We prove that if a graph has a tree-decomposition of width at most \( w \), then it has a tree-decomposition of width at most \( w \) with certain desirable properties. We will use this result in a subsequent paper to show that every 2-connected graph of large path-width has a minor isomorphic to either a large tree with a vertex attached to every vertex of the tree or a large outerplanar graph.

1 INTRODUCTION

All graphs in this paper are finite and simple; that is, they have no loops or parallel edges. Paths and cycles have no “repeated” vertices or edges. A graph \( H \) is a minor of a graph \( G \) if \( H \) can be obtained from a subgraph of \( G \) by contracting edges. An \( H \) minor is a minor isomorphic to \( H \). A tree-decomposition of a graph \( G \) is a pair \((T, X)\), where \( T \) is a tree and \( X \) is a family \((X_t : t \in V(T))\) such that:

(W1) \( \bigcup_{t \in V(T)} X_t = V(G) \), and for every edge of \( G \) with ends \( u \) and \( v \) there exists \( t \in V(T) \) such that \( u, v \in X_t \), and

(W2) if \( t_1, t_2, t_3 \in V(T) \) and \( t_2 \) lies on the path in \( T \) between \( t_1 \) and \( t_3 \), then \( X_{t_1} \cap X_{t_3} \subseteq X_{t_2} \).

The width of a tree-decomposition \((T, X)\) is \( \max\{|X_t| - 1 : t \in V(T)|\} \). The tree-width of a graph \( G \) is the least width of a tree-decomposition of \( G \). A path-decomposition of \( G \) is a tree-decomposition \((T, X)\) of \( G \) where \( T \) is a path. The path-width of \( G \) is the least width of a path-decomposition of \( G \). Robertson and Seymour [9] proved the following:

Theorem 1.1. For every planar graph \( H \) there exists an integer \( n = n(H) \) such that every graph of tree-width at least \( n \) has an \( H \) minor.

Robertson and Seymour [8] also proved an analogous result for path-width:

Theorem 1.2. For every forest \( F \), there exists an integer \( p = p(F) \) such that every graph of path-width at least \( p \) has an \( F \) minor.

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Bienstock, Robertson, Seymour and the second author [2] gave a simpler proof of Theorem 1.2 and improved the value of $p$ to $|V(F)| - 1$, which is best possible, because $K_k$ has path-width $k - 1$ and does not have any forest minor on $k + 1$ vertices. A yet simpler proof of Theorem 1.2 was found by Diestel [6].

Motivated by the possibility of extending Theorem 1.2 to matroids Seymour [5, Open Problem 2.1] asked if there was a generalization of Theorem 1.2 for 2-connected graphs with forests replaced by the two families of graphs mentioned in the abstract. In [4] we answer Seymour’s question in the affirmative:

**Theorem 1.3.** Let $P$ be a graph with a vertex $v$ such that $P - v$ is a forest, and let $Q$ be an outerplanar graph. Then there exists a number $p = p(P, Q)$ such that every 2-connected graph of path-width at least $p$ has a $P$ or $Q$ minor.

Theorem 1.3 is a generalization of Theorem 1.2. To deduce Theorem 1.2 from Theorem 1.3 given a graph $G$, we may assume that $G$ is connected, because the path-width of a graph is equal to the maximum path-width of its components. We add one vertex and make it adjacent to every vertex of $G$. Then the new graph is 2-connected, and by Theorem 1.3 it has a $P$ or $Q$ minor. By choosing suitable $P$ and $Q$, we can get an $F$ minor in $G$.

Our strategy to prove Theorem 1.3 is as follows. Let $G$ be a 2-connected graph of large path-width. We may assume that the tree-width of $G$ is bounded, for otherwise $G$ has a minor isomorphic to both $P$ and $Q$ by Theorem 1.1. So let $(T, X)$ be a tree-decomposition of $G$ of bounded width. Since the path-width of $G$ is large, it follows by a simple argument [4, Lemma 4.1] that the path-width of $T$ is large, and hence it has a subgraph $T'$ isomorphic to a subdivision of a large binary tree by Theorem 1.2. It now seems plausible that we could use $T'$ and properties (W3) and (W4) of tree-decompositions, introduced below, which we can assume by [7, 10], to show the desired conclusion. But there is a catch: for instance, a long cycle has a tree-decomposition $(T, X)$ satisfying (W3) and (W4) (and, in fact, the minimality condition used in their proof, as well as that of Bellenbaum and Diestel [11]) such that $T$ has a subgraph isomorphic to a large binary tree. And yet it feels that this is the “wrong” tree-decomposition and that the “right” tree-decomposition is one where $T$ is a path. The main result of this paper, Theorem 2.4 below, deals with converting these “branching” tree-decompositions into “non-branching” ones without increasing their width.

The paper is organized as follows. In the next section we review known results about tree-decompositions and state our main result, Theorem 2.4. In Section 3 we introduce a linear quasi-order on the class of finite trees and prove a key lemma—Lemma 3.5. In Section 4 we prove Theorem 2.4, which we restate as Theorem 4.8.

## 2 LINKED TREE-DECOMPOSITIONS

In this section we review properties of tree-decompositions established in [7, 10], and state our main result. The proof of the following easy lemma can be found, for instance, in [10].
Lemma 2.1. Let \((T, Y)\) be a tree-decomposition of a graph \(G\), and let \(H\) be a connected subgraph of \(G\) such that \(V(H) \cap Y_t \neq \emptyset \neq V(H) \cap Y_{t'}\), where \(t, t' \in V(T)\). Then \(V(H) \cap Y_t \neq \emptyset\) for every \(t \in V(T)\) on the path between \(t_1\) and \(t_2\) in \(T\).

A tree-decomposition \((T, Y)\) of a graph \(G\) is said to be linked if

(W3) for every two vertices \(t_1, t_2\) of \(T\) and every positive integer \(k\), either there are \(k\) disjoint paths in \(G\) between \(Y_{t_1}\) and \(Y_{t_2}\), or there is a vertex \(t\) of \(T\) on the path between \(t_1\) and \(t_2\) such that \(|Y_t| < k|\).

It is worth noting that, by Lemma 2.1, the two alternatives in (W3) are mutually exclusive. The following is proved in \([10]\).

Lemma 2.2. If a graph \(G\) admits a tree-decomposition of width at most \(w\), where \(w\) is some integer, then \(G\) admits a linked tree-decomposition of width at most \(w\).

Let \((T, Y)\) be a tree-decomposition of a graph \(G\), let \(t_0 \in V(T)\), and let \(B\) be a component of \(T \setminus t_0\). We say that a vertex \(v \in Y_{t_0}\) is \(B\)-tied if \(v \in Y_t\) for some \(t \in V(B)\). We say that a path \(P\) in \(G\) is \(B\)-confined if \(|V(P)| \geq 3\) and every internal vertex of \(P\) belongs to \(\bigcup_{t \in V(B)} Y_t - Y_{t_0}\). We wish to consider the following three properties of \((T, Y)\):

(W4) if \(t, t'\) are distinct vertices of \(T\), then \(Y_t \neq Y_{t'}\),

(W5) if \(t_0 \in V(T)\) and \(B\) is a component of \(T \setminus t_0\), then \(\bigcup_{t \in V(B)} Y_t - Y_{t_0} \neq \emptyset\),

(W6) if \(t_0 \in V(T)\), \(B\) is a component of \(T \setminus t_0\), and \(u, v\) are \(B\)-tied vertices in \(Y_{t_0}\), then there is a \(B\)-confined path in \(G\) between \(u\) and \(v\).

The following strengthening of Lemma 2.2 is proved in \([7]\).

Lemma 2.3. If a graph \(G\) has a tree-decomposition of width at most \(w\), where \(w\) is some integer, then it has a tree-decomposition of width at most \(w\) satisfying (W1)–(W6).

We need one more condition, which we now introduce. Let \(T\) be a tree. If \(t_1, t_2 \in V(T)\), then by \(t_1Tt_2\) we denote the vertex-set of the unique path in \(T\) with ends \(t_1\) and \(t_2\). A triad in \(T\) is a triple \(t_1, t_2, t_3\) of vertices of \(T\) such that there exists a vertex \(t\) of \(T\), called the center, such that \(t_1, t_2, t_3\) belong to different components of \(T \setminus t\). Let \((T, W)\) be a tree-decomposition of a graph \(G\), and let \(t_1, t_2, t_3\) be a triad in \(T\) with center \(t_0\). The torso of \((T, W)\) at \(t_1, t_2, t_3\) is the subgraph of \(G\) induced by the set \(\bigcup W_t\), the union taken over all vertices \(t \in V(T)\) such that either \(t \in \{t_1, t_2, t_3\}\), or for all \(i \in \{1, 2, 3\}\), the vertex \(t\) belongs to the component of \(T \setminus t_i\) containing \(t_0\). We say that the triad \(t_1, t_2, t_3\) is \(W\)-separable if, letting \(X = W_{t_1} \cap W_{t_2} \cap W_{t_3}\), the graph obtained from the torso of \((T, W)\) at \(t_1, t_2, t_3\) by deleting \(X\) can be partitioned into three disjoint non-null graphs \(H_1, H_2, H_3\) in such a way that for all distinct \(i, j \in \{1, 2, 3\}\) and all \(t \in t_jT_{t_0}\), \(|V(H_i) \cap W_t| \geq |V(H_j) \cap W_t| = |W_t| - X|/2 \geq 1\). (Let us remark that this condition implies that \(|W_{t_1}| = |W_{t_2}| = |W_{t_3}|\) and \(V(H_i) \cap W_{t_i} = \emptyset\) for \(i = 1, 2, 3\).) The last property of a tree-decomposition \((T, W)\) that we wish to consider is
(W7) if \( t_1, t_2, t_3 \) is a \( W \)-separable triad in \( T \) with center \( t \), then there exists an integer \( i \in \{1, 2, 3\} \) with \( W_{t_i} \cap W_t - (W_{t_1} \cap W_{t_2} \cap W_{t_3}) \neq \emptyset \).

The following is our main result.

**Theorem 2.4.** If a graph \( G \) has a tree-decomposition of width at most \( w \), where \( w \) is some integer, then it has a tree-decomposition of width at most \( w \) satisfying (W1)–(W7).

### 3 A QUASI-ORDER ON TREES

A quasi-ordered set is a pair \((Q, \leq)\), where \( Q \) is a set and \( \leq \) is a quasi-order; that is, a reflexive and transitive relation on \( Q \). If \( q, q' \in Q \) we define \( q < q' \) to mean that \( q \leq q' \) and \( q' \not\leq q \). We say that \( q, q' \) are \( \leq \)-equivalent if \( q \leq q' \leq q \). We say that \((Q, \leq)\) is a linear quasi-order if for every two elements \( q, q' \in Q \) either \( q \leq q' \) or \( q' \leq q \) or both. Let \((Q, \leq)\) be a linear quasi-order. If \( A, B \subseteq Q \) we say that \( B \leq\)-dominates \( A \) if the elements of \( A \) can be listed as \( a_1 \geq a_2 \geq \cdots \geq a_k \) and the elements of \( B \) can be listed as \( b_1 \geq b_2 \geq \cdots \geq b_l \) and there exists an integer \( p \) with \( 1 \leq p \leq \min\{k, l\} \) such that \( a_i \leq b_i \leq a_i \) for all \( i = 1, 2, \ldots, p \), and either \( p < \min\{k, l\} \) and \( a_{p+1} < b_{p+1} \), or \( p = k \) and \( k \leq l \).

**Lemma 3.1.** If \((Q, \leq)\) is a linear quasi-order, then \( \leq \)-domination is a linear quasi-order on the set of subsets of \( Q \).

**Proof.** It is obvious that \( \leq \)-domination is reflexive. Assume that \( B \leq\)-dominates \( A \) and \( C \leq\)-dominates \( B \). Assume that the elements of \( A \) can be listed as \( a_1 \geq a_2 \geq \cdots \geq a_k \), the elements of \( B \) can be listed as \( b_1 \geq b_2 \geq \cdots \geq b_l \), and the elements of \( C \) can be listed as \( c_1 \geq c_2 \geq \cdots \geq c_m \). By definition, there exists an integer \( p_1 \) with \( 1 \leq p_1 \leq \min\{k, l\} \) such that \( a_i \leq b_i \leq a_i \) for all \( i = 1, 2, \ldots, p_1 \), and either \( p_1 < \min\{k, l\} \) and \( a_{p_1+1} < b_{p_1+1} \), or \( p_1 = k \leq l \); and there exists an integer \( p_2 \) with \( 1 \leq p_2 \leq \min\{l, m\} \) such that \( b_i \leq c_i \leq b_i \) for all \( i = 1, 2, \ldots, p_2 \), and either \( p_2 < \min\{l, m\} \) and \( b_{p_2+1} < c_{p_2+1} \), or \( p_2 = l \leq m \). Let \( p = \min\{p_1, p_2\} \). Then \( a_i \leq c_i \leq a_i \) for all \( i = 1, 2, \ldots, p \). If either \( p_1 < \min\{k, l\} \) and \( a_{p_1+1} < b_{p_1+1} \), or \( p_2 < \min\{l, m\} \) and \( b_{p_2+1} < c_{p_2+1} \), then \( p < \min\{k, m\} \) and \( a_{p_1+1} < c_{p_2+1} \). If \( p_1 = k \leq l \) and \( p_2 = l \leq m \), then \( p = k \leq m \). Therefore, \( C \leq\)-dominates \( A \), and so \( \leq \)-domination is transitive.

Now let \( A, B \) be as above, and let \( p \) be the maximum integer such that \( p \leq \min\{k, l\} \) and \( a_i \leq b_i \leq a_i \) for all \( i = 1, 2, \ldots, p \). Then if \( p < \min\{k, l\} \), then \( A \leq\)-dominates \( B \) if \( a_{p+1} > b_{p+1} \) and \( B \leq\)-dominates \( A \) if \( a_{p+1} < b_{p+1} \). If \( p = \min\{k, l\} \) then \( A \leq\)-dominates \( B \) if \( k \geq l \) and \( B \leq\)-dominates \( A \) if \( k \leq l \). Hence, \( \leq \)-domination is linear.

We say that \( B \) strictly \( \leq \)-dominates \( A \) if \( B \leq\)-dominates \( A \) in such a way that the numberings and integer \( p \) can be chosen in such a way that either \( p < \min\{k, l\} \), or \( p = k \) and \( k < l \).

**Lemma 3.2.** Let \((Q, \leq)\) be a linear quasi-order, let \( A, B \subseteq Q \), and let \( B \leq\)-dominate \( A \). Then \( B \) strictly \( \leq \)-dominates \( A \) if and only if \( A \) does not \( \leq \)-dominate \( B \).
Proof. Let \( p \) be as in the definition of \( B \leq \)-dominates \( A \). Then \( p < \min\{k, l\} \) and \( a_{p+1} < b_{p+1} \), or \( p = k \leq l \). Assume \( B \) strictly \( \leq \)-dominates \( A \). If \( p < \min\{k, l\} \) then \( a_{p+1} < b_{p+1} \), so \( A \) does not \( \leq \)-dominate \( B \). If \( p = k < l \) then \( A \) also does not \( \leq \)-dominate \( B \). Conversely, if \( A \) does not \( \leq \)-dominate \( B \), then \( p < \min\{k, l\} \) or \( k < l \), so \( B \) strictly \( \leq \)-dominates \( A \).

Let \( G \) be a graph and let \( P \) be a subgraph of \( G \). By a \( P \)-bridge of \( G \) we mean a subgraph \( J \) of \( G \) such that either

- \( J \) is isomorphic to the complete graph on two vertices with \( V(J) \subseteq V(P) \) and \( E(J) \cap E(P) = \emptyset \), or
- \( J \) consists of a component of \( G - V(P) \) together with all edges from that component to \( P \).

We now define a linear quasi-order \( \leq \) on the class of finite trees as follows. Let \( n \geq 1 \) be an integer, and suppose that \( T \leq T' \) has been defined for all trees \( T \) on fewer than \( n \) vertices. Let \( T \) be a tree on \( n \) vertices, and let \( T' \) be an arbitrary tree. We define \( T \leq T' \) if either \( |V(T)| < |V(T')| \), or \( |V(T)| = |V(T')| \) and for every maximal path \( P' \) of \( T' \) there exists a maximal path \( P \) of \( T \) such that the set of \( P' \)-bridges of \( T' \) \( \leq \)-dominates the set of \( P \)-bridges of \( T \). It follows from Lemma 3.3 below that \( \leq \) is indeed a linear quasi-order; in particular, it is well-defined.

If \( T, T' \) are trees, \( P \) is a path in \( T \) and \( P' \) is a path in \( T' \) we define \((T, P) \leq (T', P')\) if either \( |V(T)| < |V(T')| \), or \( |V(T)| = |V(T')| \) and the set of \( P' \)-bridges of \( T' \) \( \leq \)-dominates the set of \( P \)-bridges of \( T \).

**Lemma 3.3.** (i) For every tree \( T \) there exists a maximal path \( P(T) \) in \( T \) such that \((T, P(T)) \leq (T, P)\) for every maximal path \( P \) in \( T \).
(ii) For every two trees \( T, T' \), we have \( T \leq T' \) if and only if \((T, P(T)) \leq (T', P(T'))\).
(iii) The ordering \( \leq \) is a linear quasi-order on the class of finite trees.

**Proof.** We prove all three statements simultaneously by induction. Let \( n \geq 1 \) be an integer, assume inductively that all three statements have been proven for trees on fewer than \( n \) vertices, and let \( T \) be a tree on \( n \) vertices.

(i) Statement (i) clearly holds for one-vertex trees, and so we may assume that \( n \geq 2 \). Let \( B \) be the set of all \( P \)-bridges of \( T \) for all maximal paths \( P \) of \( T \). Then every member of \( B \) has fewer than \( n \) vertices, and hence \( B \) is a linear quasi-order by \( \leq \) by the induction hypothesis applied to (iii). By Lemma 3.1 the set of subsets of \( B \) is linearly quasi-ordered by \( \leq \)-domination. It follows that there exists a maximal path \( P(T) \) in \( T \) such that the set of \( P(T) \)-bridges of \( T \) is minimal under \( \leq \)-domination.

(ii) The statement is obvious when \( |V(T)| \neq |V(T')| \), so assume \( n = |V(T)| = |V(T')| \), and let \( B \) be the set of all \( P \)-bridges of \( T \) for all maximal paths \( P \) of \( T \) and the set of all \( P' \)-bridges of \( T' \) for all maximal paths \( P' \) of \( T' \). Then as in (i) the subsets of \( B \) are linearly quasi-ordered by \( \leq \)-domination. If \( T \leq T' \), then by definition there exists a maximal path \( P \) of \( T \) such that \((T, P) \leq (T', P(T'))\). Hence \((T, P(T)) \leq (T', P(T'))\) follows from (i). If \((T, P(T)) \leq (T', P(T'))\), then by (i) \((T, P(T)) \leq (T', P')\) for every maximal path \( P' \)
in $T'$, so $T \leq T'$.

(iii) Let $T$ and $T'$ be two trees. We may assume that $n = |V(T)| = |V(T')|$. Let $B$ be as in (ii); then subsets of $B$ are linearly quasi-ordered by $\leq$-domination. Then either $(T, P(T)) \preceq (T', P(T'))$ or $(T', P(T')) \preceq (T, P(T))$, and so by (ii) $\leq$ is linear.

For a tree $T$, the path $P(T)$ from Lemma 3.3(i) will be called a spine of $T$. For later application we need the following lemma.

**Lemma 3.4.** Let $T, T'$ be trees on the same number of vertices, let $P'$ be a spine of $T'$, and let $P$ be a path in $T$. If the set of $P'$-bridges of $T'$ strictly $\leq$-dominates the set of $P$-bridges of $T$, then $T < T'$.

**Proof.** We have $(T, P) \preceq (T', P')$ and $(T', P') \not\preceq (T, P)$ by Lemma 3.2. Let $P_1$ be a maximal path that contains $P$; then $(T, P_1) \preceq (T, P)$. Therefore, $(T, P_1) \preceq (T', P')$ and $(T', P') \not\preceq (T, P_1)$. By Lemma 3.3(ii) $(T, P(T)) \preceq (T, P_1) \preceq (T', P')$ and $(T', P') \not\preceq (T, P(T))$. By Lemma 3.3(ii), $T \preceq T'$ and $T' \not\preceq T$. Therefore, $T < T'$.

By a rank we mean a class of $\leq$-equivalent trees. If $r$ is a rank we say that $T$ has rank $r$ or that the rank of $T$ is $r$ if $T \in r$. The class of all ranks will be denoted by $\mathcal{R}$.

Let $T$ be a tree, and let $t$ be a vertex of $T$. By a spine-decomposition of $T$ relative to $t$ we mean a sequence $(T_0, P_0, T_1, P_1, \ldots, T_l, P_l)$ such that

(i) $T_0 = T$,

(ii) for $i = 0, 1, \ldots, l$, $P_i$ is a spine of $T_i$, and

(iii) for $i = 1, 2, \ldots, l$, $t \notin V(P_{i-1})$ and $T_i$ is the $P_{i-1}$-bridge of $T_{i-1}$ containing $t$.

**Lemma 3.5.** Let $T$ be a tree, let $t$ be a vertex of $T$ of degree three with neighbors $t'_1, t'_2, t'_3$, and let $(T_0, P_0, T_1, P_1, \ldots, T_l, P_l)$ be a spine-decomposition of $T$ relative to $t$ with $t \in V(P_l)$. Then exactly two of $t'_1, t'_2, t'_3$ belong to $V(P_l)$, say $t'_1$ and $t'_2$. Let $r_3, r'_3$ be adjacent vertices of $T$ such that $r_3, r'_3, t'_3, t$ occur on a path of $T$ in the order listed. Thus possibly $t'_3 = r'_3$, but $t'_3 \neq r_3$. Let $T'$ be obtained from $T$ by subdividing the edge $r_3 r'_3$ twice (let $r''_3, r'''_3$ be the new vertices so that $r'_3, r''_3, r'''_3, r_3$ occur on a path of $T'$ in the order listed), deleting the edge $tt'_1$, contracting the edges $tt'_2$ and $tt'_3$ and adding an edge joining $t'_1$ and $r'''_3$. Then $T'$ has strictly smaller rank than $T$.

**Proof.** Let $T'_0 = T'$ and for $i = 1, 2, \ldots, l$, let $T'_i$ be the $P_{i-1}$-bridge of $T'_{i-1}$ containing $r''_3$. Let $P'$ be the unique maximal path in $T'$ with $V(P_l) - \{t, t'_2\} \cup \{r''_3\} \subseteq V(P')$. From the definition of a spine-decomposition and the fact that $t'_3 \notin V(P_l)$ we deduce that $r_3 \in V(T_i)$ for all $i = 0, 1, \ldots, l$. It follows that $r_3 \in V(T'_l)$ and $|V(T'_i)| = |V(T'_l)|$ for all $i = 0, 1, \ldots, l$. The $P_l$-bridge of $T_i$ that contains $r_3$ is replaced by $P'$-bridges of $T'_l$ with smaller cardinalities. Other $P_l$-bridges of $T_i$ are unchanged in $T'$. Therefore, the set of $P_l$-bridges of $T_i$ strictly $\leq$-dominates the set of $P'$-bridges of $T'_i$, and hence $T'_i < T_i$ by Lemma 3.4. This implies, by induction on $l - i$ using Lemma 3.4, that $T'_i < T_i$ for all $i = 0, 1, \ldots, l$; that is, $T'$ has smaller rank than $T$. □
4 A THEOREM ABOUT TREE-DECOMPOSITIONS

Let \((T, Y)\) be a tree-decomposition of a graph \(G\), let \(n\) be an integer, and let \(r\) be a rank. By an \((n, r)\)-cell in \((T, Y)\) we mean any component of the restriction of \(T\) to \(\{t \in V(T) : |Y_t| \geq n\}\) that has rank at least \(r\). Let us remark that if \(K\) is an \((n, r)\)-cell in \((T, Y)\) and \(r \geq r'\), then \(K\) is an \((n, r')\)-cell as well. The size of a tree-decomposition \((T, Y)\) is the family of numbers

\[
(a_{n, r} : n \geq 0, r \in \mathcal{R}),
\]

where \(a_{n, r}\) is the number of \((n, r)\)-cells in \((T, Y)\). Sizes are ordered lexicographically; that is, if

\[
(b_{n, r} : n \geq 0, r \in \mathcal{R})
\]

is the size of another tree-decomposition \((R, Z)\) of the graph \(G\), we say that (2) is smaller than (1) if there are an integer \(n \geq 0\) and a rank \(r \in \mathcal{R}\) such that \(a_{n, r} > b_{n, r}\) and \(a_{n', r'} = b_{n', r'}\) whenever either \(n' > n\), or \(n' = n\) and \(r' > r\).

Lemma 4.1. The relation “to be smaller than” is a well-ordering on the set of sizes of tree-decompositions of \(G\).

Proof. Since this ordering is clearly linear, it is enough to show that it is well-founded. Suppose for a contradiction that \(\{(a_{n, r}^{(i)} : n \geq 0, r \in \mathcal{R})\}_{i=1}^{\infty}\) is a strictly decreasing sequence of sizes, and for \(i = 1, 2, \ldots\), let \(n_i, r_i\) be such that \(a_{n_i, r_i}^{(i)} > a_{n_i, r_i}^{(i+1)}\) and \(a_{n, r}^{(i)} = a_{n, r}^{(i+1)}\) for \((n, r)\) such that either \(n > n_i\), or \(n = n_i\) and \(r > r_i\). Since \(a_{n, r}^{(1)} = 0\) for all \(r \in \mathcal{R}\) and all \(n > |V(G)|\), we may assume (by taking a suitable subsequence) that \(n_1 = n_2 = \cdots\), and that \(r_1 \leq r_2 \leq r_3 \leq \cdots\). Since clearly \(a_{n, r}^{(i)} \geq a_{n, r}^{(i)}\) for all \(n \geq 0\), all \(r \leq r'\) and all \(i = 1, 2, \ldots\), we have

\[
a_{n_1, r_1}^{(1)} > a_{n_2, r_2}^{(2)} \geq a_{n_2, r_2}^{(3)} \geq a_{n_3, r_3}^{(3)} > \cdots,
\]

a contradiction. \(\square\)

We say that a tree-decomposition \((T, W)\) of a graph \(G\) is minimal if there is no tree-decomposition of \(G\) of smaller size.

Lemma 4.2. Let \(w\) be an integer, and let \(G\) be a graph of tree-width at most \(w\). Then a minimal tree-decomposition of \(G\) exists, and every minimal tree-decomposition of \(G\) has width at most \(w\).

Proof. The existence of a minimal tree-decomposition follows from Lemma 4.1. If \(G\) has a tree-decomposition of width at most \(w\), then every minimal tree-decomposition has width at most \(w\), as desired. \(\square\)

Theorem 4.3. Let \((T, W)\) be a minimal tree-decomposition of a graph \(G\). Then \((T, W)\) satisfies (W1)–(W6).

Proof. That \((T, W)\) satisfies (W3) is shown in [10], and that it satisfies (W4), (W5) and (W6) is shown in [7]. Let us remark that [7] and [10] use a slightly different definition of minimality, but the proofs are adequate, because a minimal tree-decomposition in our sense is minimal in the sense of [7] and [10] as well. \(\square\)
Lemma 4.4. Let \((T, W)\) be a minimal tree-decomposition of a graph \(G\). Then for every edge \(tt' \in E(T)\) either \(W_t \subseteq W_{t'}\) or \(W_{t'} \subseteq W_t\).

Proof. Assume for a contradiction that there exists an edge \(tt' \in E(T)\) such that \(W_t \not\subseteq W_{t'}\) and \(W_{t'} \not\subseteq W_t\). Let \(R\) be obtained from \(T\) by subdividing the edge \(tt'\) and let \(t''\) be the new vertex. Let \(Y_r = W_t \cap W_{t'}\) and \(Y_r = W_r\) for all \(r \in V(T)\), and let \(Y = (Y_r : r \in V(R))\). Then \((R, Y)\) is a tree-decomposition of \(G\) of smaller size than \((T, W)\), contrary to the minimality of \((T, W)\). \(\square\)

Lemma 4.5. Let \((T, W)\) be a minimal tree-decomposition of a graph \(G\), let \(t \in V(T)\), let \(X \subseteq W_t\), let \(B\) be a component of \(T\setminus t\), let \(t'\) be the neighbor of \(t\) in \(B\), let \(Y = \bigcup_{r \in V(B)} W_r\), and let \(H\) be the subgraph of \(G\) induced by \(Y \cup W_t\). If \(H \setminus X = H_1 \cup H_2\), where \(V(H_1) \cap V(H_2) = \emptyset\) and both of \(V(H_1), V(H_2)\) intersect \(W_t\), then either \(W_{t'} - X \subseteq W_t \cap V(H_1)\) or \(W_{t'} - X \subseteq W_t \cap V(H_2)\).

Proof. We first prove the following claim.

Claim 4.5.1. Either \(W_t \cap W_{t'} - X \subseteq V(H_1)\) or \(W_t \cap W_{t'} - X \subseteq V(H_2)\).

To prove the claim suppose for a contradiction that there exist vertices \(v_1 \in W_t \cap W_{t'} \cap V(H_1)\) and \(v_2 \in W_t \cap W_{t'} \cap V(H_2)\). Thus both \(v_1\) and \(v_2\) are \(B\)-tied, and so by (W6), which \((T, W)\) satisfies by Theorem 4.3, there exists a \(B\)-confined path \(Q\) with ends \(v_1\) and \(v_2\). Since \(Q\) is \(B\)-confined, it is a subgraph of \(H \setminus X\), contrary to the fact that \(V(H_1) \cap V(H_2) = \emptyset\) and \(H_1 \cup H_2 = H \setminus X\). This proves Claim 4.5.1.

Since both of \(V(H_1), V(H_2)\) intersect \(W_t\), Claim 4.5.1 implies that \(W_t \not\subseteq W_{t'}\), and hence \(W_{t'} \subseteq W_t\) by Lemma 4.4. By another application of Claim 4.5.1 we deduce that either \(W_{t'} - X \subseteq W_t \cap V(H_1)\) or \(W_{t'} - X \subseteq W_t \cap V(H_2)\), as desired. \(\square\)

Lemma 4.6. Let \(k \geq 1\) be an integer, let \((T, W)\) be a minimal tree-decomposition of a graph \(G\), let \(t_1, t_2 \in V(T)\), let \(X = W_{t_1} \cap W_{t_2}\), let \(H\) be the subgraph of \(G\) induced by \(\bigcup W_t\), the union taken over all vertices \(t \in V(T)\) such that either \(t \in \{t_1, t_2\}\), or for \(i = 1, 2\) the vertex \(t\) belongs to the component of \(T\setminus t\) containing \(t_{3-i}\), let \(H \setminus X = H_1 \cup H_2\), where \(V(H_1) \cap V(H_2) = \emptyset\), and assume that \(|W_{t_i} \cap V(H_j)| = k\) and \(|W_{t_i} \cap V(H_i)| \geq k\) for all \(i, j \in \{1, 2\}\) and all \(t \in t_1 T t_2\). Let \(t, t'\) be two adjacent vertices on the path of \(T\) between \(t_1\) and \(t_2\). Then there exists an integer \(i \in \{1, 2\}\) such that \(W_{t_i} \cap V(H_i) = W_{t'} \cap V(H_i)\) and this set has cardinality \(k\).

Proof. We begin with the following claim.

Claim 4.6.1. For every \(t \in t_1 T t_2\) either \(|W_{t_1} \cap V(H_1)| = k\) or \(|W_{t_1} \cap V(H_2)| = k\).

To prove the claim let \(R\) be the subtre of \(T\) induced by vertices \(r \in V(T)\) such that either \(r \in \{t_1, t_2\}\) or \(r\) belongs to the component of \(T \setminus \{t_1, t_2\}\) that contains neighbors of both \(t_1\) and \(t_2\), let \(R_1, R_2\) be two isomorphic copies of \(R\), and for \(r \in V(R)\) let \(r_1\) and \(r_2\) denote the copies of \(r\) in \(R_1\) and \(R_2\), respectively. Assume for a contradiction that there is \(t_0 \in t_1 T t_2\) such that \(|W_{t_0} \cap V(H_i)| > k\) for all \(i \in \{1, 2\}\), and choose such a vertex with \(t_0 \in V(R)\) and \(|W_{t_0}|\) maximum. We construct a new tree-decomposition \((T', W')\) as
follows. The tree $T'$ is obtained from the disjoint union of $T \setminus (V(R) - \{t_1, t_2\})$, $R_1$ and $R_2$ by identifying $t_1$ with $(t_1)_1$, $(t_2)_1$ with $(t_1)_2$ and $(t_2)_2$ with $t_2$ (here $(t_1)_2$ denotes the copy of $t_1$ in $R_2$ and similarly for the other three quantities). The family $W' = (W'_t : t \in V(T'))$ is defined as follows:

$$W'_t = \begin{cases} 
W_t & \text{if } t \in V(T) - V(R) \\
(W_r \cap V(H_1)) \cup (W_{t_1} \cap V(H_2)) \cup X & \text{if } t = r_1 \text{ for } r \in t_1 T t_2 \\
(W_r \cap V(H_2)) \cup (W_{t_2} \cap V(H_1)) \cup X & \text{if } t = r_2 \text{ for } r \in t_1 T t_2 \\
W_r \cap V(H_1) & \text{if } t = r_1 \text{ for } r \in V(R) - t_1 T t_2 \\
W_r \cap V(H_2) & \text{if } t = r_2 \text{ for } r \in V(R) - t_1 T t_2
\end{cases}$$

Please note that the value of $W'_t$ is the same for $t = (t_2)_1$ and $t = (t_1)_2$, and hence $W'$ is well-defined. Since no edge of $G$ has one end in $V(H_1)$ and the other end in $V(H_2)$, it follows that $(T', W')$ is a tree-decomposition of $G$.

We claim that the size of $(T', W')$ is smaller than the size of $(T, W)$. Indeed, let $n_0 = |W_{t_0}|$, and let $Z = \{ t \in V(T') : |W'_t| \geq n_0 \}$. Then $n_0 > 2k + |X|$. We define a mapping $f : Z \to V(T)$ by $f(t) = t$ for $t \in Z - V(R_1) - V(R_2)$, $f(r_1) = r$ for $r \in V(R)$ such that $r_1 \in Z$ and $f(r_2) = r$ for $r \in V(R)$ such that $r_2 \in Z$. We remark that the vertex obtained by identifying $(t_2)_1$ with $(t_1)_2$ does not belong to $Z$, and hence there is no ambiguity. Then $Z$ and $f$ have the following properties:

- $|W_{f(t)}| \geq |W'_t|$ for every $t \in Z$,
- for $r \in V(R)$, at most one of $r_1, r_2$ belongs to $Z$, and
- $(t_0)_1, (t_0)_2 \notin Z$

These properties follow from the assumptions that $|W_t \cap V(H_j)| = k$ and $|W_t \cap V(H_i)| \geq k$ for all $i, j \in \{1, 2\}$ and all $t \in t_1 T t_2$. (To see the second property assume for a contradiction that for some $r \in V(R)$ both $r_1$ and $r_2$ belong to $Z$. Then $n_0 = |W_{t_0}| \geq |W_{f(r_1)}| \geq |W_{t_0}| \geq n_0$, by the maximality of $|W_{t_0}|$ and the first property, and so equality holds throughout, contrary to the construction.) It follows from the first two properties that $f$ maps injectively $(n, r)$-cells in $(T', W')$ to $(n, r)$-cells in $(T, W)$ for all $n \geq n_0$ and all ranks $r$. On the other hand, the third property implies that, letting $r_1$ denote the rank of one-vertex trees, no $(n_0, r_1)$-cell in $(T', W')$ is mapped onto the $(n_0, r_1)$-cell in $(T, W)$ with vertex-set $\{t_0\}$. Thus the size of $(T', W')$ is smaller than the size of $(T, W)$, contrary to the minimality of $(T, W)$. This proves Claim 4.6.1

Now let $t, t' \in t_1 T t_2$ be adjacent. By Lemma 4.4 we may assume that $W_t \subseteq W_{t'}$. Then $W_t \cap V(H_1) \subseteq W_{t'} \cap V(H_1)$ and $W_t \cap V(H_2) \subseteq W_{t'} \cap V(H_2)$. By Claim 4.6.1 we may assume that $|W_{t'} \cap V(H_1)| = k$. Given that $|W_t \cap V(H_1)| \geq k$ we have $W_t \cap V(H_1) = W_{t'} \cap V(H_1)$ and this set has cardinality $k$, as desired.

**Lemma 4.7.** Let $(T, W)$ be a minimal tree-decomposition of a graph $G$, let $t_1, t_2, t_3$ be a $W$-separable triad in $T$ with center $t_0$, and let $X, H, H_1, H_2$ and $H_3$ be as in the definition of $W$-separable triad. Let $k = |W_{t_1} - X|/2$ and for $i = 1, 2, 3$ let $t'_i$ denote the neighbor of $t_0$
in the component of $T \setminus t_0$ containing $t_i$. Then for all distinct $i, j \in \{1, 2, 3\}$, $V(H_i) \cap W'_{t_j} = V(H_i) \cap W_{t_0}$, and this set has cardinality $k$.

Proof. Let $X_3 = \bigcup W_t$, the union taken over all $t \in V(T)$ that do not belong to the component of $T \setminus t_3$ containing $t_0$. Since $|W_{t_0} \cap V(H_1)| \geq k$ and $|W_{t_0} \cap V(H_2)| \geq k$ by the definition of $W$-separable triad, by Lemma 4.6 applied to $t_1, t_2, H_3$ and the subgraph of $G$ induced by $V(H_1) \cup V(H_2) \cup X_3$ we deduce that $V(H_3) \cap W_{t_0} = V(H_3) \cap W'_{t_1} = V(H_3) \cap W'_{t_2}$, and this set has cardinality $k$. Similarly we deduce that $V(H_2) \cap W_{t_0} = V(H_2) \cap W'_{t_1} = V(H_2) \cap W'_{t_2}$ and $V(H_1) \cap W_{t_0} = V(H_1) \cap W'_{t_2} = V(H_1) \cap W'_{t_3}$, and that the latter two sets also have cardinality $k$.

We are finally ready to prove Theorem 4.8, which, by Lemma 4.2 is implied by the following theorem.

Theorem 4.8. Let $(T, W)$ be a minimal tree-decomposition of a graph $G$. Then $(T, W)$ satisfies (W1)–(W7).

Proof. That $(T, W)$ satisfies (W1)–(W6) follows from Theorem 4.3. Thus it remains to show that $(T, W)$ satisfies (W7). Suppose for a contradiction that $(T, W)$ does not satisfy (W7), and let $t_1, t_2, t_3$ be a $W$-separable triad in $T$ with center $t_0$ such that $W_{t_0} \subseteq X$ for every $i = 1, 2, 3$, where $X = W_{t_1} \cap W_{t_2} \cap W_{t_3}$. Let $H, H_1, H_2$ and $H_3$ be as in the definition of $W$-separable triad, and for $i \in \{1, 2, 3\}$ let $t'_i$ denote the neighbor of $t_0$ in the component of $T \setminus t_0$ containing $t_i$.

Let $n := |W_{t_1}|$, let $k := |W_{t_1} - X|/2$, and let $r_1$ be the rank of 1-vertex trees, and let $T_0$ denote the $(n, r_1)$-cell containing $t_0$. By the definition of $W$-separable triad we have $|W'_{t_i}| \geq n$ for all $i \in \{1, 2, 3\}$, and hence the degree of $t_0$ in $T_0$ is at least three and by Lemmas 4.7 and 4.9 it is at most three.

Let $(T_0, P_0, T_1, P_1, \ldots, T_l, P_l)$ be a spine-decomposition of $T_0$ relative to $t_0$ with $t_0 \in V(P_l)$. Since $P_l$ is a maximal path in $T_1$ we may assume that $t'_1, t'_2 \in V(P_l)$ and $t'_3 \notin V(P_l)$.

It follows from Lemma 4.7 that $W_{t_3} \cap W'_{t'_3} = X$. By Lemma 4.6 applied to $t_3$ and $t'_3$ and $t'_3$ and its neighbor in $t_3T'_{t_3}$ we deduce that there exists a vertex $r_3 \in t_3T'_{t_3} - \{t'_3\}$ such that either $V(H_1) \cap W_{r_3} = V(H_1) \cap W_r$ for every $r \in r_3T'_{t_3}$, or $V(H_2) \cap W_{r_3} = V(H_2) \cap W_r$ for every $r \in r_3T'_{t_3}$. Without loss of generality we may assume the latter. We may choose $r_3$ to be as close to $t_3$ as possible. The fact that $W_{t_3} \cap W_{r_3} = X$ implies that $r_3 \neq t_3$. By another application of Lemma 4.6 this time to $t_3, t'_3, r_3$ and the neighbor of $r_3$ in $r_3T'_{t_3}$, we deduce that $|V(H_1) \cap W_1| = |V(H_2) \cap W_3| = k$.

Let $r'_3$ be the neighbor of $r_3$ in $r_3T'_{t_0}$ and let the tree $T''$ be defined as follows: for every component $B$ of $T \setminus t_0T'_{t_3}$ not containing $t_1, t_2$ or $t_3$ let $r(B)r'(B)$ denote the edge connecting $B$ to $t_0T'_{t_3}$, where $r(B) \in V(B)$ and $r'(B) \in t_0T'_{r_3}$. By Lemma 4.5 there exists an integer $i \in \{1, 2, 3\}$ such that $W_{r'(B)} \subseteq W_{r'(B)} \cap V(H_i)$. Let us mention in passing that this, the choice of $r_2$ and Lemma 4.7 imply that for every such component $B$, every $(n, r_1)$-cell is either a subgraph of $B$ or is disjoint from $B$. The tree $T''$ is obtained from $T$ by, for every such component $B$ for which either $i = 2$, or $i = 3$ and $r'(B) = t_0$, deleting the edge $r(B)r'(B)$ and adding the edge $t'_1r(B)$; and for every such component $B$ for which $i = 1$ and $r'(B) = t_0$ deleting the edge $r(B)r'(B)$ and adding the
edge \( t'_2 r (B) \). Since \( W_{r' (B)} \cap (V (H_2) \cup V (H_3)) \subseteq W_{t'_2} \) by the choice of \( r_3 \) and Lemma 4.7 and \( W_{r' (B)} \cap V (H_1) \subseteq W_{t'_2} \) by Lemma 4.7 it follows that \((T'', W)\) is a tree-decomposition of \( G \).

Let \( T' \) be defined as in Lemma 3.5 starting from the tree \( T'' \), let \( t'_0 \) be the vertex that resulted from contracting the edges \( t_0 t'_2 \) and \( t_0 t'_3 \), and let \( W' = (W'_t \mid t \in V (T')) \) be defined by

\[
W'_t = \begin{cases} 
W_t & \quad \text{if } t \in V (T') - r'''T'_0 \\
W_{r_3} \cup (V (H_3) \cap W_{t_0}) & \quad \text{if } t = r''' \\
(W_{r_3} - V (H_2)) \cup (V (H_3) \cap W_{t_0}) & \quad \text{if } t = r'' \\
W_{t'_2} & \quad \text{if } t = t'_0 \\
(W_t - V (H_2)) \cup (V (H_3) \cap W_{t_0}) & \quad \text{if } t \in r'''T'_0 - \{t'_0\}
\end{cases}
\]

We claim that \((T', W')\) is a tree decomposition of \( G \). Indeed, since \( V (H_2) \cap W_r \subseteq W_{t_0} \) for all \( r \in r'''T'_0 \) it follows that \((T', W')\) satisfies (W1).

To show that \((T', W')\) satisfies (W2) let \( v \in V (G) \), let \( Z = \{ t \in V (T) : v \in W_t \} \), and let \( Z' = \{ t \in V (T') : v \in W'_t \} \). It suffices to show that \( Z' \) induces a connected subset of \( T' \), for this is easily seen to be equivalent to (W2). To that end assume first that \( v \notin W_{t_1} = W_{t'_1} = W_{t_0} \cap (V (H_2) \cup V (H_3)) \). It follows that, since \( Z \) induces a subtree of \( T \), that \( Z' \) induces a subtree of \( T' \). We assume next that \( v \in W_{t_0} \cap V (H_2) \). The choice of \( T'' \) and the definition of \( W' \) imply that no vertex in the component of \( T'' - r''' \) containing \( t'_0 \) belongs to \( Z' \). Again, it follows that \( Z' \) induces a subtree of \( T' \). Finally, let \( v \in W_{t_0} \cap V (H_3) \). Then \( t'_1 T'_0 \subseteq Z' \), and it again follows that \( Z' \) induces a subtree of \( T' \). This proves our claim that \((T', W')\) is a tree-decomposition.

We claim that the size of \((T', W')\) is smaller than the size of \((T, W)\). Let \( r \) denote the rank of \( T_0 \), and let \( T'_0 \) denote the \((n, r_1)\)-cell in \((T', W')\) containing \( t'_0 \). First, by the passing remark made a few paragraphs ago, for every integer \( m \geq n \) and every rank \( s \), to every \((m, s)\)-cell in \((T', W')\) other than \( T'_0 \) there corresponds a unique \((m, s)\)-cell in \((T, W)\). (To the \((n+1, r_1)\)-cell in \((T', W')\) with vertex-set \{r'''\} there corresponds the \((n+1, r_1)\)-cell in \((T, W)\)) with vertex-set \{t_0\}.) Second, by Lemma 3.5 the rank of \( T_0 \) is strictly larger than the rank of \( T'_0 \). Thus no \((n, r)\)-cell in \((T', W')\) corresponds to \( T_0 \). It follows that \((T', W')\) is a tree-decomposition of \( G \) of smaller size, contrary to the minimality of \((T, W)\).

\[\square\]

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