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A SURVEY OF LINKLESS EMBEDDINGS

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ABSTRACT. We announce results about flat (linkless) embeddings of graphs in 3–space. A piecewise-linear embedding of a graph in 3–space is called *flat* if every circuit of the graph bounds a disk disjoint from the rest of the graph. We have shown that:

(i) An embedding is flat if and only if the fundamental group of the complement in 3–space of the embedding of every subgraph is free.

(ii) If two flat embeddings of the same graph are not ambient isotopic, then they differ on a subdivision of K_5 or $K_{3,3}$.

(iii) Any flat embedding of a graph can be transformed to any other flat embedding of the same graph by "3–switches," an analog of 2–switches from the theory of planar embeddings. In particular, any two flat embeddings of a 4–connected graph are either ambient isotopic, or one is ambient isotopic to a mirror image of the other.

(iv) A graph has a flat embedding if and only if it has no minor isomorphic to one of seven specified graphs. These are the graphs that can be obtained from K_6 by means of $Y\Delta$– and ΔY–exchanges.

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1. INTRODUCTION

All spatial embeddings are assumed to be piecewise linear. If C, C' are disjoint simple closed curves in S^3, then their *linking number*, $\mathrm{lk}(C, C')$, is the number of times (mod 2) that C crosses over C' in a regular projection of $C \cup C'$. In this paper graphs are finite, undirected and may have loops and multiple edges. Every graph is regarded as a topological space in the obvious way. We say that an embedding of a graph G in S^3 is *linkless* if every two disjoint circuits of G have zero linking number. The following is a result of Sachs [16] and Conway and Gordon [4].

(1.1) *The graph K_6 has no linkless embedding.*

Proof. Let ϕ be an embedding of K_6 into S^3. By studying the effect of a crossing change in a regular projection, it is easy to see that the mod 2 sum $\sum \mathrm{lk}(\phi(C_1), \phi(C_2))$, where the sum is taken over all unordered pairs of disjoint circuits C_1, C_2 of K_6, is an invariant independent of the embedding. By checking an arbitrary embedding we can establish that this invariant equals 1. □

A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. Our main result is a theorem that a graph is linklessly embeddable if and only if it has no minor isomorphic to K_6 or six other closely related graphs. However, we find it much easier to work with the following stronger concept, suggested by Böhme [1] and Saran [18]. We say that an embedding ϕ of a graph G in S^3 is *flat* if for every circuit C of G there exists an open disk in S^3 disjoint from $\phi(G)$ whose boundary is $\phi(C)$. Clearly every flat embedding is linkless, but the converse is false. However, we shall see later that a graph admits a linkless embedding if and only if it admits a flat embedding, and so the classes of embeddable graphs are the same. The reason why we prefer flat embeddings is that they work better. For instance, there is a uniqueness theory parallel to the theory of planar embeddings, and a theorem which characterizes flat embeddings in terms of the fundamental group of the complement.

If G is a graph and X is a vertex or a set of vertices, we denote by $G \backslash X$ the graph obtained from G by deleting X. A graph G is *nearly–planar* if there exists a vertex v of G such that $G \backslash v$ is planar. It may be helpful to notice the following fact.

(1.2) *Every nearly–planar graph admits a flat embedding.*

Proof. Let G be nearly–planar, and let v be such that $G \backslash v$ is planar. We may assume that G is simple, because it is easy to construct a flat embedding of a graph given a flat embedding of its underlying simple graph. We embed $G \backslash v$ in the xy–plane in $R^3 \subseteq S^3$, embed v anywhere not in this plane, and embed all edges from v to the planar graph as straight line segments. It is easy to check that this defines a flat embedding. □

The following lemma was proved by Böhme [1] (see also [18]).

(1.3) *Let ϕ be a flat embedding of a graph G into S^3, and let $C_1, C_2, \ldots, C_n$ be a family of circuits of G such that for every $i \neq j$, the intersection of C_i and C_j is either connected or null. Then there exist pairwise disjoint open disks $D_1, D_2, \ldots, D_n$, disjoint from $\phi(G)$ and such that $\phi(C_i)$ is the boundary of D_i for $i = 1, 2, \ldots, n$.*

An embedding ϕ of a graph G in S^3 is *spherical* if there exists a surface $\Sigma \subseteq S^3$ homeomorphic to S^2 such that $\phi(G) \subseteq \Sigma$. Clearly if ϕ is spherical then G is planar. We illustrate the use of (1.3) with the following, which is a special case of a theorem of Wu [22].

(1.4) *Let ϕ be an embedding of a planar graph G in S^3. Then ϕ is flat if and only if it is spherical.*

Proof. Clearly if ϕ is spherical then it is flat. We prove the converse only for the case when G is 3–connected. Let $C_1, C_2, \ldots, C_n$ be the collection of face–boundaries in some planar embedding of G. These circuits satisfy the hypothesis of (1.3). Let $D_1, D_2, \ldots, D_n$ be the disks as in (1.3); then $\phi(G) \cup D_1 \cup D_2 \cup \cdots \cup D_n$ is the desired sphere. □

The paper is organized as follows. In Section 2 we present a characterization of flat embeddings in terms of the fundamental group of the complement, in Section 3 we discuss a uniqueness theory of flat embeddings, in Section 4 we state our main result, an excluded minor characterization of linklessly embeddable graphs, and finally in Section 5 we discuss three conjectures and some algorithmic aspects of flat embeddings.

2. THE FUNDAMENTAL GROUP

The following is a result of Scharlemann and Thompson [19].

(2.1) *Let ϕ be an embedding of a graph G in S^3. Then ϕ is spherical if and only if*
(i) G is planar, and
(ii) for every subgraph G' of G, the fundamental group of $S^3 - \phi(G')$ is free.

The "only if" implication is easy to see. The point of the theorem is the converse. It is easy to see that (ii) cannot be replaced by the weaker condition that the fundamental group of $S^3 - \phi(G)$ is free. We use (2.1) to prove the following generalization.

(2.2) *Let ϕ be an embedding of a graph G in S^3. Then ϕ is flat if and only if for every subgraph G' of G, the fundamental group of $S^3 - \phi(G')$ is free.*

Proof. Here we only prove "only if." Let G' be a subgraph of G such that $\pi_1(S^3 - \phi(G'))$ is not free. Choose a maximal forest F of G' and let G'' be obtained from G' by contracting all edges of F, and let ϕ'' be the induced embedding of G''. Then $\pi_1(S^3 - \phi''(G'')) = \pi_1(S^3 - \phi(G'))$ is not free, but

G'' is planar, and so ϕ'' is not flat by (2.1) and (1.4). Hence ϕ is not flat, as desired. □

Let G be a graph, and let e be an edge of G. We denote by $G\backslash e$ (G/e) the graph obtained from G by deleting (contracting) e. If ϕ is an embedding of G in S^3, then it induces embeddings of $G\backslash e$ and (up to ambient isotopy) of G/e in the obvious way. We denote these embeddings by $\phi\backslash e$ and ϕ/e, respectively.

(2.3) *Let ϕ be an embedding of a graph G in S^3, and let e be a nonloop edge of G. If both $\phi\backslash e$ and ϕ/e are flat, then ϕ is flat.*

Proof. Suppose that ϕ is not flat. By (2.2) there exists a subgraph G' of G such that $\pi_1(S^3 - \phi(G'))$ is not free. If $e \notin E(G')$, then $\phi\backslash e$ is not flat by (2.2). If $e \in E(G')$ then ϕ/e is not flat by (2.2), because $\pi_1(S^3 - (\phi/e)(G'/e)) = \pi_1(S^3 - \phi(G'))$ is not free. □

We say that a graph G is a *coforest* if every edge of G is a loop. The following follows immediately from (2.3).

(2.4) *Let ϕ be an embedding of a graph G in S^3. Then ϕ is flat if and only if the induced embedding of every coforest minor of G is flat.*

3. UNIQUENESS

We begin this section by recalling the following two classical results. The first is Kuratowski's theorem [8]. (A graph H is a *subdivision* of a graph G if H can be obtained from G by replacing edges by internally–disjoint paths.)

(3.1) *A graph is planar if and only if it has no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$.*

Let ϕ be an embedding of a graph G in S^2. Let P be a simple closed curve in S^2 meeting $\phi(G)$ in a set A containing at most two points. Let D be a chord of P (that is, a simple curve with only its distinct endpoints in common with P) and assume that every member of A is on D. Let B be the open disk of $S^2 - P$ containing the interior of D. Let ϕ' be an embedding obtained from ϕ by taking a reflection through D in B, and by leaving ϕ unchanged in $S^2 - B$. We say that ϕ' was obtained from ϕ by a *2–switch*. The second classical result is a theorem of Whitney [21], perhaps stated in a slightly unusual way.

(3.2) *Let ϕ_1, ϕ_2 be two embeddings of a graph in S^2. Then ϕ_1 can be obtained from ϕ_2 by a series of 2–switches.*

We shall see in (3.10) that a similar theorem holds for flat embeddings.

Let ϕ_1, ϕ_2 be two embeddings of a graph G in S^3. We say that ϕ_1, ϕ_2 are *ambient isotopic* if there exists an orientation preserving homeomorphism h of S^3 onto S^3 such that $\phi_1 = h\phi_2$. (We remark that by a result of Fisher [5] h can be realized by an ambient isotopy.) The following follows from (1.4) and (3.2).

(3.3) *Any two flat embeddings of a planar graph are ambient isotopic.*

(3.4) *The graphs K_5 and $K_{3,3}$ have exactly two non-ambient isotopic flat embeddings.*

Sketch of proof. Let G be $K_{3,3}$ or K_5, let e be an edge of G, and let H be $G\backslash e$. Notice that H is planar. From (1.3) it fΔollows that if ϕ is a flat embedding of G, then there is an embedded 2–sphere $\Sigma \subseteq S^3$ with $\phi(G) \cap \Sigma = \phi(H)$. If ϕ_1 and ϕ_2 are flat embeddings of G, we may assume (by replacing ϕ_2 by an ambient isotopic embedding) that this 2–sphere Σ is the same for both ϕ_1 and ϕ_2. Now ϕ_1 is ambient isotopic to ϕ_2 if and only if $\phi_1(e)$ and $\phi_2(e)$ belong to the same component of $S^3 - \Sigma$. □

As a curiosity we deduce from (3.1), (3.3) and (3.4) that a graph has a unique flat embedding if and only if it is planar.

Our next objective is to determine the relation between different flat embeddings of a given graph. We denote by $f|X$ the restriction of a mapping f to a set X.

(3.5) *Let ϕ_1, ϕ_2 be two flat embeddings of a graph G that are not ambient isotopic. Then there exists a subgraph H of G isomorphic to a subdivision of K_5 or $K_{3,3}$ for which $\phi_1|H$ and $\phi_2|H$ are not ambient isotopic.*

A question arises if there is any analogue of (3.5) when the embeddings are not necessarily flat. The following follows immediately from (2.4).

(3.6) *Let ϕ_1, ϕ_2 be two embeddings of a graph G such that they are not ambient isotopic and exactly one of them is flat. Then G has a coforest minor H such that the embeddings of H induced by ϕ_1 and ϕ_2 are not ambient isotopic.*

We do not know if (3.6) remains true when none of ϕ_1, ϕ_2 is flat.

We denote the vertex-set and edge-set of a graph G by $V(G)$ and $E(G)$ respectively. Let G be a graph and let H_1, H_2 be subgraphs of G isomorphic to subdivisions of K_5 or $K_{3,3}$. We say that H_1 and H_2 are 1-*adjacent* if there exist $i \in \{1, 2\}$ and a path P in G such that P has only its endpoints in common with H_i and such that H_{3-i} is a subgraph of the graph obtained from H_i by adding P. We say that H_1 and H_2 are 2-*adjacent* if there are seven vertices $u_1, u_2, \ldots, u_7$ of G, and thirteen paths L_{ij} of G ($1 \le i \le 4$ and $5 \le j \le 7$, or $i = 3$ and $j = 4$), such that

(i) each path L_{ij} has ends u_i, u_j,
(ii) the paths L_{ij} are mutually vertex-disjoint except for their ends,
(iii) H_1 is the union of L_{ij} for $i = 2, 3, 4$ and $j = 5, 6, 7$, and
(iv) H_2 is the union of L_{ij} for $i = 1, 3, 4$ and $j = 5, 6, 7$.

(Notice that if H_1 and H_2 are 2–adjacent, then they are both isomorphic to subdivisions of $K_{3,3}$, and that L_{34} is used in neither H_1 nor H_2.) We denote by $\mathcal{K}(G)$ the simple graph with vertex-set all subgraphs of G isomorphic to subdivisions of K_5 or $K_{3,3}$ in which two distinct vertices are adjacent if they are either 1–adjacent or 2–adjacent. The following is easy to see, using (3.4).

(3.7) *Let ϕ_1, ϕ_2 be two flat embeddings of a graph G, and let H, H' be two adjacent vertices of $\mathcal{K}(G)$. If $\phi_1|H$ is ambient isotopic to $\phi_2|H$, then $\phi_1|H'$ is ambient isotopic to $\phi_2|H'$.*

We need the following purely graph-theoretic lemma.

(3.8) *If G is a 4–connected graph, then $\mathcal{K}(G)$ is connected.*

We prove (3.8) in [12] by proving a stronger result, a necessary and sufficient condition for $H, H' \in V(\mathcal{K}(G))$ to belong to the same component of $\mathcal{K}(G)$ in an arbitrary graph G. The advantage of this approach is that it permits an inductive proof using the techniques of deleting and contracting edges.

If ϕ is an embedding of a graph G in S^3 we denote by $-\phi$ the embedding of G obtained by composing ϕ with the antipodal map. The following is our uniqueness theorem.

(3.9) *Let G be a 4–connected graph and let ϕ_1, ϕ_2 be two flat embeddings of G. Then ϕ_1 is ambient isotopic to either ϕ_2 or $-\phi_2$.*

Proof. If G is planar then ϕ_1 is ambient isotopic to ϕ_2 by (3.3). Otherwise there exists, by (3.1), a subgraph H of G isomorphic to a subdivision of K_5 or $K_{3,3}$. By replacing ϕ_2 by $-\phi_2$ we may assume by (3.4) that $\phi_1|H$ is ambient isotopic to $\phi_2|H$. From (3.7) and (3.8) we deduce that $\phi_1|H'$ is ambient isotopic to $\phi_2|H'$ for every $H' \in V(\mathcal{K}(G))$. By (3.5) ϕ_1 and ϕ_2 are ambient isotopic, as desired. □

Actually, the 4–connectedness is not necessary for (3.9). It turns out that what is necessary and sufficient for the conclusion of (3.9) is, roughly, that no two subgraphs isomorphic to subdivisions of K_5 or $K_{3,3}$ are "separated" by a separation of order at most 3. Let us call such graphs *Kuratowski 4–connected.*

We now state a generalization of (3.9). Let ϕ be a flat embedding of a graph G, and let $\Sigma \subseteq S^3$ be a surface homeomorphic to S^2 meeting $\phi(G)$ in a set A containing at most three points. In one of the open balls into which Σ divides S^3, say B, choose an open disk D with boundary a simple closed curve ∂D such that $A \subseteq \partial D \subseteq \Sigma$. Let ϕ' be an embedding obtained from ϕ by taking a reflection of ϕ through D in B, and leaving ϕ unchanged in $\Sigma - B$. We say that ϕ' is obtained from ϕ by a *3–switch.* The following analog of (3.2) generalizes (3.9).

(3.10) *Let ϕ_1, ϕ_2 be two flat embeddings of a graph G in S^3. Then ϕ_2 can be obtained from ϕ_1 by a series of 3–switches.*

4. THE PETERSEN FAMILY

Let G be a graph and let v be a vertex of G of valency 3 with distinct neighbors. Let H be obtained from G by deleting v and adding an edge between every pair of neighbors of v. We say that H was obtained from G by a $Y\Delta$-*exchange* and that G was obtained from H by a ΔY-*exchange*. We say that two graphs are $Y\Delta$-*equivalent* if one can be obtained from a graph isomorphic to the other by a sequence of the following operations and their inverses:

(i) Deleting a vertex of valency ≤ 1,
(ii) suppressing a vertex of valency 2 (that is, contracting an edge incident to it),
(iii) deleting a parallel edge or a loop,
(iv) $Y\Delta$-exchange.

(4.1) *If G, H are $Y\Delta$-equivalent, then G has a flat embedding if and only if H does.*

It follows from (4.1) and (1.2) that if a graph is $Y\Delta$-equivalent to a nearly-planar graph, then it admits a flat embedding. The converse is false, because $K_{5,5}$ minus a perfect matching is a counterexample.

The *Petersen family* is the set of all graphs that can be obtained from K_6 by doing $Y\Delta$- and ΔY-exchanges. There are (up to isomorphism) exactly seven such graphs, one of which is the Petersen graph. The Petersen family is depicted in Figure 1. The following is our main theorem.

(4.2) *For a graph G, the following conditions are equivalent.*
(i) G has a flat embedding,
(ii) G has a linkless embedding,
(iii) G has no minor isomorphic to a member of the Petersen family.

Here (i) $\Rightarrow$ (ii) is trivial. Sachs [16] has in fact shown that no member of the Petersen family has a linkless embedding, from which (ii) $\Rightarrow$ (iii) follows because the property of having a linkless embedding is closed under taking minors. (Sachs stated his result in a weaker form, but the proof is adequate.) The hard part is that (iii) $\Rightarrow$ (i), which we now briefly sketch.

Sketch of the proof that in (4.2), (iii) $\Rightarrow$ (i). Suppose that G is a minor-minimal graph with no flat embedding. It can be shown that G is "basically 5-connected", which is a certain weaker form of 5-connectivity (see the next section for a precise definition). From (4.1) we may assume that G has no triangles. Suppose that there are edges e, f of G and an end v of e not adjacent to either end of f such that $G\backslash v, G\backslash e/f, G/e/f$ are all Kuratowski 4-connected. Since G is minor-minimal with no flat embedding, there are flat embeddings ϕ_1, ϕ_2, ϕ_3 of $G\backslash e, G/e, G/f$, respectively. By (3.9), since $\phi_3\backslash e$ and ϕ_1/f are both flat embeddings of the Kuratowski 4-connected graph $G\backslash e/f$, we may assume that $\phi_3\backslash e \simeq \phi_1/f$, and similarly that $\phi_3/e \simeq \phi_2/f$. (Here and later $\simeq$ means "ambient isotopic to.") From the first equation there is a 1-edge uncontraction of $\phi_3\backslash e$ which yields an embedding ambient isotopic to ϕ_1, and similarly there is a 1-edge uncontraction of ϕ_3/e yielding ϕ_2. These two uncontractions can be viewed as "local" operations at a vertex common to $\phi_3\backslash e$ and ϕ_3/e, and it

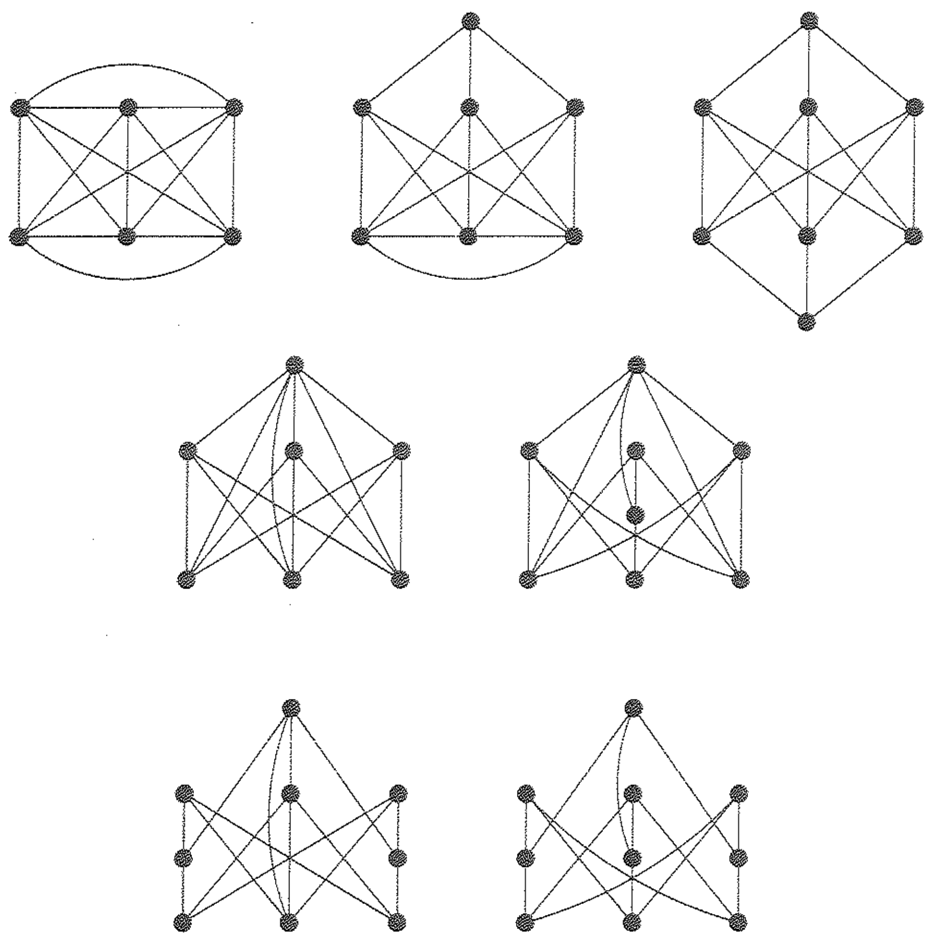

Figure 1: The Petersen Family

can be argued (the details are quite complicated, see [14]) that they are the "same" uncontraction operation. Let ϕ be obtained from ϕ_3 by performing this uncontraction; then $\phi\backslash e \simeq \phi_1$ and $\phi/e \simeq \phi_2$. Since ϕ_1 and ϕ_2 are flat, so is ϕ by (2.3), a contradiction since G has no flat embedding. Thus no two such edges e, f exist. But now a purely graph-theoretic argument [13] (using the non-existence of such edges e, f, the high connectivity of G and that G is not nearly-planar) implies that G has a minor in the Petersen family. □

There have been a number of other attempts [10, 18, 2] at proving (iii) ⇒ (i) and (iii) ⇒ (ii). However, none of them is correct. The question whether (iii) ⇒ (i) was first raised by Sachs [16], and that (i) and (ii) are equivalent was conjectured by Böhme [1].

We mention the following corollary, which is vaguely related to the so-called "strong embedding conjecture." Let ψ be an embedding of a graph G in a surface (=compact 2-manifold without boundary) Σ. We say that ψ is k-*representative* if every non-null-homotopic closed curve in Σ meets $\psi(G)$ at least k times. The strong embedding conjecture states that every 2-connected graph has a 2-representative embedding in some surface. It is also possible that every 3-connected graph has such an embedding in a nonorientable surface. From (4.2) we deduce the following.

(4.3) *If a graph G admits a 3-representative embedding into some nonorientable surface, then G has a minor isomorphic to a member of the Petersen family other than $K_{4,4}^-$ ($K_{4,4}$ minus an edge).*

Proof. Let ψ be a 3-representative embedding of G in a nonorientable surface Σ. By [15, Proposition 7.3] we may assume (by taking a minor of G) that G is 3-connected. We first show that G has a minor isomorphic to a member of the Petersen family. By (4.2) it suffices to show that G has no flat embedding. Suppose for a contradiction that G has a flat embedding ϕ into S^3. Let $C_1, C_2, \ldots, C_n$ be the collection of face-boundaries in the embedding ψ; since G is 3-connected and ψ is 3-representative, $C_1, C_2, \ldots, C_n$ are circuits and satisfy the hypothesis of (1.3). Let $D_1, D_2, \ldots, D_n$ be the disks as in (1.3). Then $\psi(G) \cup D_1 \cup D_2 \cup \ldots \cup D_n$ is homeomorphic to Σ, a contradiction because Σ has no embedding in S^3.

Thus G has a minor isomorphic to a member of the Petersen family, and so we may assume that it has a minor isomorphic to $K_{4,4}^-$ and to no other member of the Petersen family. Now it is easy to show, using the splitter theorem [20] of the second author, that G is isomorphic to $K_{4,4}^-$. But $K_{4,14}^-$has no 3-representative embedding in any nonorientable surface, and the theorem follows. □

Conversely, every member of the Petersen family except $K_{4,4}^-$ admits a 3-representative embedding in the projective plane.

5. REMARKS

It would be nice to have a structural description of all linklessly embeddable graphs. Let us say that a graph G has a *hamburger structure* if either $|V(G)| \leq 4$ or there are vertices $v_1, v_2, \ldots, v_5$ of G and three subgraphs G_1, G_2, G_3 of G such that

(i) $G_1 \cup G_2 \cup G_3 = G$,
(ii) $V(G_i) \cap V(G_j) = \{v_1, v_2, \ldots, v_5\}$ for $i \neq j$, and
(iii) each G_i can be embedded in a closed disk with vertices $v_1, v_2, \ldots, v_5$ (in this order) on the boundary.

It is not difficult to see that if G has a hamburger structure, then it has a flat embedding. We say that a graph G is *basically* 5*–connected* if G is simple, 3–connected, and cannot be expressed as a union of two subgraphs G_1 and G_2, where $E(G_1) \cap E(G_2) = \emptyset$, and either
(i) $|V(G_1) \cap V(G_2)| = 3$ and $|E(G_1)|, |E(G_2)| \geq 4$, or
(ii) $|V(G_1) \cap V(G_2)| = 4$ and $|E(G_1)|, |E(G_2)| \geq 7$.

(5.1) Conjecture. *Let G be a basically 5–connected, triangle–free linklessly embeddable graph. Then either there are two vertices u, v of G such that $G\backslash\{u,v\}$ is planar, or else G has a hamburger structure.*

From (4.1) we see that the requirement that G be triangle–free is not restrictive. One can also modify the definition of "hamburger structure" so that (5.1) could be true for all basically 5–connected linklessly embeddable graphs. The point of (5.1) is that if $G\backslash\{u,v\}$ is planar then there is a simple polynomial–time algorithm to test if G has a flat embedding. The algorithm is based on a study of homotopy of paths joining the neighbors of u and v in $G\backslash\{u,v\}$.

A second relevant conjecture is the following, due to Jorgensen [7].

(5.2) Conjecture. *Let G be a 6–connected graph with no minor isomorphic to K_6. Then G is nearly–planar.*

This was motivated by Hadwiger's conjecture [6]. One case of the latter states that every loopless graph with no minor isomorphic to K_6 is 5–colorable. Mader [9] showed that every minor-minimal counterexample G is 6–connected, in which case (5.2) and the Four Color Theorem would imply that G is 5–colorable, a contradiction. However, we believe that we have now obtained a proof of this case of Hadwiger's conjecture, without proving (5.2). We do not even know if (5.2) holds for linklessly embeddable graphs.

Our third conjecture relates linklessly embeddable graphs and a graph parameter $\mu(G)$ introduced by Colin de Verdière in [3]. We refer the reader to that paper for a definition of $\mu(G)$ (an English translation appears in this volume), which is in terms of the multiplicities of the second largest eigenvalues of certain matrices associated with G.

(5.3) Conjecture. *A graph G has a flat embedding if and only if $\mu(G) \leq 4$.*

The "if" part of (5.3) follows from our main result, and so the problem is about the converse.

Finally, let us mention two algorithmic aspects of flat embeddings. In [19] Scharlemann and Thompson describe an algorithm to test if a given embedding is spherical. Using their algorithm, (1.4) and (2.4), we can test if a given embedding is flat, by testing the flatness of all coforest minors. At the moment there is no known *polynomial-time* algorithm to test if an embedding of a given coforest is flat, because it includes testing if a given knot is trivial. On the other hand, we can test if a given graph G has a flat embedding in time $O(|V(G)|^3)$. This is done by testing the absence of minors isomorphic to members of the Petersen family, using (4.2) and the algorithm [11] of the first two authors.

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