that no block of $S$ is a cycle. By a *segment of $S$* we mean a maximal subpath $P$ of $S$ such that every internal vertex of $P$ has degree two in $S$. It follows that the segments of $S$ are uniquely determined. Now if $B$ is an $S$-bridge of $G$, then we say that $B$ is *unstable* if some segment of $S$ includes all the attachments of $B$, and otherwise we say that $B$ is *stable*. Our next lemma says that it is possible to make all bridges stable by making the following "local" changes. Let $G$ and $S$ be as before, let $P$ be a segment of $S$ of length at least two, and let $Q$ be a path in $H$ with ends $x, y \in V(P)$ and otherwise disjoint from $S$. Let $S'$ be obtained from $S$ by replacing the path $xPy$ (the subpath of $P$ with ends $x$ and $y$) by $Q$; then we say that $S'$ was obtained from $S$ by *rerouting* $P$ along $Q$, or simply that $S'$ was obtained from $S$ by *rerouting*. Please note that $P$ is required to have length at least two, and hence this relation is not symmetric. We say that the rerouting is *proper* if all the attachments of the $S$-bridge that contains $Q$ belong to $P$. The following lemma is essentially due to Tutte.

**Lemma 2.1** *Let $G$ be a graph, and let $S$ be a subgraph of $G$ such that no block of $S$ is a cycle. Then there exists a subgraph $S'$ of $G$ obtained from $S$ by a sequence of proper reroutings such* *that if an $S'$-bridge $B$ of $G$ is unstable, say all its attachments belong to a segment $P$ of $S'$, then there exist vertices $x, y \in V(P)$ such that some component of $G \backslash \{x, y\}$ includes a vertex of $B$ and is disjoint from $S \backslash V(P)$.*

**Proof.** We may choose a subgraph $S'$ of $G$ obtained from $S$ by a sequence of proper reroutings such that the number of vertices that belong to stable $S'$-bridges is maximum, and, subject to that, $|V(S')|$ is minimum. We will show that $S'$ is as desired. To that end we may assume that $B$ is an $S'$-bridge of $G$ with all its attachments in a segment $P$ of $S'$.

Let $v_0, v_1, \dots, v_k$ be distinct vertices of $P$, listed in order of occurrence on $P$ such that $v_0$ and $v_k$ are the ends of $P$ and $\{v_1, \dots, v_{k-1}\}$ is the set of all internal vertices of $P$ that are attachments of a stable $S'$-bridge. We claim that if $u, v$ are two attachments of $B$, then no $v_i$ belongs to the interior of $uPv$. To prove this suppose to the contrary that $v_i$ is an internal vertex of $uPv$. But then replacing $uPv$ by an induced subpath of $B$ with ends $u, v$ and otherwise disjoint from $S'$ is a proper rerouting that produces a graph $S''$ with strictly more vertices belonging to stable $S''$-bridges, contrary to the choice of $S'$. This proves our claim that no $v_i$ belongs to the interior of $uPv$. But then for some $i = 1, 2, \dots, k$ the path $v_{i-1}Pv_i$ includes all attachments of $B$. Since $G$ has no parallel edges, the same argument (using the minimality of $|V(S')|$) now shows that $|V(B)| \geq 3$. Consequently some component $J$ of $G \backslash \{v_{i-1}, v_i\}$ includes a vertex of $B$. It follows that $B \backslash \{v_{i-1}, v_i\}$ is a subgraph of $J$. Now $B$ has all its attachments in $v_{i-1}Pv_i$, the interior of $v_{i-1}Pv_i$ includes no attachment of a stable $S'$-bridge, and (by what we have shown about $B$) every unstable $S'$-bridge with an attachment in the interior of $v_{i-1}Pv_i$ has all its attachments in $v_{i-1}Pv_i$. It follows that $J$ is disjoint from $S \backslash V(P)$, as desired. □

We deduce the following corollary.

**Theorem 2.2** *Let $G$ be a 3-connected graph, and let $S$ be a subgraph of $G$ with at least two segments such that no block of $S$ is a cycle. Then there exists a subgraph $S'$ of $G$ obtained from $S$ by a sequence of proper reroutings such that every $S'$-bridge is stable.*

We will need the following lemma, a special case of [8, Lemma 3.1]. A *linkage* in a graph is a set $\mathcal{P}$ of disjoint paths. If $A, B$ are sets such that each $P \in \mathcal{P}$ has one end in $A$ and the other in $B$, then we say that $\mathcal{P}$ is a *linkage from $A$ to $B$*. The *graph of the linkage* $\mathcal{P}$ is the union of all $P \in \mathcal{P}$. Occasionally we will use $\mathcal{P}$ in reference to the graph of $\mathcal{P}$; thus we will use $V(\mathcal{P})$ to denote the vertex-set of the graph of $\mathcal{P}$ and we will also speak of $\mathcal{P}$-bridges. A path is *trivial* if it has exactly one vertex and *non-trivial* otherwise. By a $\mathcal{P}$-path we mean a non-trivial path with both ends in $V(\mathcal{P})$ and otherwise disjoint from the graph of $\mathcal{P}$.

**Lemma 2.3** *Let $k \geq 1$ be an integer, let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ be a linkage in a graph $G$, where $P_i$ has distinct ends $u_i$ and $v_i$, and let every $\mathcal{P}$-bridge of $G$ be stable. Assume that $G$*

*cannot be drawn in a disk with* $u_1, u_2, \ldots, u_k, v_k, v_{k-1}, \ldots, v_1$ *drawn on the boundary of the disk in order and the paths* $P_1$ *and* $P_k$ *also drawn on the boundary, and assume also that there is no set* $X \subseteq V(G)$ *of size at most three such that some component of* $G \backslash X$ *is disjoint from* $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k\}$. *Then either*

(i) *there exist integers* $i, j \in \{1, 2, \ldots, k\}$ *with* $|i-j| > 1$ *and a* $\mathcal{P}$*-path* $Q$ *in* $G$ *with one end in* $V(P_i)$ *and the other end in* $V(P_j)$, *or*

(ii) *there exist an integer* $i \in \{1, 2, \ldots, k-1\}$ *and two disjoint* $\mathcal{P}$*-paths* $Q_1$, $Q_2$ *in* $G$ *such that* $Q_j$ *has ends* $x_j, y_j$, *the vertices* $u_i, x_1, x_2, v_i$ *occur on* $P_i$ *in the order listed and* $u_{i+1}, y_2, y_1, v_{i+1}$ *occur on* $P_{i+1}$ *in the order listed, or*

(iii) *there exist an integer* $i = 2, 3, \ldots, k-1$ *and three* $\mathcal{P}$*-paths* $Q_0$, $Q_1$, $Q_2$ *such that* $Q_j$ *has ends* $x_j$ *and* $y_j$, *we have* $x_0, y_0 \in V(P_i)$, *the vertices* $x_1, x_2$ *are internal vertices of* $x_0 P_i y_0$, $y_1 \in V(P_{i-1})$, $y_2 \in V(P_{i+1})$, *and the paths* $Q_0$, $Q_1$, $Q_2$ *are pairwise disjoint, except possibly for* $x_1 = x_2$.

We need a slight variation of the previous lemma. We omit its proof, because it is completely analogous.

**Lemma 2.4** *Let* $k \geq 3$ *be an integer, let* $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ *be a linkage in a graph* $G$, *where* $P_i$ *has distinct ends* $u_i$ *and* $v_i$, *and let every* $\mathcal{P}$*-bridge of* $G$ *be stable. Assume that* $G$ *cannot be drawn in a cylinder with* $u_1, u_2, \ldots, u_k$ *drawn on one boundary component in the cyclic order listed and* $v_k, v_{k-1}, \ldots, v_1$ *drawn on the other boundary component in the order listed, assume also that there is no set* $X \subseteq V(G)$ *of size at most three such that some component of* $G \backslash X$ *is disjoint from* $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k\}$, *and finally assume that if* $k = 3$, *then no* $\mathcal{P}$*-bridge has vertices of attachment on all three members of* $\mathcal{P}$. *Then either*

(i) *there exist integers* $i, j \in \{1, 2, \ldots, k\}$ *with* $|i-j| > 1$ *and* $\{i, j\} \neq \{1, k\}$ *and a* $\mathcal{P}$*-path* $Q$ *in* $G$ *with one end in* $V(P_i)$ *and the other end in* $V(P_j)$, *or*

(ii) *there exist an integer* $i \in \{1, 2, \ldots, k-1\}$ *and two disjoint* $\mathcal{P}$*-paths* $Q_1$, $Q_2$ *in* $G$ *such that* $Q_j$ *has ends* $x_j y_j$, *the vertices* $u_i, x_1, x_2, v_i$ *occur on* $P_i$ *in the order listed and* $u_{i+1}, y_2, y_1, v_{i+1}$ *occur on* $P_{i+1}$ *in the order listed, or*

(iii) *there exist an integer* $i = 1, 2, \ldots, k$ *and three* $\mathcal{P}$*-paths* $Q_0$, $Q_1$, $Q_2$ *such that* $Q_j$ *has ends* $x_j$ *and* $y_j$, *we have* $x_0, y_0 \in V(P_i)$, *the vertices* $x_1, x_2$ *are internal vertices of* $x_0 P_i y_0$, $y_1 \in V(P_{i-1})$, $y_2 \in V(P_{i+1})$, *and the paths* $Q_0$, $Q_1$, $Q_2$ *are pairwise disjoint, except possibly for* $x_1 = x_2$, *where* $P_0$ *means* $P_k$ *and* $P_{k+1}$ *means* $P_1$.

We finish the section by introducing several notions and a theorem from [14]. We will make use of them in the last two sections. Let $\Omega$ be a cyclic permutation of the elements of some set; we denote this set by $V(\Omega)$. A *society* is a pair $(G, \Omega)$, where $G$ is a graph, and $\Omega$ is a cyclic permutation with $V(\Omega) \subseteq V(G)$. A society $(G, \Omega)$ is *rural* if $G$ can be drawn in a disk with $V(\Omega)$ drawn on the boundary of the disk in the order given by $\Omega$. A *cross* in $(G, \Omega)$ is a pair of disjoint non-trivial paths $P_1$ and $P_2$ with ends $u_1$, $v_1$ and $u_2$,$v_2$ respectively, so that $u_1, u_2, v_1, v_2 \in V(\Omega)$ appear in $\Omega$ in this or reverse order, and $P_1$ and $P_2$ are otherwise disjoint from $V(\Omega)$.

A *separation* of a graph $G$ is a pair $(A, B)$ such that $A \cup B = V(G)$ and there is no edge with one end in $A - B$ and the other end in $B - A$. The order of $(A, B)$ is $|A \cap B|$. We say that a society $(G, \Omega)$ is 4-*connected* if there is no separation $(A, B)$ of $G$ of order at most three with $V(\Omega) \subseteq A$ and $B - A \neq \emptyset$.

The next theorem follows from Theorems (2.3) and (2.4) in [14].

**Theorem 2.5** *Let $(G, \Omega)$ be a 4-connected society with no cross. Then $(G, \Omega)$ is rural.*

# 3 Linear decompositions

In this section we show that it suffices to prove Theorem 1.2 for graphs that have a "linear decomposition" of bounded "adhesion". A linear decomposition is really a tree-decomposition, where the underlying tree is a path, but it is more convenient to number the sets by integers rather than vertices of a path. Thus a linear decomposition of a graph $G$ is a family of sets $\mathcal{W} = (W_0, W_1, \dots, W_l)$ such that

(L1) $\bigcup_{i=0}^{l} W_i = V(G)$, and every edge of $G$ has both ends in some $W_i$, and

(L2) if $0 \leq i \leq j \leq k \leq l$, then $W_i \cap W_k \subseteq W_j$.

We say that the *length* of $\mathcal{W}$ is $l$.

In the proof of Theorem 1.2 we will need linear decompositions satisfying the following additional properties:

(L3) there is an integer $q$ such that $|W_{i-1} \cap W_i| = q$ for all $i = 1, 2, \dots, l$,

(L4) for every $i = 1, 2, \dots, l-1$, $W_{i-1} \neq W_{i-1} \cap W_i \neq W_i$,

(L5) there exists a linkage from $W_0 \cap W_1$ to $W_{l-1} \cap W_l$ of cardinality $q$.

If a linear decomposition satisfies (L3), then we say that it has *adhesion* $q$. A linkage as in (L5) will be called a *foundational linkage* and its members will be called *foundational paths.* We will need more properties, but first we show that we can assume that our graph has a linear decomposition satisfying (L1)–(L5). In the proof we will need the following additional properties of tree-decompositions:

(W3) for every two vertices $t, t'$ of $T$ and every positive integer $k$, either there are $k$ disjoint paths in $G$ between $Y_t$ and $Y_{t'}$, or there is a vertex $t''$ of $T$ on the path between $t$ and $t'$ such that $|Y_{t''}| < k$, (W4) if $t, t'$ are distinct vertices of $T$, then $Y_t \neq Y_{t'}$, and

(W5) if $t_0 \in V(T)$ and $W$ is a component of $T - t_0$, then $\bigcup_{t \in V(W)} Y_t \setminus Y_{t_0} \neq \emptyset$.

**Lemma 3.1** *For all integers $k, l, p, w \geq 0$ there exists an integer $N$ with the following property. If $G$ is a $p$-connected graph of tree-width at most $w$ with at least $N$ vertices, then either $G$ has a minor isomorphic to $K_{p,k}$, or $G$ has a linear decomposition of length at least $l$ and adhesion at most $w$ satisfying* (L1)–(L5).

**Proof.** Let $k, l, w \geq 0$ be given integers. We will use the proof technique of [10, Theorem 3.1] with the constants $n_1, n_6, n_7, n_8$ and $n_9$ redefined as follows: Let $n_1 := w$, $n_6 := l$, $n_7 := n_6^{n_1+1}$, $n_8 := \binom{n_1}{p}(k-1)$, and

$$n_9 := 2 + n_8 + n_8(n_8 - 1) + \cdots + n_8(n_8 - 1)^{\lceil n_7/2 \rceil - 2}.$$

We will show that $N := n_1 n_9$ satisfies the lemma.

To this end let $G$ be as stated. The argument in Claims (1)–(4) of [10, Theorem 3.1] shows that $G$ either has a minor isomorphic to $K_{p,k}$, or a tree-decomposition $(T, Y)$ satisfying (W1)–(W5) such that $T$ has a path $R$ that includes distinct vertices $r_1, r_2, \ldots, r_l$, appearing on $R$ in the order listed, such that for some integer $q$ with $p \leq q \leq w$ we have that $|Y_{r_i}| = q$ for all $i = 1, 2, \ldots, l$ and $|Y_r| \geq q$ for every $r \in V(R)$ between $r_1$ and $r_l$.

It is easy to see that there exist subtrees $T_0, T_1, \ldots, T_l$ of $T$ such that

(i) $T_0 \cup T_1 \cup \cdots \cup T_l = T$,

(ii) $T_i$ and $T_j$ are disjoint whenever $|i - j| \geq 2$, and

(iii) $V(T_{i-1}) \cap V(T_i) = \{r_i\}$ for all $i = 1, 2, \ldots, l$.

For $i = 0, 1, \ldots, l$ let $W_i$ be the union of $Y_t$ over all $t \in V(T_i)$. We claim that $(W_0, W_1, \ldots, W_l)$ is a linear decomposition of $G$ satisfying (L1)–(L5).

Property (L1) is satisfied by (W1) and (i). If $0 \leq i < j < k \leq l$, then for every $t \in V(T_i)$ and $t' \in V(T_k)$ the path from $t$ to $t'$ in $T$ contains the path from $r_{i+1}$ to $r_k$. Therefore, by (W2) and (iii), we have $Y_t \cap Y_{t'} \subseteq Y_{r_j}$ and, consequently, $W_i \cap W_k \subseteq Y_{r_j} \subseteq W_j$. Thus (L2) is satisfied. Similarly, we have $W_{i-1} \cap W_i = Y_{r_i}$, and, therefore, we have $|W_{i-1} \cap W_i| = q$, implying (L3). For $1 < i \leq l$ we have $|Y_{r_{i-1}}| = |Y_{r_i}| = q$, and $Y_{r_i} \neq Y_{r_{i-1}}$ by (W4). Therefore $W_{i-1} - W_i \supseteq Y_{r_{i-1}} - Y_{r_i} \neq \emptyset$. By construction, $T_0 \setminus r_1$ is the union of components of $T \setminus r_1$ disjoint from $R$. It follows from (W5) that $W_0 - W_1 = W_0 - Y_{r_1} \neq \emptyset$. By symmetry, $W_i - W_{i-1} \neq \emptyset$ for every $1 \leq i \leq l$, and (L4) holds. Finally, by (W3) and the choice of $r_1, r_2, \ldots, r_l$, there exists a linkage from $W_0 \cap W_1 = Y_{r_1}$ to $W_{l-1} \cap W_l = Y_{r_l}$, implying (L5). □

Let $\mathcal{P}$ be a foundational linkage for a linear decomposition $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ of a graph $G$, and let $i \in \{1, 2, \ldots, l-1\}$. We say that distinct foundational paths $P, P' \in \mathcal{P}$ are *bridge adjacent* in $W_i$ if there exists a $\mathcal{P}$-bridge in $G[W_i]$ with an attachment in both $P$ and

$P'$. Given a fixed integer $p$ we wish to consider the following properties of $\mathcal{W}$ and $\mathcal{P}$. In our applications we will always have $p = 6$.

(L6) for all $i \in \{1, 2, \ldots, l-1\}$ and all non-trivial paths $P \in \mathcal{P}$, if some $\mathcal{P}$-bridge of $G[W_i]$ has at least one attachment in $P$ and no attachment in a non-trivial foundational path other than $P$, then $P$ is bridge adjacent in $W_i$ to at least $p-2$ trivial members of $\mathcal{P}$,

(L7) for every $P \in \mathcal{P}$, if there exists an index $i \in \{1, 2, \ldots, l-1\}$ such that $P[W_i]$ is a trivial path, then $P[W_k]$ is a trivial path for all $k = 1, 2, \ldots, l-1$,

(L8) for every two distinct paths $P, P' \in \mathcal{P}$, if there exists an integer $k \in \{1, \ldots, l-1\}$ such that $P$ and $P'$ are bridge adjacent in $W_k$, then they are bridge adjacent in $W_{k'}$ for all $k' \in \{1, \ldots, l-1\}$.

With respect to condition (L8) it may be helpful to point out that for all $i = 1, 2, \ldots, l$ we have $W_{i-1} \cap W_i \subseteq V(\mathcal{P})$, and hence each $\mathcal{P}$-bridge $H$ of $G$ satisfies $V(H) \subseteq W_k$ for some $k \in \{0, 1, \ldots, l\}$, even though this index $k$ need not be unique.

**Lemma 3.2** *Let $p \geq 0$ be an integer, and let $\mathcal{W}$ be a linear decomposition of a $p$-connected graph satisfying* (L1)–(L5)*. Then $\mathcal{W}$ has a foundational linkage $\mathcal{P}$ satisfying* (L6)*.*

**Proof.** Let $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ be as stated. By (L5) there exists a linkage $\mathcal{P}$ from $W_0 \cap W_1$ to $W_{l-1} \cap W_l$ of cardinality $q$. Let $S$ be the union of all non-trivial paths in $\mathcal{P}$, and let $H$ be obtained from $G[W_1 \cup W_2 \cup \cdots \cup W_{l-1}]$ by deleting all trivial paths in $\mathcal{P}$. By Lemma 2.1 applied to $H$ and $S$ we may assume (by changing $\mathcal{P}$) that $S$ satisfies the conclusion of that lemma. We claim that the linkage $\mathcal{P}$ then satisfies (L6). To prove this claim suppose that $i \in \{1, 2, \ldots, l-1\}$ and some $S$-bridge $B$ of $H[W_i]$ has all its attachments in $V(P)$ for some non-trivial $P \in \mathcal{P}$; then there are vertices $x, y \in V(P)$ such that some component $J$ of $H \backslash \{x, y\}$ has at least three vertices, includes a vertex of $B$ and is disjoint from $V(S) - V(P)$. Since $G$ is $p$-connected the set $V(J)$ has at least $p-2$ neighbors among the trivial paths in $\mathcal{P}$. Hence $P$ is bridge adjacent in $W_i$ to those trivial paths, as required. This proves that $\mathcal{P}$ satisfies (L6). $\square$

We will make use of the following easy lemma, whose proof we omit.

**Lemma 3.3** *Let $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ be a linear decomposition of a graph $G$ of length $l \geq 2$, and let $i \in \{1, 2, \ldots, l\}$. Then $\mathcal{W}' := (W_0, W_1, \ldots, W_{i-2}, W_{i-1} \cup W_i, W_{i+1}, W_{i+2}, \ldots, W_l)$ is also a linear decomposition of $G$. Furthermore, if $\mathcal{W}$ satisfies any of the properties* (L3)–(L8)*, then so does $\mathcal{W}'$.*

If $\mathcal{W}$ and $\mathcal{W}'$ are as in Lemma 3.3, then we say that $\mathcal{W}'$ was obtained from $\mathcal{W}$ by an *elementary contraction.* Let $\mathcal{P}$ be a foundational linkage for $\mathcal{W}$. If $i \notin \{1, l\}$, then let $\mathcal{P}' := \mathcal{P}$. If $i = 1$, then let $\mathcal{P}'$ be the linkage obtained from $\mathcal{P}$ by restricting each $P \in \mathcal{P}$ to $W_2 \cup W_3 \cup \ldots \cup W_l$, and if $i = l$, then let $\mathcal{P}'$ be obtained by restricting $\mathcal{P}$ to $W_1 \cup W_2 \cup \ldots \cup W_{l-1}$. Then $\mathcal{P}'$ is a foundational linkage for $\mathcal{W}'$. It will be referred to as the *corresponding restriction* of $\mathcal{P}$. If a linear decomposition $\mathcal{W}''$ is obtained from $\mathcal{W}$ by a sequence of elementary contractions, then we say that $\mathcal{W}''$ is obtained from $\mathcal{W}$ by a *contraction*. We will also need the following lemma about sequences of sets.

**Lemma 3.4** *Let $l, n, \lambda \geq 0$ be integers such that $\lambda \geq l^{n+1} n!$. For all sequences $S_1, S_2, \ldots, S_\lambda$ of subsets of $\{1, \ldots, n\}$ there exist integers $1 \leq i_0 < i_1 < \cdots < i_l \leq \lambda + 1$ such that*

$$S_{i_0} \cup S_{i_0+1} \cup \cdots \cup S_{i_1-1} = S_{i_1} \cup S_{i_1+1} \cup \cdots \cup S_{i_2-1} = \cdots = S_{i_{l-1}} \cup \cdots \cup S_{i_l-1}.$$

**Proof.** We proceed by induction on $n$. The lemma clearly holds when $n = 0$, and so we assume that $n > 0$ and that the lemma holds for all smaller values of $n$. If $l$ consecutive sets $S_i$ are empty, say $S_i, S_{i+1}, \ldots, S_{i+l-1}$, then the lemma holds with $i_j = i + j$ for $j = 0, 1, \ldots, l$. Thus we may assume that this is not the case, and hence there is an integer $x \in \{1, 2, \ldots, n\}$ such that at least $\lambda' := \lambda/(ln) \geq l^n (n-1)!$ of the sets $S_i$ include the element $x$. Thus $\{1, \ldots, \lambda\}$ can be partitioned into consecutive intervals $I_1, I_2, \ldots, I_{\lambda'}$ such that each interval includes an index $i$ with $x \in S_i$. For $i = 1, 2, \ldots, \lambda'$ let $S'_i$ be the union of $S_j - \{x\}$ over all $j \in I_i$. By the induction hypothesis applied to the sets $S'_i$ there exist required indices $1 \leq i'_0 < i'_1 < \cdots < i'_l \leq \lambda' + 1$ for the sets $S'_i$. For $j = 0, 1, \ldots, l$ let $i_j := \min I_{i'_j}$. It follows from the construction that these indices satisfy the conclusion of the lemma. $\square$

**Lemma 3.5** *For every triple of integers $l, p, q \geq 0$ there exists an integer $\lambda$ such that the following holds. If a graph $G$ has a linear decomposition $\mathcal{W}$ of length $\lambda + 1$ and adhesion $q$ and a foundational linkage $\mathcal{P}$ satisfying* (L1)–(L6)*, then it has a linear decomposition $\mathcal{W}'$ of length $l$ and adhesion $q$ obtained from $\mathcal{W}$ by a contraction such that $\mathcal{W}'$ and the corresponding restriction of $\mathcal{P}$ satisfy* (L1)–(L8).

**Proof.** Let $l, q \geq 0$ be given, let $s := \binom{q}{2}$, and let $\mu := l^{s+1} s!$. We will show that $\lambda := \mu^{q+1} q!$ satisfies the conclusion of the lemma.

Let $\mathcal{W} = (W_0, W_1, \ldots, W_{\lambda+1})$ be as stated. We wish to apply Lemma 3.4 with $q$ playing the role of $n$ and $\mu$ playing the role of $l$. For $i = 1, 2, \ldots, \lambda$ let $S_i$ be the set of all $P \in \mathcal{P}$ such that $P[W_i]$ is a non-trivial path. By Lemma 3.4 there exist indices $1 \leq i_0 < i_1 < \cdots < i_\mu \leq \lambda + 1$ as stated in that lemma. Let $i_{-1} := 0$ and $i_{\mu+1} := \lambda + 1$ and for $t = -1, 0, \ldots, \mu$ define

$$W'_{t+1} := W_{i_t} \cup W_{i_t+1} \cup \cdots \cup W_{i_{t+1}-1}.$$

By Lemma 3.3 $\mathcal{W}' := (W'_0, W'_1, \ldots, W'_{\mu+1})$ is a linear decomposition of $G$ satisfying (L1)–(L6). It follows from the construction that it also satisfies (L7).

To construct a linear decomposition satisfying (L1)–(L8) we apply the same argument again, as follows. For a 2-element subset $X \subseteq \mathcal{P}$ let $S_X$ be the set of integers $j \in \{1, 2, \ldots, q\}$ such that some $\mathcal{P}$-bridge $H$ of $G$ has attachments in $P$ for both elements $P \in X$ and satisfies $V(H) \subseteq W_j$. By applying Lemma 3.4 with $n := \binom{q}{2}$ and $\lambda$ replaced by $\mu$ to the linear decomposition $\mathcal{W}'$ and using the same construction we arrive at the desired linear decomposition of $G$. $\square$

Let $\mathcal{W}$ be a linear decomposition of a graph $G$ of length $l \geq 2$ with foundational linkage $\mathcal{P}$ satisfying (L1)–(L8). We define the *auxiliary graph* of the pair $(\mathcal{W}, \mathcal{P})$ to be the graph with vertex-set $\mathcal{P}$ in which two paths $P, P' \in \mathcal{P}$ are adjacent if they are bridge adjacent in $W_i$ for some (and hence every) $i \in \{1, 2, \ldots, l-1\}$.

We will need one more property of a linear decomposition $\mathcal{W}$ and its foundational linkage $\mathcal{P}$. The parameter $p$ is the same as in (L6).

(L9) Let $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}$ such that $|\mathcal{P}_1| + |\mathcal{P}_2| \leq p$ and each member of $\mathcal{P}_1$ is non-trivial. Then there exists a linkage $\mathcal{Q}$ in $G$ of cardinality $|\mathcal{P}_1|$ from $W_0 \cap W_1 \cap V(\mathcal{P}_1)$ to $W_{l-1} \cap W_l \cap V(\mathcal{P}_1)$ such that its graph is a subgraph of $H := G[W_0 \cup W_l] \cup \bigcup_{P \in \mathcal{P} - \mathcal{P}_2} P$.

Our objective is to show that if a graph has a linear decomposition satisfying (L1)–(L8), then it also has one satisfying (L9). For the proof we need a definition and a lemma. Let $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ be a linear decomposition of a graph $G$ with foundational linkage $\mathcal{P}$ satisfying (L1)–(L8). We say that a set $\mathcal{P}'$ of components of $\mathcal{P}$ is *well-connected* if for every two paths $P, P' \in \mathcal{P}'$ there exists a path $\mathcal{Q}$ in the auxiliary graph of $(\mathcal{W}, \mathcal{P})$ such that every internal vertex of $\mathcal{Q}$ is a non-trivial foundational path belonging to $\mathcal{P}'$. The lemma we need is the following.

**Lemma 3.6** *Let $l, s, q \geq 0$ be integers, and let $G$ be a graph with a linear decomposition $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ of length $l$, adhesion $q$ and foundational linkage $\mathcal{P}$ satisfying* (L1)–(L8). *Let $\mathcal{Q}$ be a well-connected set of foundational paths, and let $X_{ij} := (W_{i-1} \cap W_i \cap V(\mathcal{Q})) \cup (W_j \cap W_{j+1} \cap V(\mathcal{Q}))$. Then for every two integers $i, j$ with $1 \leq i \leq i + 2q \leq j < l$ and every two sets $A, B \subseteq X_{ij}$ of size $s$ there exist $s$ disjoint paths, each with no internal vertex in any $W_k$ for $k \in \{0, 1, \ldots, l\} - \{i, i+1, \ldots, j\}$.*

**Proof.** Let $H$ be the subgraph of $G$ obtained by deleting $W_j - A - B$ for all $j \in \{0, 1, \ldots, l\} - \{i, i+1, \ldots, j\}$. If the paths do not exist, then by Menger's theorem there exists a set $Y \subseteq V(H)$ of size at most $s - 1$ such that $H \backslash Y$ has no path from $A$ to $B$. We may assume that $A \cap B = \emptyset$, for otherwise we may proceed by induction by deleting $A \cap B$. Since $|W_{i-1} \cap W_i| = |W_j \cap W_{j+1}| = q$ we deduce that $s \leq q$. Let $Z$ be the union of the vertex-sets of the trivial paths in $\mathcal{P}$. By (L7) and the fact that $W_i \cap W_{i+1} \subseteq V(\mathcal{P})$ for all $i = 1, 2, \ldots, l-1$, the sets $W_{i+1} - Z, W_{i+3} - Z, \ldots, W_{i+2q-1} - Z$ are pairwise disjoint, and so one of them, say

$W_m - Z$, is disjoint from $Y$. For $x \in X_{ij}$ let $P_x$ be the member of $\mathcal{Q}$ that includes $x$. If $x \in W_{i-1} \cap W_i$, then let $P'_x$ denote the restriction of $P_x$ to $W_i \cup W_{i+1} \cup \cdots \cup W_{m-1}$, and if $x \in W_l \cap W_{l+1}$, then let $P'_x$ denote the restriction of $P_x$ to $W_{m+1} \cup W_{m+2} \cup \cdots \cup W_l$. Since the paths $P'_x$ are pairwise vertex-disjoint, there exist $a \in A$ and $b \in B$ such that $P'_a$ and $P'_b$ are disjoint from $Y$. Since $\mathcal{Q}$ is well-connected it follows that $P'_a \cup G[W_m] \cup P'_b$ includes a path in $H$ from $a$ to $b$ with no internal vertex in $Z$. That path is disjoint from $Y$, a contradiction. □

**Lemma 3.7** *Let $p, q \geq 0$ and $l \geq 3$ be integers, and let $G$ be a $p$-connected graph with a linear decomposition $\mathcal{W} = (W_0, W_1, \ldots, W_{l+4q+2})$ of length $l + 4q + 2$, adhesion $q$ and foundational linkage $\mathcal{P}$ satisfying* (L1)–(L8). *Let $\mathcal{W}' := (W'_0, W'_1, \ldots, W'_l)$, where $W'_0 := W_0 \cup W_1 \cup \cdots \cup W_{2q+1}$, $W'_i := W_{i+2q+1}$ for $i = 1, 2, \ldots, l-1$ and $W'_l := W_{l+2q+1} \cup W_{l+2q+2} \cup \cdots \cup W_{l+4q+2}$, and let $\mathcal{P}'$ be the corresponding restriction of $\mathcal{P}$. Then $\mathcal{W}'$ is a linear decomposition of $G$ of length $l$ and adhesion $q$, and $\mathcal{P}'$ is a foundational linkage for $\mathcal{W}'$ such that conditions* (L1)–(L9) *hold.*

**Proof.** The linear decomposition $\mathcal{W}'$ satisfies (L1)–(L8) by Lemma 3.3, and so it remains to show that it satisfies (L9). Since $l \geq 3$ we may choose an index $s$ with $2q+2 < s < l+2q+1$. Let $\mathcal{P}_1 \subseteq \mathcal{P}_2$ be two sets of foundational paths such that every member of $\mathcal{P}_1$ is non-trivial and $|\mathcal{P}_1| + |\mathcal{P}_2| \leq p$. Let $H := G[W'_0 \cup W'_l] \cup \bigcup_{P \in \mathcal{P} - \mathcal{P}_2} P$. We must show that there exist $|\mathcal{P}_1|$ disjoint paths in $H$ from $X_0 := W'_0 \cap W'_1 \cap V(\mathcal{P}_1)$ to $X_l := W'_{l-1} \cap W'_l \cap V(\mathcal{P}_1)$. Since $G$ is $p$-connected and $|W_j \cap W_{j+1} \cap V(\mathcal{P}_2)| = |\mathcal{P}_2|$ we deduce that there exists a linkage of size $|\mathcal{P}_1|$ from $X_0$ to $X_l$ in $G \backslash (W_s \cap W_{s+1} \cap V(\mathcal{P}_2))$. Let us choose such linkage, say $\mathcal{Q}$, such that it uses the least number of edges not in $H$. We will prove that $\mathcal{Q}$ is as desired. To do so we may assume for a contradiction that $\mathcal{Q}$ uses an edge $e \in E(G) - E(H)$. By considering the linear decomposition $(W'_l, W'_{l-1}, \ldots, W'_0)$ we may assume that $e$ has both ends in $W_i$ for some $i \in \{2q+2, 2q+3, \ldots, s\}$.

By an *annex* we mean a maximal well-connected set of foundational paths that includes at least one non-trivial foundational path. Let $\mathcal{R}$ be an annex. We define $H_1(\mathcal{R})$ to be the subgraph of $J := G[W_1 \cup W_2 \cup \cdots \cup W_s]$ consisting of the graph of $\mathcal{R}$ restricted to $J$ and all $\mathcal{R}$-bridges that are the subgraphs of $J$ and have all vertices of attachment in $V(\mathcal{R})$. We define $H_0(\mathcal{R})$ analogously as a subgraph of $G[W_1 \cup W_2 \cup \cdots \cup W_{2q+1}]$. It follows that $e$ is an edge of $H_1(\mathcal{R})$ for some maximal well-connected set $\mathcal{R}$ of foundational paths. Let us assume that $e$ belongs to $H_1(\mathcal{R})$ for some annex $\mathcal{R}$. Thus we fix $\mathcal{R}$ and denote $H_0(\mathcal{R})$ and $H_1(\mathcal{R})$ by $H_0$ and $H_1$, respectively. We will modify the linkage $\mathcal{Q}$ within $H_1$, and will obtain a contradiction to its choice that way.

Let $\mathcal{Q}'$ be the subset of $\mathcal{Q}$ consisting of those paths that use at least one vertex of $H_1$. For $Q \in \mathcal{Q}'$ let $a(Q)$ be its end in $X_0$, let $d(Q)$ be its end in $X_l$, and let $b(Q)$ and $c(Q)$ be two vertices of $Q \cap H_1$ such that the subpath of $Q$ from $b(Q)$ to $c(Q)$ is maximum and $a(Q), b(Q), c(Q), d(Q)$ occur on $Q$ in the order listed. It follows that $b(Q), c(Q)$ belong to

$(W_0 \cap W_1) \cup (W_0' \cap W_1') \cup (W_s \cap W_{s+1})$, but if one of them belongs to $W_0' \cap W_1'$, then it is equal to $a(Q)$.

If $b(Q) \in W_0 \cap W_1$ or $b(Q) \in W_0' \cap W_1'$ we define $b'(Q) := b(Q)$ and let $B(Q)$ be the null graph; otherwise $b(Q)$ belongs to a foundational path $P \notin \mathcal{P}_2$, and we define $b'(Q)$ to be the unique member of $W_{2q+1} \cap W_{2q+2} \cap V(P)$, and we let $B(Q) := P[W_{2q+2} \cup W_{2q+3} \cup \cdots \cup W_s]$. We define $c'(Q)$ and $C(Q)$ analogously. By Lemma 3.6 applied to $\mathcal{W}$ and $\mathcal{P}$ with $i = 0$ and $j = 2q+1$ there exists a linkage $\mathcal{S}$ in $H_0$ of size $|\mathcal{Q}'|$ from $\{b'(Q) : Q \in \mathcal{Q}'\}$ to $\{c'(Q) : Q \in \mathcal{Q}'\}$. The fact that $\mathcal{R}$ was chosen to be a maximal well-connected set implies that members of this linkage are disjoint from the members of $\mathcal{Q} - \mathcal{Q}'$. For each $Q \in \mathcal{Q}'$ we delete the interior of the subpath of $Q$ between $b(Q)$ and $c(Q)$, and add the linkage $\mathcal{S}$ and the paths $B(Q)$ and $C(Q)$ for all $Q \in \mathcal{Q}'$. Thus we obtain a new linkage with the same properties as $\mathcal{Q}$, but with fewer edges not in $H$, contrary to the choice of $\mathcal{Q}$. This completes the case when $e$ belongs to $H_1(\mathcal{R})$ for some annex $\mathcal{R}$, and so from now on we may assume the opposite.

Let $K$ denote the union of the trivial paths in $\mathcal{P}$. Since $e$ belongs to $H_1(\mathcal{R})$ for no annex $\mathcal{R}$ it follows that the $K$-bridge $B$ of $H_1$ containing $e$ includes no non-trivial foundational path. Let $Q \in \mathcal{Q}$ be the path containing $e$, and let $b, c \in V(Q)$ be such that $bQc$ is a maximal subpath of $B$ containing $e$. Since $Q$ is disjoint from $W_s \cap W_{s+1} \cap V(\mathcal{P}_2)$, and hence from the the trivial paths in $\mathcal{P}_2$, we deduce that $b, c \notin V(\mathcal{P}_2)$. It follows more generally (from the fact that $e$ belongs to $H_1(\mathcal{R})$ for no annex $\mathcal{R}$) that every $K$-bridge $B'$ of $H_1$ that has $b$ and $c$ as attachments includes no non-trivial foundational path. Consequently, if $B'$ includes a non-trivial subpath of some member of $\mathcal{Q}$, then this subpath uses two vertices of $V(K)$. On the other hand the foundational paths with vertex-sets $\{b\}$ and $\{c\}$ are adjacent in the auxiliary graph, and hence for each $i = 1, 2, \ldots, q$ there exists a $K$-bridge of $G[W_i]$ whose attachments include $b$ and $c$. By the conclusion of the sentence before the previous one we deduce that there is $i \in \{1, 2, \ldots, q\}$ such that $W_i$ includes no non-trivial subpath of a member of $\mathcal{Q}$. Thus we can replace $bQc$ by a subpath of $W_i$, contrary to the choice of $\mathcal{Q}$. This completes the proof that $\mathcal{W}'$ and $\mathcal{P}'$ satisfy (L9). $\square$

We are now ready to state the main result of this section.

**Theorem 3.8** *For all integers $k, l, p, w \geq 0$ there exists an integer $N$ with the following property. If $G$ is a $p$-connected graph of tree-width at most $w$ with at least $N$ vertices, then either $G$ has a minor isomorphic to $K_{p,k}$, or $G$ has a linear decomposition of length at least $l$ and adhesion at most $w$ satisfying* (L1)–(L9).

**Proof.** Let $k, l, p, w \geq 0$ be integers, and let $l_1 := l + 4w + 2$. Let $l_2$ be the minimum value of $\lambda$ such that Lemma 3.5 holds for $l = l_1$, $p$ and all $q \leq w$. Finally, let $N$ be such that Lemma 3.1 holds for $l = l_2$, $k, p$, and $w$. We claim that $N$ satisfies the theorem. To prove the claim let $G$ be a $p$-connected graph of tree-width at most $w$ with at least $N$ vertices. By Lemma 3.1 it has either a minor isomorphic to $K_{p,k}$, or a linear decomposition $\mathcal{W}_2$ of length at least $l_2$ and adhesion $q \leq w$ satisfying (L1)–(L5), and so we may assume the latter. By Lemma 3.2 there is a foundational linkage $\mathcal{P}_1$ satisfying (L6). By Lemma 3.5 the graph $G$ has a linear decomposition $\mathcal{W}_1$ of length $l_1$ and adhesion $q$ such that $\mathcal{W}_1$ and $\mathcal{P}_1$ satisfy (L1)–(L8). Finally, by Lemma 3.7 there exist a linear decomposition $\mathcal{W}$ of length $l$ and adhesion $q$ and a foundational linkage satisfying (L1)–(L9). □

We will need the following special case.

**Corollary 3.9** *For all integers $l, w \geq 0$ there exists an integer $N$ with the following property. If $G$ is a 6-connected graph of tree-width at most $w$ with at least $N$ vertices, then either $G$ has a minor isomorphic to $K_6$, or $G$ has a linear decomposition of length at least $l$ and adhesion at most $w$ satisfying* (L1)–(L9) *for $p = 6$.*

# 4 Analyzing the auxiliary graph

Let $G$ be a 6-connected graph with no $K_6$ minor, and let $\mathcal{W}$ and $\mathcal{P}$ be as before and satisfy (L1)–(L9). In this section we establish several properties of the auxiliary graph of the pair $(\mathcal{W}, \mathcal{P})$. The first main result is Lemma 4.6 stating that if $\mathcal{W}$ is sufficiently long, then every component of the subgraph of the auxiliary graph induced by the non-trivial foundational paths is either a path or a cycle. The second main result of this section, Lemma 4.10, allows us to modify the pair $(\mathcal{W}, \mathcal{P})$ such that in the new pair every non-trivial $\mathcal{P}$-bridge attaches to exactly two non-trivial foundational paths.

Let $k, l \geq 3$ be integers. For $i \in \{1, 2, \ldots, k\}$ let $P_i$ be a path with vertices $v_1^i, \ldots, v_l^i$ in order. We define the *linked k-cylinder of length l* to be the graph with vertex-set $\bigcup_{i=1}^k V(P_i)$ and edge-set $\bigcup_{i=1}^k E(P_i) \cup \{v_j^i v_j^{i+1} : 1 \leq i \leq k, 1 \leq j \leq l\} \cup \{q_1, q_2\}$, where the index notation is taken modulo $k$ and the edges $q_1$ and $q_2$ have no common end and each have one end in $\{v_1^1, v_1^2, \ldots, v_1^k\}$ and the other end in $\{v_l^1, v_l^2, \ldots, v_l^k\}$. Figure 1 shows a linked 3-cylinder of length six.

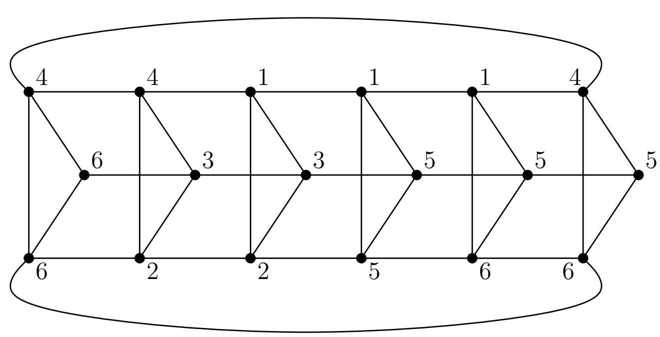


Figure 1: Finding a $K_6$ minor in a linked 3-cylinder of length six.

**Lemma 4.1** *For all integers $k \geq 3$, a linked $k$-cylinder of length twelve has a $K_6$ minor.*

**Proof.** By finding two suitable paths with vertex-sets in $\{v_j^i : 1 \leq i \leq k, 1 \leq j \leq 3\}$, and two paths with vertex-sets in $\{\{v_j^i : 1 \leq i \leq k, 10 \leq j \leq 12\}$, we see that a linked $k$-cylinder of length twelve has a minor isomorphic to a linked 3-cylinder of length six with the additional property that the ends of the edge $q_i$ are $v_1^i$ and $v_6^i$ for $i = 1, 2$. This graph has a $K_6$ minor as indicated in Figure 1. □

**Lemma 4.2** *Let $l \geq 2$ and $q \geq 3$ be integers, and let $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ be a linear decomposition of length $l$ and adhesion $q$ of a graph $G$, and let $\mathcal{P}$ be a foundational linkage for $\mathcal{W}$ such that* (L1)–(L5) *and* (L9) *hold. If for at least $48\binom{q}{3}$ indices $i \in \{1, 2, \ldots, l-1\}$ there exists a $\mathcal{P}$-bridge in $G[W_i]$ with attachments on at least three non-trivial paths in $\mathcal{P}$, then $G$ has a $K_6$ minor.*

**Proof.** Let $l, q$ be integers and $\mathcal{W} = (W_0, \ldots, W_l)$ and $\mathcal{P}$ be given. If there exist $48\binom{q}{3}$ distinct indices $i$ with $1 \leq i \leq l-1$ such that $G[W_i]$ contains a $\mathcal{P}$-bridge attaching to at least three non-trivial foundational paths, then there exist 48 distinct indices $i$ and three distinct non-trivial foundational paths $P_1, P_2, P_3 \in \mathcal{P}$ such that $G[W_i]$ contains a $\mathcal{P}$-bridge attaching to $P_j$ for $j = 1, 2, 3$. Then there exists a subset of indices $I \subseteq \{1, \ldots, l-1\}$ with $|I| = 24$ such that $|i - j| > 2$ for all distinct $i, j \in I$, and furthermore, $G[W_i]$ contains a bridge $B_i$ attaching to $P_j$ for all $i \in I$ and $j = 1, 2, 3$. By property (L9), there exist two disjoint paths $Q_1$ and $Q_2$ each with one end in $V(P_1 \cup P_2 \cup P_3) \cap W_1 \cap W_2$ and one end in $V(P_1 \cup P_2 \cup P_3) \cap W_{l-1} \cap W_l$. Moreover, the paths $Q_1$ and $Q_2$ do not have an internal vertex in either $B_i \setminus V(\mathcal{P})$ or $P_j$ for all $i \in I$ and $1 \leq j \leq 3$. It follows that $G$ has a minor isomorphic to a linked 3-cylinder of length twelve since each pair of successive bridges $B_i$ can be contracted to a single cycle of length three. By Lemma 4.1 the graph $G$ has a $K_6$ minor, as desired. □

The following will be a hypothesis common to several forthcoming lemmas. In order to avoid unnecessary repetition we give it a name.

**Hypothesis 4.3** Let $p = 6$, $l \geq 2$ and $q \geq 6$ be integers, let $G$ be a 6-connected graph with no $K_6$ minor, and let $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ be a linear decomposition of $G$ of length $l$ and adhesion $q$ with a foundational linkage $\mathcal{P}$ such that conditions (L1)–(L9) hold.

**Lemma 4.4** *Assume Hypothesis 4.3. Then there do not exist $6\binom{q}{6}$ distinct indices $i$ with $1 \leq i \leq l-1$ such that $G[W_i]$ contains a non-trivial $\mathcal{P}$-bridge attaching only to trivial foundational paths.*

**Proof.** Let $G$, $\mathcal{W}$, $\mathcal{P}$, $q$, and $l$ be as stated. If the conclusion of the lemma does not hold, then there exist six distinct indices $i$ such that $G[W_i]$ contains a non-trivial $\mathcal{P}$-bridge $B_i$

attaching to the same subset of six trivial foundational paths. By contracting the internal vertices of each $B_i$ to a single vertex, we see $G$ would have a $K_6$ minor, a contradiction. □

**Lemma 4.5** *Assume Hypothesis 4.3. If $l > 6\binom{q}{6}$, then $\mathcal{P}$ includes at least one non-trivial path.*

**Proof.** Let $G$, $\mathcal{W}$, $\mathcal{P}$, $q$, and $l$ be as stated, and suppose for a contradiction that every path in $\mathcal{P}$ is trivial. For every $i$, $1 \leq i \leq l-1$, $G[W_i]$ contains a non-trivial bridge $B_i$, as $W_i \nsubseteq W_{i+1}$, $W_i \nsubseteq W_{i-1}$ by (L4), in contradiction with Lemma 4.4. □

Let $\mathcal{W}$ be a linear decomposition of a graph $G$ and let $\mathcal{P}$ be a foundational linkage such that $\mathcal{W}$ and $\mathcal{P}$ satisfy (L1)–(L8). By a *core* of the pair $(\mathcal{W}, \mathcal{P})$ we mean a component of the graph obtained from the auxiliary graph of $(\mathcal{W}, \mathcal{P})$ by deleting all trivial foundational paths. The next lemma is the first main result of this section.

**Lemma 4.6** *Assume Hypothesis 4.3. If $l \geq 48$, then every core of the pair $(\mathcal{W}, \mathcal{P})$ is a path or a cycle.*

**Proof.** Let $G$, $\mathcal{W}$, $\mathcal{P}$, $q$, and $l$ be as stated. Suppose for a contradiction that there exists a non-trivial foundational path $P_1 \in \mathcal{P}$ adjacent in the auxiliary graph to three non-trivial paths $P_2, P_3, P_4 \in \mathcal{P}$. By property (L9), there exist two disjoint paths $Q_1$ and $Q_2$ each with one end in $V(P_2 \cup P_3 \cup P_4) \cap W_0 \cap W_1$ and one end in $V(P_2 \cup P_3 \cup P_4) \cap W_{l-1} \cap W_l$. Furthermore, $Q_1$ and $Q_2$ avoid any internal vertex of $P_i$ for $1 \leq i \leq 4$ as well as any internal vertex of a $\mathcal{P}$-bridge in $G[W_j]$ for $1 \leq j \leq l-1$. For all $i \in \{1, 2, \dots, 24\}$, we contract to a single vertex $b_i$ the set of vertices consisting of $P_1[W_{2i-1}]$ and the internal vertices of every non-trivial bridge attaching to $P_1$ in $G[W_{2i-1}]$. Note that no vertex of $Q_i$ for $i = 1, 2$ is contained in the contracted set of $b_{2j-1}$ for any $1 \leq j \leq 24$. Each vertex $b_i$ has a neighbor in each of $P_2$, $P_3$, and $P_4$. Also, the neighbors of $b_i$ and $b_j$ are distinct for $i \neq j$. It follows that $G$ has a minor isomorphic to a linked 3-cylinder of length twelve, contrary to Lemma 4.1. □

**Lemma 4.7** *Assume Hypothesis 4.3. If $l \geq 12$, then every non-trivial path in $\mathcal{P}$ is adjacent in the auxiliary graph to at most three trivial paths in $\mathcal{P}$.*

**Proof.** Let $G$, $\mathcal{W}$, $\mathcal{P}$, $q$, and $l$ be as stated. Assume, to reach a contradiction, that $P_1 \in \mathcal{P}$ is a non-trivial path and is adjacent to four trivial foundational paths in the auxiliary graph. Let the vertices comprising the four trivial foundational paths be $v_1, v_2, v_3, v_4$. For each $i \in \{1, 2, \dots, 6\}$ we contract to a single vertex $b_i$ the vertex set containing $P_1[W_{2i-1}]$ and the internal vertices of all non-trivial bridges of $G[W_{2i-1}]$ attaching to $P_1$. It follows that $G$ has as a minor isomorphic to the graph with vertex set $\{v_i : 1 \leq i \leq 4\} \cup \{b_i : 1 \leq i \leq 6\}$ and edges $\{v_ib_j : 1 \le i \le 4, 1 \le j \le 6\} \cup \{b_ib_{i+1} : 1 \le i \le 5\}$. This graph has a $K_6$ minor, and hence so does $G$, a contradiction. □

**Corollary 4.8** *Assume Hypothesis 4.3. If $l \ge 12$, then every member of $\mathcal{P}$ is an induced path.*

**Proof.** If some non-trivial $P \in \mathcal{P}$ is not induced, then by (L6) the path $P$ is adjacent to at least 4 trivial foundational paths in the auxilliary graph, contrary to Lemma 4.7. □

**Lemma 4.9** *Assume Hypothesis 4.3. If $l \ge 12$, then no non-trivial foundational path is adjacent in the auxiliary graph to three or more trivial foundational paths.*

**Proof.** Let $G$, $\mathcal{W}$, $\mathcal{P}$, $q$, and $l$ be as stated. As above, assume to reach a contradiction, that $P_1 \in \mathcal{P}$ is a non-trivial path and is adjacent to three trivial foundational paths in the auxiliary graph. By the 6-connectivity of $G$, $P_1$ must be adjacent to another foundational path in the auxiliary graph. By Lemma 4.7, such a path, call it $P_2$, must be non-trivial. For each $i$, $1 \le i \le 6$, we contract to a single vertex the vertex set containing $P_1[W_{2i-1}]$ and the internal vertices of any non-trivial bridge of $G[W_{2i-1}]$ attaching to $P_1$. It follows that $G$ has a minor isomorphic to the graph in Figure 2, which has a $K_6$ minor as indicated in that figure, a contradiction. □.

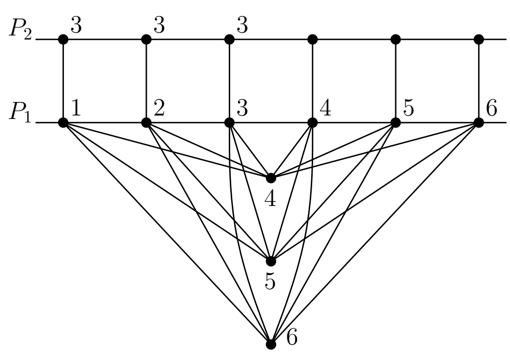


Figure 2: Finding a $K_6$ minor when a non-trivial foundational path is bridge adjacent to three trivial foundational paths.

In the next lemma, the second main result of this section, we show that we can assume that our linear decomposition $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ and foundational linkage $\mathcal{P}$ satisfy the following property.

(L10) For all $i \in \{1, 2, \ldots, l-1\}$, every non-trivial $\mathcal{P}$-bridge of $G[W_i]$ attaches to exactly two non-trivial foundational paths.

**Lemma 4.10** *Assume Hypothesis 4.3. If* $l \geq \left(6\binom{q}{6} + 48\binom{q}{3}\right) l'$, *then there exist a contraction* $\mathcal{W}'$ *of* $\mathcal{W}$ *of length* $l'$ *and adhesion* $q$ *and a foundational linkage* $\mathcal{P}'$ *for* $\mathcal{W}'$ *satisfying* (L1) - (L10).

**Proof.** By Lemma 4.4 and Lemma 4.2 and our choice of $l$, there exists an index $\alpha$ such that for all $i \in \{1, 2, \ldots, l'-1\}$, $G[W_{\alpha+i}]$ contains neither a non-trivial $\mathcal{P}$-bridge attaching only to trivial foundational paths nor a $\mathcal{P}$-bridge attaching to three or more non-trivial foundational paths. Moreover, Lemma 4.7 and property (L6) imply that no non-trivial bridge attaches to exactly one non-trivial foundational path. The lemma follows from considering the contraction $\mathcal{W}' = \left(\bigcup_{i=0}^{\alpha} W_i, W_{\alpha+1}, W_{\alpha+2}, \ldots, W_{\alpha+l'-1}, \bigcup_{i=\alpha+l'}^{l} W_i\right)$ of $\mathcal{W}$ and the corresponding restriction of $\mathcal{P}$. □

# 5 Finding and eliminating a pinwheel

Let us assume Hypothesis 4.3. In the previous section we have shown that $\mathcal{W}$ and $\mathcal{P}$ can be chosen so that for every $i \in \{1, 2, \ldots, l-1\}$, every non-trivial $\mathcal{P}$-bridge $B$ of $G[W_i]$ attaches to exactly two non-trivial foundational paths. The main result of this section will be used in Section 6 to show that if $G$ is not an apex graph then $\mathcal{W}$ and $\mathcal{P}$ can be chosen so that every such bridge attaches to no trivial foundational path. The proof technique is different, and relies on a theory of "non-planar extensions" of planar graphs, developed in [18].

A *pinwheel with* $t$ *vanes* is the graph defined as follows. Let $C^1$ and $C^2$ be two disjoint cycles of length $2t$, where the vertices of $C^i$ are $v_1^i, v_2^i, \ldots, v_{2t}^i$ in order. Let $w_1, w_2, \ldots, w_t, x$ be $t+1$ distinct vertices. The pinwheel with $t$ vanes has vertex-set $V(C^1) \cup V(C^2) \cup \{w_1, w_2, \ldots, w_t, x\}$ and edge-set

$$\begin{aligned} &E(C^1) \cup E(C^2) \cup \{v_{2j}^1 v_{2j}^2 : 1 \leq j \leq t\} \\ &\qquad \cup \{w_j v_{2j-1}^i : 1 \leq j \leq t, i = 1, 2\} \cup \{x w_j : 1 \leq j \leq t\} \end{aligned}$$

The cycles $C^1$ and $C^2$ form the *rings* of the pinwheel. A pinwheel with four vanes is pictured in Figure 3. A *Möbius pinwheel with* $t$ *vanes* is obtained from a pinwheel with $t$ vanes by deleting the edges $v_{2t}^1 v_1^1$ and $v_{2t}^2 v_1^2$ and adding the edges $v_{2t}^1 v_1^2$ and $v_{2t}^2 v_1^1$. The cycle formed by $V(C^1) \cup V(C^2)$ in a Möbius pinwheel is the *ring* of the Möbius pinwheel. A Möbius pinwheel with 4 vanes contains $K_6$ as a minor as shown on Figure 3.

**Lemma 5.1** *Let* $q$, $l$, *and* $p = 6$, $t \geq 4$ *be positive integers. Let* $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ *be a linear decomposition of a 6-connected graph* $G$ *of length* $l$ *and adhesion* $q$ *with foundational linkage* $\mathcal{P}$ *satisfying* (L1)–(L9). *Let* $P_1, P_2, P_3, Q \in \mathcal{P}$ *be distinct, let* $Q$ *be trivial, and let* $P_i$ *be non-trivial for* $i = 1, 2, 3$. *Furthermore, let* $P_2$ *be adjacent to* $P_1$, $P_3$, *and* $Q$ *in the auxiliary graph. If* $l \geq 4t + 1$, *then* $G$ *has a subgraph isomorphic to a subdivision of a pinwheel or a Möbius pinwheel with* $t$ *vanes.*

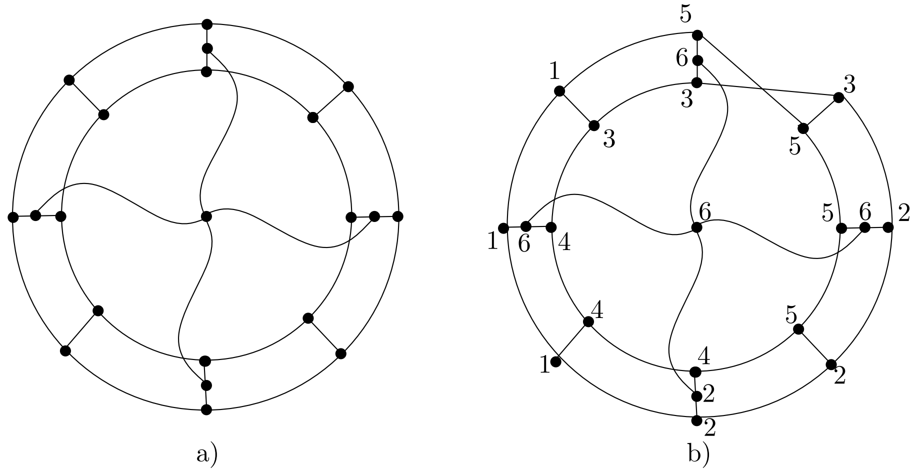


Figure 3: (a) A pinwheel with four vanes, (b) A Möbius pinwheel with 4 vanes and a $K_6$ minor in it.

**Proof.** Let $V(Q) = \{x\}$, let $P_i \cap W_0 \cap W_1 = \{s_i\}$ for $i = 1, 3$, and let $P_i \cap W_{l-1} \cap W_l = \{t_i\}$ for $i = 1, 3$. Let $\bar{\mathcal{P}} = \mathcal{P} - \{P_1, P_2, P_3, Q\}$. By property (L9), there exist two disjoint paths $R_1$ and $R_2$ in $G[W_0 \cup W_l] \cup \bigcup_{P \in \bar{\mathcal{P}}} P$ each with one end in $\{s_1, s_3\}$ and one end in $\{t_1, t_3\}$. The rings of our pinwheel will be formed by $R_1 \cup R_2 \cup P_1 \cup P_3$. If the paths $R_1$ and $R_2$ cross, i.e. the ends of $R_1$ are $s_1$ and $t_3$ and the ends of $R_2$ are $s_3$ and $t_1$, we construct a Möbius pinwheel. Otherwise, we simply construct a pinwheel on $t$ vanes.

Note that for every $j = 1, \ldots, l-1$ there exists a path $S_j$ with one end in $W_j \cap V(P_1)$ and the other end in $W_j \cap V(P_3)$, such that $V(S_j) \subseteq W_j$, and $S_j$ is internally disjoint from $\bigcup_{P \in \mathcal{P} - P_2} P$. Also, for every $j = 1, \ldots, l-1$ there exists a vertex $v_j \in W_j$ and three paths $T_j^1, T_j^2$ and $T_j^3$, internally disjoint from each other and from $\bigcup_{P \in \mathcal{P} - P_2} P$, satisfying the following. Each of $T_j^1, T_j^2$ and $T_j^3$ has one end $v_j$, the second end of $T_j^1$ is in $V(P_1)$, the second end of $T_j^3$ is in $V(P_3)$ and the second end of $T_j^2$ is $x$. The paths $S_j, T_j^1, T_j^2$ and $T_j^3$ are internally disjoint from the rings of our pinwheel by construction, and the paths, corresponding to the sets $W_i$ with non-consecutive indices, are also disjoint. Therefore we can use the paths corresponding to the sets $W_i$ with odd indices to construct a subgraph of $G$ isomorphic to a subdivision of a pinwheel or a Möbius pinwheel, with rings of the pinwheel as prescribed above. $\square$

As we have seen above a Möbius pinwheel with sufficiently many vanes contains a $K_6$ minor. A pinwheel is, however, an apex graph. In order to prove that graphs containing a subdivision of a pinwheel with many vanes satisfy Theorem 1.2, we will need the following lemma concerning subdivisions of apex graphs contained in larger non-apex graphs. The proof of the lemma will appear in [18].

**Lemma 5.2** *Let $J$ be an internally 4-connected triangle-free planar graph, and let $F \subseteq E(J)$ be such that no two edges of $F$ are incident with the same face of $J$. Let $J'$ be obtained from $J$ by subdividing each edge in $F$ exactly once, and let $H$ be the graph obtained from $J'$ by adding a new vertex $v \notin V(J)$ and joining it by an edge to all the new vertices of $J'$. Let a subdivision of $H$ be isomorphic to a subgraph of $G$, and let $u \in V(G)$ correspond to the vertex $v$. If $G \backslash u$ is internally 4-connected and non-planar, then there exists an edge $e \in E(H)$ incident with $v$ such that either*

*(i) there exist vertices $x, y \in V(J')$ not belonging to the same face of $J'$ such that $(H \backslash e)+xy$ is isomorphic to a minor of $G$, or*

*(ii) there exist vertices $x_1, x_2, x_3, x_4 \in V(J)$ appearing on some face of $J$ in order such that $(H \backslash e) + x_1x_3 + x_2x_4$ is isomorphic to a minor of $G$.*

**Lemma 5.3** *If a 5-connected graph $G$ with no $K_6$ minor contains a subdivision of a pinwheel with 20 vanes as a subgraph, then $G$ is apex.*

**Proof.** We will show that for every positive integer $t$ every 5-connected non-apex graph $G$ containing a subdivision of a pinwheel with $4t$ vanes contains a Möbius pinwheel with $t-1$ vanes as a minor. A Möbius pinwheel with 4 vanes contains a $K_6$ minor, as observed above, and so the lemma will follow.

We apply Lemma 5.2, where the graphs $H$ and $J$, the vertex $v \in V(H)$ and the set of edges $F \subseteq E(J)$ are defined as follows. Let $H$ be the pinwheel with $4t$ vanes, and let $v$ be the "hub" of the pinwheel (denoted by $x$ in the definition of a pinwheel). Using notation from the definition of pinwheel, let the graph $J$ consist of two $C^1$ and $C^2$ be two disjoint cycles of length $8t$ with the vertices of $C^i = \{v^i_j : 1 \leq j \leq 8t\}$ for $i = 1, 2$ and $v^i_j$ adjacent to $v^i_{j+1}$ and $v^{i+1}_j$ for all $1 \leq j \leq 8t$ and $i = 1, 2$ with the subscript addition taken modulo $8t$ and the superscript addition taken modulo 2. Finally, let $F = \{v^1_{2j-1}v^2_{2j-1} : 1 \leq j \leq 4t\}$.

Suppose that outcome (ii) of Lemma 5.2 holds (the case when outcome (i) holds is analogous). If the boundary of the face of $J$ containing the vertices $x_1, x_2, x_3$ and $x_4$ is not one of the cycles $C_1$ and $C_2$, then without loss of generality we have $x_1 = v^1_1, x_2 = v^2_1, x_3 = v^2_2$ and $x_4 = v^1_2$. Clearly, for every edge $e \in E(H)$ incident to $v$ the graph $(H \backslash e) + x_1x_3 + x_2x_4$ contains a Möbius pinwheel with $4t - 1$ vanes as a subgraph.

Therefore, by symmetry, we assume that the vertices $x_1, x_2, x_3$ and $x_4$ are contained in $C_1$, i.e. $x_i = v^1_{k_i}$ for $i = 1, 2, 3, 4$, where, without loss of generality, $t \leq k_1, k_2, k_3, k_4 \leq 4t$. Then the subgraph $J_0$ of $J + x_1x_3 + x_2x_4$ induced on $\{v^j_i : t \leq i \leq 4t, j = 1, 2\}$ contains two disjoint paths, one with ends $v^1_t$ and $v^2_{4t}$, and another with ends $v^2_t$ and $v^1_{4t}$. Now consider the graph $(H \backslash e) + x_1x_3 + x_2x_4$, where $e \in E(H)$ is an edge incident to $v$, and delete all the edges of subdivision of $J_0$ from this graph, except for those that belong to the paths constructed above. If is easy to see that the resulting graph contains a subdivision of a Möbius pinwheel with $t-1$ vanes, as claimed. □

The next corollary follows immediately from Lemmas 5.1 and 5.3.

**Corollary 5.4** *Assume Hypothesis 4.3. If* $l \geq 81$ *and some non-trivial foundational path is adjacent in the auxiliary graph to two non-trivial and at least one trivial foundational path, then* $G$ *is apex.*

# 6 Taming the bridges

In Lemma 4.10 we have modified $\mathcal{W}$ and $\mathcal{P}$ so that for every $i \in \{1, 2, \ldots, l-1\}$ every non-trivial $\mathcal{P}$-bridge $B$ of $G[W_i]$ attaches to exactly two non-trivial foundational paths. In Corollary 5.4 we have shown that $B$ attaches to no trivial foundational path, unless $G$ is apex. Let us recall that a core is a component of the subgraph of the auxiliary graph restricted to non-trivial foundational paths. In this section we show that the graph consisting of all paths of a core of $(\mathcal{W}, \mathcal{P})$ and all bridges that attach to two paths of the core can be drawn in either a disk or a cylinder, depending on whether the core is a path or a cycle.

The following lemma follows easily from the definition of properties (L1)–(L5) and (L9). Let us recall that rerouting was defined prior to Lemma 2.1.

**Lemma 6.1** *Let* $l \geq 2$, $q \geq 0$, *and* $p \geq 0$ *be integers, and let* $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ *be a linear decomposition of length* $l$ *and adhesion* $q$ *of a graph* $G$, *and let* $\mathcal{P}$ *be a foundational linkage for* $\mathcal{W}$ *such that* (L1)–(L5) *and* (L9) *hold. Let* $i$ *be fixed with* $1 \leq i \leq l-1$ *and let* $Q$ *be a path in* $G[W_i]$ *with ends* $x$ *and* $y$ *such that* $x, y \in V(P)$ *for some* $P \in \mathcal{P}$ *and* $Q$ *is otherwise disjoint from* $V(\mathcal{P})$. *Let* $P'$ *be obtained by rerouting* $P$ *along* $Q$. *Then the linkage* $\mathcal{P}' = (\mathcal{P} - \{P\}) \cup \{P'\}$ *satisfies* (L1) - (L5) *and* (L9).

Let $G$ be a graph and $\mathcal{W} = (W_0, \ldots, W_l)$ be a linear decomposition of length $l$ and adhesion $q$ of $G$, and let $\mathcal{P}$ be a foundational linkage such that (L1)–(L5) hold. Let $i \in \{1, 2, \ldots, l-1\}$, let $P, P' \in \mathcal{P}$ be two non-trivial foundational paths, let $W_{i-1} \cap W_i \cap V(P) = \{x\}$, $W_{i-1} \cap W_i \cap V(P') = \{x'\}$, $W_i \cap W_{i+1} \cap V(P) = \{y\}$, and $W_i \cap W_{i+1} \cap V(P') = \{y'\}$. If the paths $Q_1$ and $Q_2$ are internally disjoint from $V(\mathcal{P})$, the vertices $x$, $u_1$, $u_2$, $y$ occur on $P$ in that order, and $x'$, $v_2$, $v_1$, $y'$ occur on $P'$ in that order, then we say that the foundational paths $P$ and $P'$ *twist*.

Let $P_1$, $P_2$ and $P_3$ be three non-trivial foundational paths and let $Q_1$, $Q_2$, and $Q_3$ be three internally disjoint paths such that $Q_j$ is also internally disjoint from each member of $\mathcal{P}$ for each $j \in \{1, 2, 3\}$. Let the ends of $Q_j$ be $x_j$, $y_j$ for $1 \leq j \leq 3$. The paths $Q_1$, $Q_2$, and $Q_3$ form a $P_1$-*tunnel* if $x_1, y_1 \in V(P_1)$, the vertices $x_2, x_3 \in V(x_1 P_1 y_1) - \{x_1, y_1\}$ and $y_j \in V(P_j)$ for $j = 2, 3$. The path $Q_1$ is called the *arch* of the tunnel.

**Lemma 6.2** *Let $l \geq 2$, $q \geq 3$, and $p = 6$ be integers, and let $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ be a linear decomposition of length $l$ and adhesion $q$ of a graph $G$, and let $\mathcal{P}$ be a foundational linkage for $\mathcal{W}$ such that* (L1)–(L5) *and* (L9) *hold. If there exist $48\binom{q}{3}$ distinct indices $i \in \{1, 2, \ldots, l-1\}$ such that $G[W_i]$ contains a $P$-tunnel for some non-trivial foundational path $P \in \mathcal{P}$, then $G$ has a $K_6$ minor.*

**Proof.** Let $l$, $q$, $p$, $\mathcal{W}$ and $\mathcal{P}$ be given. Assume, to reach a contradiction, that there exist $48\binom{q}{3}$ indices $i \in \{1, 2, \ldots, l-1\}$ such that $G[W_i]$ has a $P_i$-tunnel for some non-trivial foundational path $P_i \in \mathcal{P}$. Reroute the paths $P_i$ along the arches of the $P_i$-tunnels to get a linkage $\mathcal{P}'$. By Lemma 6.1 $\mathcal{W}$ and $\mathcal{P}'$ satisfy (L1)–(L5) and (L9). Moreover, for each of the above $48\binom{q}{3}$ distinct indices $i$ there exists a non-trivial $\mathcal{P}'$-bridge in $G[W_i]$ that attaches to at least three non-trivial foundational paths. It follows from Lemma 4.2 that $G$ has a $K_6$ minor, as desired. □

**Lemma 6.3** *Let $l \geq 2$, $q \geq 3$, and $p = 6$ be integers, and let $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ be a linear decomposition of length $l$ and adhesion $q$ of a graph $G$, and let $\mathcal{P}$ be a foundational linkage for $\mathcal{W}$ such that* (L1)–(L5) *and* (L9) *hold. If there exist $12\binom{q}{2}$ distinct indices $i \in \{1, 2, \ldots, l-1\}$ such that $G[W_i]$ contains a pair of twisting non-trivial foundational paths, then $G$ has a $K_6$ minor.*

**Proof.** Let $l$, $q$, $p$, $\mathcal{W}$ and $\mathcal{P}$ be given. Assume there exist $12\binom{q}{2}$ distinct indices $i \in \{1, 2, \ldots, l-1\}$ such that $G[W_i]$ contains a pair of twisting non-trivial foundational paths. It follows that there exists a subset $\mathcal{I} \subseteq \{1, 2, \ldots, l-1\}$ of cardinality 12 and non-trivial paths $P_1, P_2 \in \mathcal{P}$ such that $P_1$ and $P_2$ twist in $G[W_i]$ for all $i \in \mathcal{I}$. We use the twisting paths to contract three disjoint $K_4$ subgraphs onto $P_1$ and $P_2$ to find a minor isomorphic to the graph in Figure 4. The edges $r_1$ and $r_2$ in the figure exist by applying property (L9) to the ends of $P_1$ and $P_2$. The numbering in Figure 4 shows a $K_6$ minor, implying that $G$ also has a $K_6$ minor, as desired. □

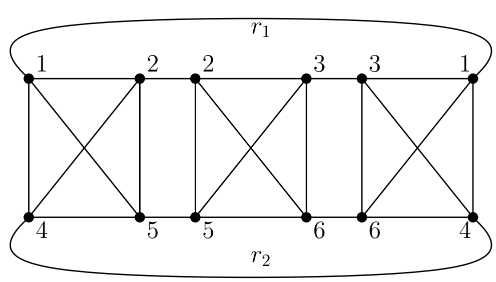


Figure 4: Finding a $K_6$ minor when there exist a pair of non-trivial foundational paths that twist in twelve distinct $W_i$. The edges $r_1$ and $r_2$ are depicted as not crossing, however, if they cross the graph still contains $K_6$ as a minor.

**Lemma 6.4** *Let $G$ be a 6-connected graph with no $K_6$ minor. Let $l \geq 2$, $q \geq 3$, and $p = 6$ be integers, let $\mathcal{W} = (W_0, W_1, \dots, W_l)$ be a linear decomposition of length $l$ and adhesion $q$ of $G$, and let $\mathcal{P}$ be a foundational linkage for $\mathcal{W}$ such that* (L1)–(L9) *hold. If there exist $40\binom{q}{3}$ distinct indices $i \in \{1, 2, \dots, l-1\}$ such that $G[W_i]$ contains a non-trivial $\mathcal{P}$-bridge attaching to a trivial foundational path, then $G$ is apex.*

**Proof.** Let $l$, $q$, $p$, $\mathcal{W}$ and $\mathcal{P}$ be given. Assume that there exist $40\binom{q}{3}$ distinct indices $i \in \{1, 2, \dots, l-1\}$ such that $G[W_i]$ contains a non-trivial $\mathcal{P}$-bridge attaching to a trivial foundational path. By (L10) each such bridge attaches to two non-trivial foundational paths. Therefore, there exist distinct non-trivial paths $P, P' \in \mathcal{P}$ and a trivial path $Q \in \mathcal{P}$ such that $G[W_i]$ contains a $\mathcal{P}$-bridge attaching to $P, P'$ and $Q$ for at least 40 distinct indices $i \in \{1, 2, \dots, l-1\}$. The argument used in the proof of Lemma 5.1 implies that $G$ contains a subgraph isomorphic to a subdivision of a pinwheel with 20 vanes or a Möbius pinwheel with 20 vanes. Note that the Möbius pinwheel with 20 vanes contains a $K_6$ minor, and, thus, $G$ is apex by Lemma 5.3, as desired. □

Let us assume Hypothesis 4.3, and let $\mathcal{C}$ be a core of $(\mathcal{W}, \mathcal{P})$. We define *the $i^{th}$ section of* $\mathcal{C}$, denoted by $G(\mathcal{C}, i)$, to be the subgraph of $G[W_i]$, obtained from the union of the paths in $\mathcal{C}$ and all $\mathcal{P}$-bridges of $G[W_i]$ that attach to a member of $\mathcal{C}$ by deleting the trivial foundational paths. By Lemma 4.6 the graph $\mathcal{C}$ is a path or a cycle. Let $P_1, P_2, \dots, P_t$ be the vertices of $\mathcal{C}$, listed in order, let $W_{i-1} \cap W_i \cap V(P_j) = \{u_j\}$ and let $W_i \cap W_{i+1} \cap V(P_j) = \{v_j\}$. If $\mathcal{C}$ is a path, then we say that $\mathcal{C}$ is *flat in* $W_i$ if $G(\mathcal{C}, i)$ can be drawn in a disk with the vertices $u_1, u_2, \dots, u_t, v_t, v_{t-1}, \dots, v_1$ drawn on the boundary of the disk in order, and the paths $P_1$ and $P_t$ also drawn on the boundary of the disk. If $\mathcal{C}$ is a cycle, then we say that $\mathcal{C}$ is *flat in* $W_i$ if $G(\mathcal{C}, i)$ can be drawn in a cylinder with the vertices $u_1, u_2, \dots, u_t$ drawn on one of the boundary components of the cylinder in the clockwise order listed, and $v_1, v_2, \dots, v_t$ drawn on the other boundary component in the clockwise order listed. Our next objective is to find a linear decomposition $\mathcal{W} = (W_0, W_1, \dots, W_l)$ and a foundational linkage $\mathcal{P}$ such that

(L11) Every core of $(\mathcal{W}, \mathcal{P})$ is flat in $W_i$ for every $i \in \{1, 2, \dots, l-1\}$.

(L12) For every $i \in \{1, 2, \dots, l-1\}$, no non-trivial $\mathcal{P}$-bridge of $G[W_i]$ attaches to a trivial foundational path.

**Lemma 6.5** *Let $G$ be a 6-connected non-apex graph not containing $K_6$ as a minor. Let $p = 6$, $l \geq 2$, $q \geq 6$ be integers, and let $\mathcal{W} = (W_1, W_2, \dots, W_l)$ be a linear decomposition of $G$ of adhesion $q$ and length $l$ satisfying* (L1)–(L10)*. If $l > \left(88\binom{q}{3} + 12\binom{q}{2}\right) l'$, then there exists a contraction $\mathcal{W}'$ of $\mathcal{W}$ of length $l'$ such that $\mathcal{W}'$ and the corresponding restriction of $\mathcal{P}$ satisfy* (L1)–(L12).

**Proof.** Let $G$, $p$, $q$, $l$, $\mathcal{W}$, and $\mathcal{P}$ be given. By our choice of $l$ and Lemmas 6.3, 6.2 and 6.4, there exists an index $\alpha$ such that for all $i \in \{0, 1, \dots, l'\}$ the graph $G[W_{\alpha+i}]$ does not contain a $P$-tunnel for any $P$ in $\mathcal{P}$, nor does it contain a pair of non-trivial twisting foundational paths, nor does it contain a non-trivial bridge attaching to a trivial foundational path. We claim that the contraction $\left(\bigcup_{i=0}^{\alpha-1} W_i, W_\alpha, W_{\alpha+1}, \ldots, W_{\alpha+l'}, \bigcup_{i=\alpha+l'+1}^{l} W_i\right)$ of $\mathcal{W}$ is as desired. Condition (L12) follows from the construction, and hence it suffices to prove (L11).

Fix an index $i \in \{0, 1 \ldots, l'\}$ and a core $\mathcal{C}$ of the auxiliary graph. We wish to apply Lemma 2.3 or 2.4, depending on whether $\mathcal{C}$ is a path or cycle, to the graph $H := G(\mathcal{C}, \alpha + i)$ and linkage $\mathcal{C}$. Let $P_j, u_j, v_j$ for $j \in \{1, 2, \ldots, t\}$ be as in the definition of flat. By Corollary 4.8 and (L10) every $\mathcal{C}$-bridge of $H$ is stable, and by (L10) no $\mathcal{C}$-bridge of $H$ attaches to three or more members of $\mathcal{C}$. If there exists a set $X \subseteq V(H)$ of size at most three such that some component $J$ of $G \setminus X$ is disjoint from $\{u_1, u_2, \ldots, u_t, v_1, v_2, \ldots, v_t\}$, then by 6-connectivity of $G$ the vertices of $J$ include a neighbor of at least three distinct trivial paths of $\mathcal{P}$. We conclude that some member of $\mathcal{C}$ is adjacent in the auxiliary graph to at least three trivial foundational paths, contrary to Lemma 4.9. Thus no such set $X$ exists. Next we show that none of the outcomes (i)–(iii) of Lemmas 2.3 and 2.4 hold. Outcome (i) does not hold by the definition of $\mathcal{C}$, and outcomes (ii) and (iii) do not hold by the choice of $\alpha$ and $i$. Thus it follows from Lemma 2.3 if $\mathcal{C}$ is a path or Lemma 2.4 if $\mathcal{C}$ is a cycle that $H$ can be drawn in a disk or a cylinder as described in that lemma, which is precisely the definition of $\mathcal{C}$ being flat in $W_{\alpha+i}$. Thus $\mathcal{W}'$ satisfies (L11) as well. □

# 7 Controlling the boundary of a planar graph

Let $G$ be a simple plane graph with the infinite region bounded by a cycle $C$, and such that the degree of every vertex in $V(G) - V(C)$ is at least six. DeVos and Seymour [2] proved that $|V(G)| \leq |V(C)|^2/12 + O(|V(C)|)$. In this section we digress to prove a similar result under the weaker hypothesis that $G$ has deficiency at most five, where the *deficiency* of a plane graph $G$ with the infinite region bounded by a cycle $C$ is defined as $\sum_{v \in V(G)-V(C)} \max\{6 - \deg(v), 0\}$. We denote the deficiency of $G$ by $\mathrm{def}(G)$. The proof is an adaptation of the argument from [2], but we include it, because the details are different. We begin with a couple of definitions and a lemma.

A *quilt* is a simple plane graph $G$ with the infinite region bounded by a cycle $C$, such that $G$ has deficiency at most five and every finite region of $G$ is bounded by a triangle. If exactly one vertex of $C$ has degree three, and all other vertices have degree exactly four, then we say that $C$ is a *convenient graph*. Otherwise, a convenient graph is a subpath of $C$ with at least one edge, with both ends of degree exactly three, and all internal vertices of degree exactly four.

**Lemma 7.1** *Every quilt with no vertices of degree two has a convenient graph.*

**Proof.** Let $G$ be a quilt with no vertices of degree two, and let the deficiency of $G$ be $d$. Consider the planar graph $G'$ obtained by adding a vertex $v$ to $G$ adjacent to every vertex of $C$. Let $|V(G)| = n$ and $|V(C)| = m$. Then

$$\begin{aligned} 6(n+1) - 12 &= \sum_{v \in V(G')} \deg_{G'}(v) \\ &= \sum_{v \in V(C)} (\deg_G(v) + 1) + m + \sum_{v \in V(G)-V(C)} \deg_G(v) \\ &\geq \sum_{v \in V(C)} \deg_G(v) + 6(n-m) - d + 2m. \end{aligned}$$

It follows that $\sum_{v \in V(C)} \deg_G(v) \leq 4m - 6 + d$. Since $d \leq 5$ we deduce that there are strictly more vertices in $C$ of degree three than of degree at least five. Thus, a convenient graph exists. □

The main theorem of this section follows easily from the next lemma. If $G$ is a quilt, we define $\mu(G)$ to be 1 if $G$ has a vertex of degree two, and otherwise we define $\mu(G)$ to be the minimum number of edges in a convenient graph. Thus $\mu(G)$ is at least one, and at most the length of the cycle bounding the infinite region of $G$.

**Lemma 7.2** *Let $G$ be a quilt on at least four vertices with the infinite region bounded by a cycle of length $k$. Then $|V(G)| \leq k^2/2 + k/2 + \mu(G) + \operatorname{def}(G) - 6$.*

**Proof.** Let $G$ and $k$ be as stated. We proceed by induction on $|V(G)|$. If $G$ has exactly four vertices, then it is isomorphic to $K_4$, or $K_4$ minus an edge. We have $k = 3$, $\mu(G) = 1$, $\operatorname{def}(G) = 3$, or $k = 4$, $\mu(G) = 1$, $\operatorname{def}(G) = 0$, and the lemma holds. Thus we may assume that $G$ has at least five vertices, and that the lemma holds for all quilts on fewer than $|V(G)|$ vertices. Let $C$ be the cycle bounding the infinite region of $G$. If $C$ has a chord, then the chord divides $G$ into two quilts $G_1$ and $G_2$ in the obvious way. Let the infinite region of $G_i$ have length $k_i$. Assume first that $G_2$ has exactly three vertices. Then by induction

$$\begin{aligned} |V(G)| = |V(G_1)| + 1 &\leq k_1^2/2 + k_1/2 + \mu(G_1) + \operatorname{def}(G_1) - 6 + 1 \\ &= k^2/2 + k/2 + \mu(G_1) - k + 1 + \operatorname{def}(G_1) - 6 \\ &\leq k^2/2 + k/2 + \mu(G) + \operatorname{def}(G) - 6, \end{aligned}$$

as desired. Thus we may assume that both $G_1$ and $G_2$ have at least four vertices. Since $k_1, k_2 \geq 3$ we have $3(k_1 + k_2) \leq k_1 k_2 + 9$, and hence by induction

$$\begin{aligned} |V(G)| &= |V(G_1)| + |V(G_2)| - 2 \\ &\leq k_1^2/2 + k_1/2 + k_1 + \operatorname{def}(G_1) - 6 + k_2^2/2 + k_2/2 + k_2 + \operatorname{def}(G_2) - 6 - 2 \\ &= (k_1 + k_2 - 2)^2/2 + (k_1 + k_2 - 2)/2 + \operatorname{def}(G_1) + \operatorname{def}(G_2) - k_1 k_2 + 3k_1 + 3k_2 - 15 \\ &\leq k^2 + k/2 + \mu(G) + \operatorname{def}(G) - 6, \end{aligned}$$

as desired. Thus we may assume that $C$ has no chord. In particular, $G$ has no vertex of degree two.

By Lemma 7.1 the quilt $G$ has a convenient graph. Let $P$ be a convenient graph with the smallest number of edges. Let us assume first that $P$ has exactly one edge. Then $P$ is a path with ends $u$ and $v$, say. Since $C$ does not have any chords and $G$ has at least five vertices, the graph $G' := G\backslash\{u, v\}$ is a quilt. If $G'$ has exactly three vertices, then $G$ is the wheel on five vertices, $k = 4$, $\mu(G) = 1$, $\mathrm{def}(G) = 2$, and the lemma holds. Thus we may assume that $G'$ has at least four vertices, and hence by induction

$$\begin{aligned}|V(G)| = |V(G')| + 2 &\leq (k-1)^2/2 + (k-1)/2 + \mu(G') + \mathrm{def}(G') - 6 + 2 \\ &= k^2/2 + k/2 + \mu(G') - k + 2 + \mathrm{def}(G') - 6 \\ &\leq k^2/2 + k/2 + \mu(G) + \mathrm{def}(G) - 6,\end{aligned}$$

as desired. Thus we may assume that $P$ has at least two edges. If $P = C$, then let $u$ be the unique vertex of $C$ of degree three; otherwise $P$ is a path, and we let $u$ be an end of $P$. Let $u'$ be the unique neighbor of $u$ that does not belong to $C$. Then $G' := G\backslash u$ is a quilt on at least four vertices with the infinite region bounded by a cycle $C'$, where $C'$ has length $k$. Since $C$ has no chords and $G$ has at least five vertices we deduce that $\deg_{G'}(u') \geq 3$. If equality holds, then $u$ has degree four in $G$, and hence $\mathrm{def}(G') = \mathrm{def}(G) - 2$. Otherwise $\mu(G') \leq \mu(G) - 1$. In either case we have by induction

$$\begin{aligned}|V(G)| = |V(G')| + 1 &\leq k^2/2 + k/2 + \mu(G') + \mathrm{def}(G') - 6 + 1 \\ &\leq k^2/2 + k/2 + \mu(G) + \mathrm{def}(G) - 6,\end{aligned}$$

as desired. □

**Theorem 7.3** *Let $G$ be a simple graph drawn in a disk, let $X$ be the set of vertices of $G$ drawn on the boundary of the disk, and assume that $\sum_{v\in V(G)-X} \max\{6 - \deg(v), 0\} \leq 5$. If $|X| \geq 3$, then $|V(G)| \leq |X|^2/2 + 3|X|/2 - 1$.*

**Proof.** Let $G$ and $X$ be as stated. We may assume, by adding edges to $G$, that $G$ is a quilt with the infinite region bounded by a cycle with vertex set $X$. By Lemma 7.2 we have $|V(G)| \leq |X|^2/2 + |X|/2 + \mu(G) + \mathrm{def}(G) - 6 \leq |X|^2/2 + 3|X|/2 - 1$, as desired. □

# 8 Cylindrical tube

Lemma 4.5 guarantees the existence of a non-empty core in a sufficiently long linear decomposition of any $K_6$-minor-free 6-connected graph $G$ of bounded tree-width, assuming that such a decomposition satisfies conditions (L1)-(L9). Lemma 4.6 implies that, under the same conditions, each core is a path or a cycle. In this section we handle the case when some core of a linear decomposition of the graph $G$ is a cycle.

Before introducing the main result of this section, we need to present one more definition and a related lemma. Let $k$, $l$ be positive integers, $k, l \geq 3$. A *double crossed $k$-cylinder of length $l$* is the graph defined as follows. Let $P_1, \ldots, P_k$ be $k$ vertex disjoint paths with the vertex set of $P_i = \{v_j^i : 1 \leq j \leq l\}$ for all $1 \leq i \leq k$ with $v_j^i$ adjacent to $v_{j+1}^i$ for all $1 \leq j \leq l-1$. The double crossed $k$-cylinder of length $l$ has vertex set $\{v_j^i : 1 \leq j \leq l, 1 \leq i \leq k\}$ and edge set

$$\left(\bigcup_{i=1}^{k} E(P_i)\right) \cup \{v_j^i v_j^{i+1} : 1 \leq i \leq k, 1 \leq j \leq l\} \cup \{q_1, q_2, r_1, r_2\},$$

where the superscript addition is taken modulo $k$. Furthermore, the ends of $q_i$ are $u_i, v_i \in \{v_1^j : 1 \leq j \leq k\}$ for $i = 1, 2$ and the vertices $u_1, u_2, v_1, v_2$ occur in that order in the cyclic order $(v_1^1, v_1^2, \ldots, v_k^1)$. Similarly, the edges $r_1$ and $r_2$ cross in the cyclic order $(v_l^1, v_l^2, \ldots, v_l^k)$. Explicitly, the ends of $r_i$ are $x_i, y_i \in \{v_l^j : 1 \leq j \leq k\}$ for $i = 1, 2$ and occur in the order $x_1, x_2, y_1, y_2$ in the cyclic order $(v_l^1, v_l^2, \ldots, v_l^k)$.

**Lemma 8.1** *Let $t$ and $l$ be integers, $t \geq 5$, $l \geq 16$. A double crossed $t$-cylinder of length $l$ contains $K_6$ as a minor.*

**Proof.** Let $G$ be a doubled crossed $t$-cylinder of length $l$ with vertex set $\{v_j^i : 1 \leq j \leq l, 1 \leq i \leq t\}$. By possibly routing the crossing edges $q_1$ and $q_2$ in the first five cycles on vertices $\{v_j^i : 1 \leq j \leq 5, 1 \leq i \leq t\}$ and routing the edges $r_1$ and $r_2$ on the final five cycles with vertex set $\{v_j^i : l-5 \leq j \leq l, 1 \leq i \leq t\}$, we see that $G$ contains as a minor a doubled crossed 5-cylinder $G'$ of length 6 and moreover, with the additional property that the ends of $q_1$ are $v_1^1$ and $v_1^3$ and the ends of $q_2$ are $v_1^2$ and $v_1^4$. Similarly, the edges $r_1$ and $r_2$ of $G'$ have ends $v_6^1$, $v_6^3$ and $v_6^2$, $v_6^4$, respectively. The graph $G$ then contains $K_6$ as a minor, as indicated in Figure 5. □

We now give the main result of this section.

**Lemma 8.2** *Let $p = 6$, $l \geq 2$, and $q \geq 6$ be integers. Let $G$ be a 6-connected graph with no $K_6$ minor, and let $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ be a linear decomposition of $G$ of length $l$ and adhesion $q$ with a foundational linkage $\mathcal{P}$ satisfying* (L1)–(L12)*. Further, assume that some core of $(\mathcal{W}, \mathcal{P})$ is a cycle. If $l \geq 2q + 32$, then $G$ is apex.*

**Proof.** Let $p$, $l$, $q$, and $\mathcal{W}$ be given, let $\mathcal{C}$ be a core of $(\mathcal{W}, \mathcal{P})$ that is a cycle, and assume for a contradiction that $G$ is not apex. Let $P_1, P_2, \ldots, P_t$ be the vertices of $\mathcal{C}$ listed in order. For $i = 1, 2, \ldots, l-1$ let $H_i$ denote the graph $G(\mathcal{C}, i)$, and for $j = 1, 2, \ldots, t$ let $u_j$ be the unique element of $V(P_j) \cap W_q \cap W_{q+1}$ and $v_j$ the unique element of $V(P_j) \cap W_{q+32} \cap W_{q+33}$. Let

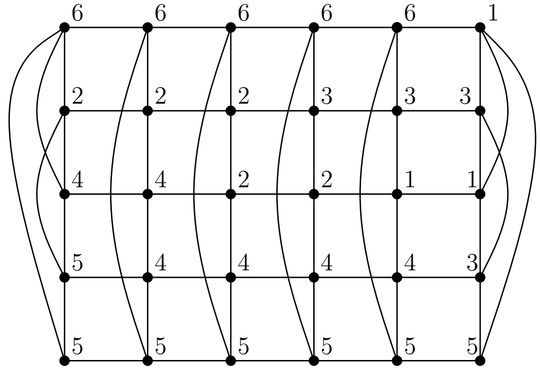


Figure 5: A double crossed 5-cylinder of length 6 contains $K_6$ as a minor

$A = \{u_1, u_2, \ldots, u_t\}$, $B = \{v_1, v_2, \ldots, v_t\}$, let $K$ denote the graph $H_{q+1} \cup H_{q+2} \cup \ldots \cup H_{q+32}$, and let $L$ denote the graph $G \setminus (V(K) - A - B)$. Since $G$ is not apex and $\mathcal{C}$ is a cycle, by Corollary 5.4 the core $\mathcal{C}$ forms a component of the auxiliary graph. Therefore, we have $K \cup L = G$ and $V(K \cap L) = A \cup B$.

We claim that $L$ does not include two disjoint paths from $A$ to $B$. Indeed, otherwise by contracting $P_i[W_{q+2j}]$ to a single vertex for $1 \le i \le t$ and $0 \le j \le 11$, we see that $G$ contains a linked $t$-cylinder of length twelve. Lemma 4.1 then contradicts our choice of $G$. Thus there exist subgraphs $L_1, L_2$ of $L$ such that $L_1 \cup L_2 = L$, $A \subseteq V(L_1)$, $B \subseteq V(L_2)$ and $|V(L_1 \cap L_2)| \le 1$. Now property (L9) applied to $\mathcal{C}$ and a subset of $\mathcal{C}$ of size two implies that $t \ge 5$.

Let $\Omega_1$ be the cyclic permutation $(u_1, u_2, \ldots, u_t)$, and let $\Omega_2$ be the cyclic permutation $(v_1, v_2, \ldots, v_t)$. Thus $(L_1, \Omega_1)$ and $(L_2, \Omega_2)$ are societies. Let $X = V(L_1 \cap L_2)$. By (L11) the graph $K$ can be drawn in a cylinder with $u_1, u_2, \ldots, u_t$ drawn in one boundary component in the clockwise order listed, and $v_1, v_2, \ldots, v_t$ drawn in the other boundary component in the clockwise order listed. Thus if both societies $(L_1 \setminus X, \Omega_1 \setminus X)$ and $(L_2 \setminus X, \Omega_2 \setminus X)$ are rural, then $G$ is apex, so we may assume that $(L_1 \setminus X, \Omega_1 \setminus X)$ is not rural and hence by Theorem 2.5 it has a cross. The society $(L_2, \Omega_2)$ is not rural by Theorem 7.3, because each vertex of $V(L_2) - B - X$ has degree at least 6 and $|V(L_2)| \ge qt \ge t^2 = |B|^2$, because $V(L_2)$ includes each of the pairwise disjoint sets $W_i \cap W_{i+1} \cap V(\mathcal{C})$ for $i = q+32, q+33, \ldots, 2q+31$. Likewise, $(L_2, \Omega_2)$ has a cross by Theorem 7.3.

We have shown that there exist four pairwise disjoint paths, two of them forming a cross in $(L_1, \Omega_1)$ and two forming a cross in $(L_2, \Omega_2)$. Let $j \in \{0, 1, \ldots, 15\}$. By the definition of core the graph $G(\mathcal{C}, q+2j+1)$ has internally disjoint paths $Q_1, Q_2, \ldots, Q_t$ such that $Q_i$ has one end in $P_i$, the other end in $P_{i+1}$ (where $P_{t+1}$ means $P_1$), and is otherwise disjoint from $\mathcal{C}$. Since for $j \ne j'$ the graphs $G(\mathcal{C}, q+2j+1)$ and $G(\mathcal{C}, q+2j'+1)$ are vertex disjoint, we conclude that $G$ contains as a minor a double crossed $t$-cylinder of length at least 16. This observation contradicts Lemma 8.1 and completes the proof of the lemma. □

# 9 Planar strip

We now examine the case when some core of the auxiliary graph is a path.

**Lemma 9.1** *Let $p = 6$, $l \geq 2$ and $q \geq 6$ be integers. Let $G$ be a 6-connected graph with no $K_6$ minor, and let $\mathcal{W} = (W_0, W_1, \ldots, W_l)$ be a linear decomposition of $G$ of length $l$ and adhesion $q$ with a foundational linkage $\mathcal{P}$ satisfying (L1) - (L12). Further, assume that some core of $(\mathcal{W}, \mathcal{P})$ is a path. If $l \geq \max\{4q + 11, 48\}$, then $G$ is an apex graph.*

**Proof.** Let $p$, $l$, $q$, and $\mathcal{W}$ be given, let $\mathcal{C}$ be a core of $(\mathcal{W}, \mathcal{P})$ that is a path, and assume for a contradiction that $G$ is not apex. Let $P_1, P_2, \ldots, P_t$ be the vertices of $\mathcal{C}$ listed in order. As in the proof of Lemma 8.2, for $i = 1, 2, \ldots, l-1$ let $H_i$ denote the graph $G(\mathcal{C}, i)$, and for $j = 1, 2, \ldots, t$ let $u_j$ be the unique element of $V(P_j) \cap W_0 \cap W_1$ and $v_j$ the unique element of $V(P_j) \cap W_{l-1} \cap W_l$. Let $A = \{u_1, u_2, \ldots, u_t\}$, $B = \{v_1, v_2, \ldots, v_t\}$, and let $\mathcal{Q}$ denote the set of trivial foundational paths adjacent in the auxiliary graph to paths in $\mathcal{C}$. Let $K$ denote the subgraph of $G$ induced on $V(H_1 \cup H_2 \cup \ldots \cup H_{l-1}) \cup V(\mathcal{Q})$, and let $L$ denote the graph $G \setminus (V(K) - A - B - V(\mathcal{Q}))$. Note that $K \cup L = G$ and $V(K) \cap V(L) = A \cup B \cup V(\mathcal{Q})$.

We claim that either $P_1$ or $P_t$ is adjacent in the auxiliary graph to at least two paths in $\mathcal{Q}$. Suppose for a contradiction that both $P_1$ and $P_t$ are adjacent to at most one such path. We assume that $P_i$ is adjacent to exactly one trivial foundational path $S_i \in \mathcal{Q}$ for $i = 1, i = t$. The argument is similar in the case when one or both of $P_1$ and $P_t$ are not adjacent to any paths in $\mathcal{Q}$. Note that by (L12) and Corollary 5.4 all the neighbors of $V(S_1)$ and $V(S_2)$ lie on $P_1 \cup P_2$. If $S_1 \neq S_t$, we let $\{s_i\} = V(S_i)$ for $i = 1$, $i = t$ and $K' = K$. If $S_1 = S_t$ with $V(S_1) = V(S_t) = \{s\}$, let $K'$ be obtained from $K$ by deleting $s$, and adding new vertices $s_1$ and $s_2$, where $s_1$ is adjacent to every neighbor of $s$ on $P_1$, and $s_t$ is adjacent to every neighbor of $s$ on $P_t$. By property (L11), the graph $K'$ is planar and embeds in a disk with exactly the vertices $\{s_1, s_t\} \cup A \cup B$ on the boundary. Moreover, every vertex not on the boundary of the disk has degree at least six. This is a contradiction to Theorem 7.3, as $|V(K')| \geq lt > (2t + 2)^2$, because $l \geq 4q + 11$.

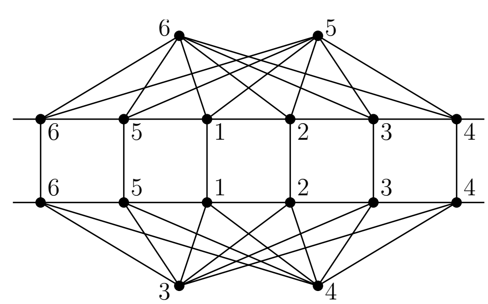


Figure 6: Finding a $K_6$ minor when there exist four distinct trivial foundational paths with neighbors in $\mathcal{C}$.

Using the above claim and Lemma 4.2 we assume without loss of generality that $P_1$ is adjacent in the auxiliary graph to exactly two paths in $\mathcal{Q}$, say $Q_1$ and $Q_2$. Let $V(Q_1) = \{q_1\}$ and $V(Q_2) = \{q_2\}$. We claim that the graph $G' = G \setminus \{q_1, q_2\}$ is planar and that $P_1$ is a subset of a facial boundary of $G'$. Suppose that $P_t$ is adjacent to at least two paths in $\mathcal{Q} - \{Q_1, Q_2\}$. Then $G$ contains as a minor the graph in Figure 6. The horizontal paths in the figure correspond to contractions of $P_1$ and $P_t$ and the vertical edges correspond to paths in $H_{2i+1}$ for $i = 1, 2, \ldots, 6$ with ends on $P_1$ and $P_t$, which exist by the definition of $\mathcal{C}$. The graph in Figure 6 contains a $K_6$ minor, as indicated, a contradiction. Therefore $P_t$ is adjacent to at most one path in $\mathcal{Q} - \{Q_1, Q_2\}$. By (L11), (L12) and Corollary 5.4, the graph $K$ is planar and embeds in the disk with $P_1$ forming part of its boundary. Let $\Omega$ be a cyclic permutation of the set $V(\Omega) = A \cup B \cup (V(\mathcal{Q}) - \{q_1, q_2\})$ ordered $u_t, u_{t-1}, \ldots, u_1, v_1, \ldots, v_t$ followed by the element of $V(\mathcal{Q}) - \{q_1, q_2\}$ if $V(\mathcal{Q}) - \{q_1, q_2\} \neq \emptyset$. If the society $(L, \Omega)$ contains a cross, then $G$ contains as a minor one of the configurations pictured in Figure 7. As each of this configurations contains a $K_6$ minor as indicated in Figure 7, we conclude by Theorem 2.5 that $(L, \Omega)$ is rural. Combined with the planarity of $K$ this implies our claim that $G'$ is planar and $P_1$ is a subset of a facial boundary.

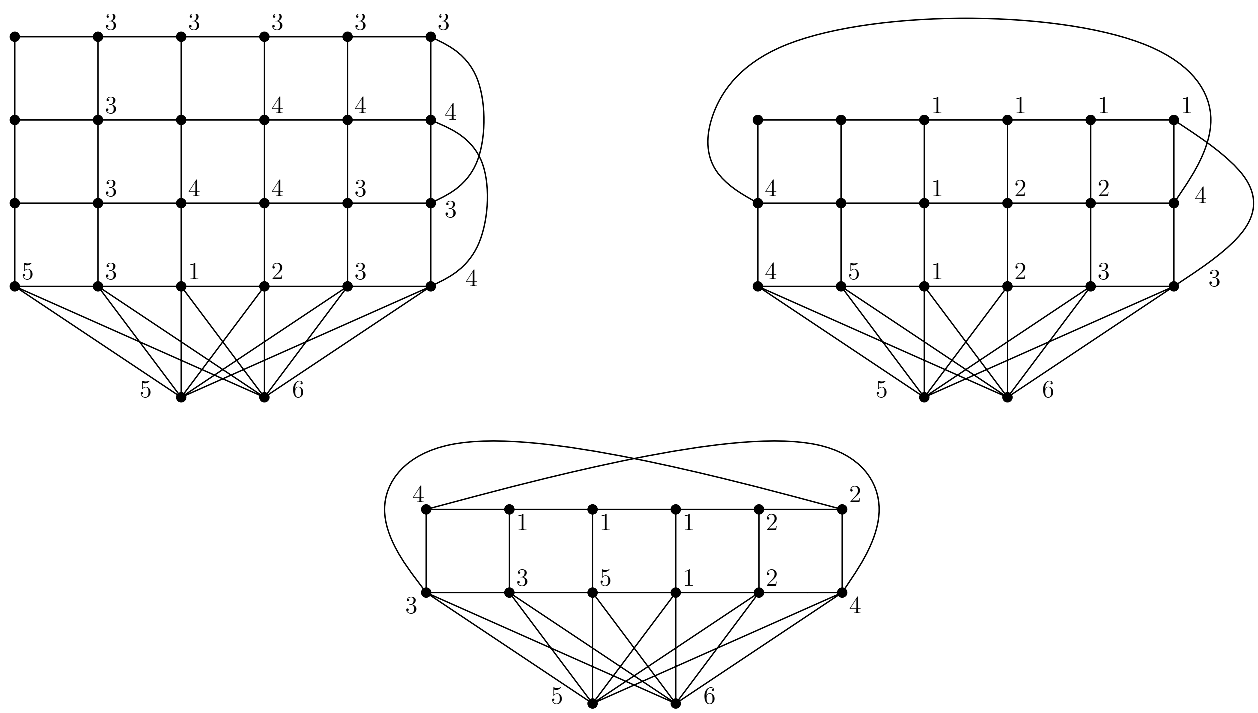


Figure 7: Finding $K_6$ minor when the society $(L, \Omega)$ is not rural.

Let $\mathcal{P}_2 = \{Q_1, Q_2, P_1, P_2\}$. By property (L9), there exist two disjoint paths $R_1$ and $R_2$ in $G[W_0 \cup W_l] \cup \bigcup_{P \in \mathcal{P} - \mathcal{P}_2} P$ linking the set $\{u_1, u_2\}$ to the set $\{v_1, v_2\}$. By the claim in the previous paragraph we assume without loss of generality that $R_i$ has ends $u_i$ and $v_i$ for $i = 1, 2$, and that $R_1 \cup P_1$ forms a facial cycle of $G'$. As $G$ is not apex, both $q_1$ and $q_2$ must have some neighbor not contained in $R_1 \cup P_1$. Let $q_i'$ be such a neighbor of $q_i$ for $i = 1, 2$. The cycle $R_1 \cup P_1$ is a facial cycle in the 4-connected planar graph $G'$, and hence there is a unique $(R_1 \cup P_1)$-bridge in $G - \{q_1, q_2\}$. It follows that for each $q_i'$ there exists a path from $q_i'$ to $R_2 \cup P_2$ avoiding $R_1 \cup P_1$. Let $R_i'$ for $i = 1, 2$ be such paths from $q_i'$ to $R_2 \cup P_2$. Since $l \geq 48$ there exists an index $\alpha$ such that $W_{\alpha+i}$ is disjoint from $R_1'$ and $R_2'$ for $0 \leq i \leq 14$. By considering $P_1$ and $P_2$ and the bridges attaching to $P_1$ and $P_2$ in $H_\alpha, H_{\alpha+1}, \ldots, H_{\alpha+14}$, we see that $G$ contains as a minor the graph in Figure 8, and consequently, a $K_6$ minor, as indicated in Figure 8. This contradiction completes the proof of the lemma. □

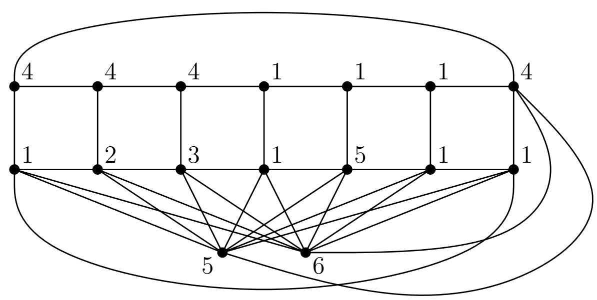


Figure 8: A configuration giving a $K_6$ minor when the trivial foundational paths $Q_1$ and $Q_2$ have a neighbor not contained in the boundary of the face defined by $R_1 \cup P_1$

Lemma 9.1 represents the final step in our analysis of the structure of the auxiliary graph. We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let $w \geq 1$ be an integer. Let $l_1 = \max\{4w + 11, 2w + 32, 58\}$, let $l_2 = \left(88\binom{w}{3} + 12\binom{w}{2}\right) l_1$, and let $l_3 = \left(6\binom{w}{6} + 48\binom{w}{3}\right) l_2$. By Corollary 3.9 there exists an integer $N$ such that every 6-connected graph $G$ of tree-width at most $w$ with no $K_6$ minor has a linear decomposition of length at least $l_3$ and adhesion at most $w$ satisfying properties (L1)–(L9) for $p = 6$. We claim that such an integer $N$ satisfies Theorem 1.2.

Let $G$ be a 6-connected graph of tree-width at most $w$ with at least $N$ vertices and no $K_6$ minor. By Lemma 4.10 the graph $G$ has a linear decomposition of length at least $l_2$ and adhesion at most $w$ satisfying properties (L1)–(L10), and thus by Lemma 6.5 the graph $G$ has a linear decomposition $\mathcal{W}$ of length at least $l_1$ and adhesion at most $w$ and a foundational linkage $\mathcal{P}$ satisfying properties (L1)–(L12). By Lemma 4.5 $\mathcal{P}$ includes a non-trivial foundational path. By Lemma 4.9 every non-trivial foundational path of $\mathcal{P}$ attaches to at most 2 trivial foundational paths in the auxiliary graph. Therefore, by the 6-connectivity of $G$, every core of $(\mathcal{W}, \mathcal{P})$ has at least two vertices, and by Lemma 4.6 every core is a path or a cycle. If some core of $(\mathcal{W}, \mathcal{P})$ is a cycle, then $G$ is apex by Lemma 8.2. Otherwise, $G$ is apex by Lemma 9.1. □

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