NON-BRANCHNG TREE-DECOMPOSITIONS[1]

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Abstract

We prove that if a graph has a tree-decomposition of width at most w, then it has a tree-decomposition of width at most w with certain desirable properties. We will use this result in a subsequent paper to show that every 2-connected graph of large path-width has a minor isomorphic to either a large tree with a vertex attached to every vertex of the tree or a large outerplanar graph.

1 INTRODUCTION

All *graphs* in this paper are finite and siimple; that is, they have no loops or parallel edges. *Paths* and *cycles* have no "repeated" vertices or edges. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. An H *minor* is a minor isomorphic to H. A tree-decomposition of a graph G is a pair (T, X), where T is a tree and X is a family $(X_t : t \in V(T))$ such that:

(W1) $\bigcup_{t\in\ V((T)} X_t = V(G)$, and for every edge of G with ends u and v there exists $t \in V(T)$ such that $u, v \in X_t$, and

(W2) if $t_1, t_2, t_3 \in V(T)$ and t_2 lies on the path in T between t_1 and t_3, then $X_{t_1} \cap X_{t_3} \subseteq X_{t_2}$.

The *width* of a tree-decomposition (T, X) is $\max\{|X_t| - 1 : t \in V(T)\}$. The *tree-width* of a graph G is the least width of a tree-decomposition of G. A *path-decomposition* of G is a tree-decomposition (T, X) of G where T is a path. The *path-width* of G is the least width of a path-decomposition of G. Robertson and Seymour [9] proved the following:

Theorem 1.1. *For every planar graph H there exists an integer $n = n(H)$ such that every graph of tree-width at least n has an H minor.*

Robertson and Seymour [8] also proved an analogous result for path-width:

Theorem 1.2. *For every forest F, there exists an integer $p = p(F)$ such that every graph of path-width at least p has an F minor.*

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Bienstock, Robertson, Seymour and the second author [2] gave a simpler proof of Theorem 1.2 and improved the value of p to $|V(F)|-1$, which is best possible, because K_k has path-width $k-1$ and does not have any forest minor on $k+1$ vertices. A yet simpler proof of Theorem 1.2 was found by Diestel [6].

Motivated by the possibility of extending Theorem 1.2 to matroids Seymour [5, Open Problem 2.1] asked if there was a generalization of Theorem 1.2 for 2-connected graphs with forests replaced by the two families of graphs mentioned in the abstract. In [4] we answer Seymour's question in the affirmative:

Theorem 1.3. *Let P be a graph with a vertex v such that $P\backslash v$ is a forest, and let Q be an outerplanar graph. Then there exists a number $p = p(P, Q)$ such that every 2-connected graph of path-width at least p has a P or Q minor.*

Theorem 1.3 is a generalization of Theorem 1.2. To deduce Theorem 1.2 from Theorem 1.3, given a graph G, we may assume that G is connected, because the path-width of a graph is equal to the maximum path-width of its components. We add one vertex and make it adjacent to every vertex of G. Then the new graph is 2-connected, and by Theorem 1.3, it has a P or Q minor. By choosing suitable P and Q, we can get an F minor in G.

Our strategy to prove Theorem 1.3 is as follows. Let G be a 2-connected graph of large path-width. We may assume that the tree-width of G is bounded, for otherwise G has a minor isomorphic to both P and Q by Theorem 1.1. So let (T, X) be a tree-decomposition of G of bounded width. Since the path-width of G is large, it follows by a simple argument [4, Lemma 4.1] that the path-width of T is large, and hence it has a subgraph T' isomorphic to a subdivision of a large binary tree by Theorem 1.2. It now seems plausible that we could use T' and properties (W3) and (W4) of tree-decompositions, introduced below, which we can assume by [7, 10], to show the desired conclusion. But there is a catch: for instance, a long cycle has a tree-decomposition (T, X) satisfying (W3) and (W4) (and, in fact, the minimality condition used in their proof, as well as that of Bellenbaum and Diestel [1]) such that T has a subgraph isomorphic to a large binary tree. And yet it feels that this is the "wrong" tree-decomposition and that the "right" tree-decomposition is one where T is a path. The main result of this paper, Theorem 2.4 below, deals with converting these "branching" tree-decompositions into "non-branching" ones without increasing their width.

The paper is organized as follows. In the next section we review known results about tree-decompositions and state our main result, Theorem 2.4. In Section 3 we introduce a linear quasi-order on the class of finite trees and prove a key lemma—Lemma 3.5. In Section 4 we prove Theorem 2.4, which we restate as Theorem 4.8.

2 LINKED TREE-DECOMPOSITIONS

In this section we review properties of tree-decompositions established in [7, 10], and state our main result. The proof of the following easy lemma can be found, for instance, in [10].

Lemma 2.1. *Let (T, Y) be a tree-decomposition of a graph G, and let H be a connected subgraph of G such that $V(H) \cap Y_{t_1} \neq \emptyset \neq V(H) \cap Y_{t_2}$, where $t_1, t_2 \in V(T)$. Then $V(H) \cap Y_t \neq \emptyset$ for every $t \in V(T)$ on the path between t_1 and t_2 in T.*

A tree-decomposition (T, Y) of a graph G is said to be *linked* if

(W3) for every two vertices t_1, t_2 of T and every positive integer k, either there are k disjoint paths in G between Y_{t_1} and Y_{t_2}, or there is a vertex t of T on the path between t_1 and t_2 such that $|Y_t| < k$.

It is worth noting that, by Lemma 2.1, the two alternatives in (W3) are mutually exclusive. The following is proved in [10].

Lemma 2.2. *If a graph G admits a tree-decomposition of width at most w, where w is some integer, then G admits a linked tree-decomposition of width at most w.*

Let (T, Y) be a tree-decomposition of a graph G, let $t_0 \in V(T)$, and let B be a component of $T \backslash t_0$. We say that a vertex $v \in Y_{t_0}$ is *B-tied* if $v \in Y_t$ for some $t \in V(B)$. We say that a path P in G is *B-confined* if $|V(P)| \geq 3$ and every internal vertex of P belongs to $\bigcup_{t \in V(B)} Y_t - Y_{t_0}$. We wish to consider the following three properties of (T, Y):

(W4) if t, t' are distinct vertices of T, then $Y_t \neq Y_{t'}$,

(W5) if $t_0 \in V(T)$ and B is a component of $T \backslash t_0$, then $\bigcup_{t \in V(B)} Y_t - Y_{t_0} \neq \emptyset$,

(W6) if $t_0 \in V(T)$, B is a component of $T \backslash t_0$, and u, v are B-tied vertices in Y_{t_0}, then there is a B-confined path in G between u and v.

The following strengthening of Lemma 2.2 is proved in [7].

Lemma 2.3. *If a graph G has a tree-decomposition of width at most w, where w is some integer, then it has a tree-decomposition of width at most w satisfying* (W1)–(W6).

We need one more condition, which we now introduce. Let T be a tree. If $t_1, t_2 \in V(T)$, then by $t_1 T t_2$ we denote the vertex-set of the unique path in T with ends t_1 and t_2. A *triad in T* is a triple t_1, t_2, t_3 of vertices of T such that there exists a vertex t of T, called the *center*, such that t_1, t_2, t_3 belong to different components of $T \backslash t$. Let (T, W) be a tree-decomposition of a graph G, and let t_1, t_2, t_3 be a triad in T with center t_0. The *torso of (T, W) at t_1, t_2, t_3* is the subgraph of G induced by the set $\bigcup W_t$, the union taken over all vertices $t \in V(T)$ such that either $t \in \{t_1, t_2, t_3\}$, or for all $i \in \{1, 2, 3\}$, the vertex t belongs to the component of $T \backslash t_i$ containing t_0. We say that the triad t_1, t_2, t_3 is *W-separable* if, letting $X = W_{t_1} \cap W_{t_2} \cap W_{t_3}$, the graph obtained from the torso of (T, W) at t_1, t_2, t_3 by deleting X can be partitioned into three disjoint non-null graphs H_1, H_2, H_3 in such a way that for all distinct $i, j \in \{1, 2, 3\}$ and all $t \in t_j T t_0$, $|V(H_i) \cap W_t| \geq |V(H_i) \cap W_{t_j}| = |W_{t_j} - X|/2 \geq 1$. (Let us remark that this condition implies that $|W_{t_1}| = |W_{t_2}| = |W_{t_3}|$ and $V(H_i) \cap W_{t_i} = \emptyset$ for $i = 1, 2, 3$.) The last property of a tree-decomposition (T, W) that we wish to consider is

(W7) if t_1, t_2, t_3 is a W–separable triad in T with center t, then there exists an integer $i \in \{1, 2, 3\}$ with $W_{t_i} \cap W_t - (W_{t_1} \cap W_{t_2} \cap W_{t_3}) \neq \emptyset$.

The following is our main result.

Theorem 2.4. *If a graph G has a tree-decomposition of width at most w, where w is some integer, then it has a tree-decomposition of width at most w satisfying* (W1)–(W7).

3 A QUASI-ORDER ON TREES

A *quasi-ordered set* is a pair $(Q, \leq)$, where Q is a set and $\leq$ is a *quasi-order*; that is, a reflexive and transitive relation on Q. If $q, q' \in Q$ we define $q < q'$ to mean that $q \leq q'$ and $q' \not\leq q$. We say that q, q' are $\leq$-*equivalent* if $q \leq q' \leq q$. We say that $(Q, \leq)$ is a *linear quasi-order* if for every two elements $q, q' \in Q$ either $q \leq q'$ or $q' \leq q$ or both. Let $(Q, \leq)$ be a linear quasi-order. If $A, B \subseteq Q$ we say that B $\leq$-*dominates* A if the elements of A can be listed as $a_1 \geq a_2 \geq \cdots \geq a_k$ and the elements of B can be listed as $b_1 \geq b_2 \geq \cdots \geq b_l$, and there exists an integer p with $1 \leq p \leq \min\{k, l\}$ such that $a_i \leq b_i \leq a_i$ for all $i = 1, 2, \ldots, p$, and either $p < \min\{k, l\}$ and $a_{p+1} < b_{p+1}$, or $p = k$ and $k \leq l$.

Lemma 3.1. *If $(Q, \leq)$ is a linear quasi-order, then $\leq$-domination is a linear quasi-order on the set of subsets of Q.*

Proof. It is obvious that $\leq$-domination is reflexive. Assume that B $\leq$-dominates A and C $\leq$-dominates B. Assume that the elements of A can be listed as $a_1 \geq a_2 \geq \cdots \geq a_k$, the elements of B can be listed as $b_1 \geq b_2 \geq \cdots \geq b_l$, and the elements of C can be listed as $c_1 \geq c_2 \geq \cdots \geq c_m$. By definition, there exists an integer p_1 with $1 \leq p_1 \leq \min\{k, l\}$ such that $a_i \leq b_i \leq a_i$ for all $i = 1, 2, \ldots, p_1$, and either $p_1 < \min\{k, l\}$ and $a_{p_1+1} < b_{p_1+1}$, or $p_1 = k \leq l$; and there exists an integer p_2 with $1 \leq p_2 \leq \min\{l, m\}$ such that $b_i \leq c_i \leq b_i$ for all $i = 1, 2, \ldots, p_2$, and either $p_2 < \min\{l, m\}$ and $b_{p_2+1} < c_{p_2+1}$, or $p_2 = l \leq m$. Let $p = \min\{p_1, p_2\}$. Then $a_i \leq c_i \leq a_i$ for all $i = 1, 2, \ldots, p$. If either $p_1 < \min\{k, l\}$ and $a_{p_1+1} < b_{p_1+1}$, or $p_2 < \min\{l, m\}$ and $b_{p_2+1} < c_{p_2+1}$, then $p < \min\{k, m\}$ and $a_{p+1} < c_{p+1}$. If $p_1 = k \leq l$ and $p_2 = l \leq m$, then $p = k \leq m$. Therefore, C $\leq$-dominates A, and so $\leq$-domination is transitive.

Now let A, B be as above, and let p be the maximum integer such that $p \leq \min\{k, l\}$ and $a_i \leq b_i \leq a_i$ for all $i = 1, 2, \ldots, p$. Then if $p < \min\{k, l\}$, then A $\leq$-dominates B if $a_{p+1} > b_{p+1}$ and B $\leq$-dominates A if $a_{p+1} < b_{p+1}$. If $p = \min\{k, l\}$ then A $\leq$-dominates B if $k \geq l$ and B $\leq$-dominates A if $k \leq l$. Hence, $\leq$-domination is linear. □

We say that B *strictly* $\leq$-*dominates* A if B $\leq$-dominates A in such a way that the numberings and integer p can be chosen in such a way that either $p < \min\{k, l\}$, or $p = k$ and $k < l$.

Lemma 3.2. *Let $(Q, \leq)$ be a linear quasi-order, let $A, B \subseteq Q$, and let B $\leq$-dominate A. Then B strictly $\leq$-dominates A if and only if A does not $\leq$-dominate B.*

Proof. Let p be as in the definition of B $\leq$-dominates A. Then $p < \min\{k, l\}$ and $a_{p+1} < b_{p+1}$, or $p = k \leq l$. Assume B strictly $\leq$-dominates A. If $p < \min\{k, l\}$ then $a_{p+1} < b_{p+1}$, so A does not $\leq$-dominate B. If $p = k < l$ then A also does not $\leq$-dominate B. Conversely, if A does not $\leq$-dominate B, then $p < \min\{k, l\}$ or $k < l$, so B strictly $\leq$-dominates A. □

Let G be a graph and let P be a subgraph of G. By a P-bridge of G we mean a subgraph J of G such that either

- J is isomorphic to the complete graph on two vertices with $V(J) \subseteq V(P)$ and $E(J) \cap E(P) = \emptyset$, or
- J consists of a component of $G - V(P)$ together with all edges from that component to P.

We now define a linear quasi-order $\leq$ on the class of finite trees as follows. Let $n \geq 1$ be an integer, and suppose that $T \leq T'$ has been defined for all trees T on fewer than n vertices. Let T be a tree on n vertices, and let T' be an arbitrary tree. We define $T \leq T'$ if either $|V(T)| < |V(T')|$, or $|V(T)| = |V(T')|$ and for every maximal path P' of T' there exists a maximal path P of T such that the set of P'-bridges of T' $\leq$-dominates the set of P-bridges of T. It follows from Lemma 3.3 below that $\leq$ is indeed a linear quasi-order; in particular, it is well-defined.

If T, T' are trees, P is a path in T and P' is a path in T' we define $(T, P) \preceq (T', P')$ if either $|V(T)| < |V(T')|$, or $|V(T)| = |V(T')|$ and the set of P'-bridges of T' $\leq$-dominates the set of P-bridges of T.

Lemma 3.3. *(i) For every tree T there exists a maximal path $P(T)$ in T such that $(T, P(T)) \preceq (T, P)$ for every maximal path P in T.*
(ii) For every two trees T, T', we have $T \leq T'$ if and only if $(T, P(T)) \preceq (T', P(T'))$.
(iii) The ordering $\leq$ is a linear quasi-order on the class of finite trees.

Proof. We prove all three statements simultaneously by induction. Let $n \geq 1$ be an integer, assume inductively that all three statements have been proven for trees on fewer than n vertices, and let T be a tree on n vertices.
(i) Statement (i) clearly holds for one-vertex trees, and so we may assume that $n \geq 2$. Let $\mathcal{B}$ be the set of all P-bridges of T for all maximal paths P of T. Then every member of $\mathcal{B}$ has fewer than n vertices, and hence $\mathcal{B}$ is a linear quasi-order by $\leq$ by the induction hypothesis applied to (iii). By Lemma 3.1 the set of subsets of $\mathcal{B}$ is linearly quasi-ordered by $\leq$-domination. It follows that there exists a maximal path $P(T)$ in T such that the set of $P(T)$-bridges of T is minimal under $\leq$-domination.
(ii) The statement is obvious when $|V(T)| \neq |V(T')|$, so assume $n = |V(T)| = |V(T')|$, and let $\mathcal{B}$ be the set of all P-bridges of T for all maximal paths P of T and the set of all P'-bridges of T' for all maximal paths P' of T'. Then as in (i) the subsets of $\mathcal{B}$ are linearly quasi-ordered by $\leq$-domination. If $T \leq T'$, then by definition there exists a maximal path P of T such that $(T, P) \preceq (T', P(T'))$. Hence $(T, P(T))) \preceq (T', P(T'))$ follows from (i). If $(T, P(T))) \preceq (T', P(T'))$, then by (i) $(T, P(T))) \preceq (T', P')$ for every maximal path P'

in T', so $T \le T'$.
(iii) Let T and T' be two trees. We may assume that $n = |V(T)| = |V(T')|$. Let $\mathcal{B}$ be as in (ii); then subsets of $\mathcal{B}$ are linearly quasi-ordered by $\le$-domination. Then either $(T, P(T)) \preceq (T', P(T'))$ or $(T', P(T')) \preceq (T, P(T))$, and so by (ii) $\le$ is linear. □

For a tree T, the path $P(T)$ from Lemma 3.3(i) will be called a *spine of* T. For later application we need the following lemma.

Lemma 3.4. *Let T, T' be trees on the same number of vertices, let P' be a spine of T', and let P be a path in T. If the set of P'-bridges of T' strictly $\le$-dominates the set of P-bridges of T, then $T < T'$.*

Proof. We have $(T, P) \preceq (T', P')$ and $(T', P') \not\preceq (T, P)$ by Lemma 3.2. Let P_1 be a maximal path that contains P; then $(T, P_1) \preceq (T, P)$. Therefore, $(T, P_1) \preceq (T', P')$ and $(T', P') \not\preceq (T, P_1)$. By Lemma 3.3(i), $(T, P(T)) \preceq (T, P_1) \preceq (T', P')$ and $(T', P') \not\preceq (T, P(T))$. By Lemma 3.3(ii), $T \le T'$ and $T' \not\le T$. Therefore, $T < T'$. □

By a *rank* we mean a class of $\le$-equivalent trees. If r is a rank we say that T *has rank* r or that the *rank of* T *is* r if $T \in r$. The class of all ranks will be denoted by $\mathcal{R}$.

Let T be a tree, and let t be a vertex of T. By a spine-decomposition of T relative to t we mean a sequence $(T_0, P_0, T_1, P_1, \ldots, T_l, P_l)$ such that

(i) $T_0 = T$,

(ii) for $i = 0, 1, \ldots, l$, P_i is a spine of T_i, and

(iii) for $i = 1, 2, \ldots, l$, $t \notin V(P_{i-1})$ and T_i is the P_{i-1}-bridge of T_{i-1} containing t.

Lemma 3.5. *Let T be a tree, let t be a vertex of T of degree three with neighbors t'_1, t'_2, t'_3, and let $(T_0, P_0, T_1, P_1, \ldots, T_l, P_l)$ be a spine-decomposition of T relative to t with $t \in V(P_l)$. Then exactly two of t'_1, t'_2, t'_3 belong to $V(P_l)$, say t'_1 and t'_2. Let r_3, r'_3 be adjacent vertices of T such that r_3, r'_3, t'_3, t occur on a path of T in the order listed. Thus possibly $t'_3 = r'_3$, but $t'_3 \ne r_3$. Let T' be obtained from T by subdividing the edge $r_3r'_3$ twice (let r''_3, r'''_3 be the new vertices so that r'_3, r''_3, r'''_3, r_3 occur on a path of T' in the order listed), deleting the edge tt'_1, contracting the edges tt'_2 and tt'_3 and adding an edge joining t'_1 and r'''_3. Then T' has strictly smaller rank than T.*

Proof. Let $T'_0 = T'$ and for $i = 1, 2, \ldots, l$, let T'_i be the P_{i-1}-bridge of T'_{i-1} containing r'''_3. Let P' be the unique maximal path in T' with $V(P_l) - \{t, t'_2\} \cup \{r'_3\} \subseteq V(P')$. From the definition of a spine-decomposition and the fact that $t'_3 \notin V(P_l)$ we deduce that $r_3 \in V(T_i)$ for all $i = 0, 1, \ldots, l$. It follows that $r_3 \in V(T'_i)$ and $|V(T_i)| = |V(T'_i)|$ for all $i = 0, 1, \ldots, l$. The P_l-bridge of T_l that contains r_3 is replaced by P'-bridges of T'_l with smaller cardinalities. Other P_l-bridges of T_l are unchanged in T'. Therefore, the set of P_l-bridges of T_l strictly $\le$-dominates the set of P'-bridges of T'_l, and hence $T'_l < T_l$ by Lemma 3.4. This implies, by induction on $l - i$ using Lemma 3.4, that $T'_i < T_i$ for all $i = 0, 1, \ldots, l$; that is, T' has smaller rank than T. □

4 A THEOREM ABOUT TREE-DECOMPOSITIONS

Let (T, Y) be a tree-decomposition of a graph G, let n be an integer, and let r be a rank. By an (n, r)*–cell* in (T, Y) we mean any component of the restriction of T to $\{t \in V(T) : |Y_t| \geq n\}$ that has rank at least r. Let us remark that if K is an (n, r)-cell in (T, Y) and $r \geq r'$, then K is an (n, r')-cell as well. The *size* of a tree-decomposition (T, Y) is the family of numbers

$$(a_{n,r} : n \geq 0, r \in \mathcal{R}), \tag{1}$$

where $a_{n,r}$ is the number of (n, r)-cells in (T, Y). Sizes are ordered lexicographically; that is, if

$$(b_{n,r} : n \geq 0, r \in \mathcal{R}) \tag{2}$$

is the size of another tree-decomposition (R, Z) of the graph G, we say that (2) is *smaller than* (1) if there are an integer $n \geq 0$ and a rank $r \in \mathcal{R}$ such that $a_{n,r} > b_{n,r}$ and $a_{n',r'} = b_{n',r'}$ whenever either $n' > n$, or $n' = n$ and $r' > r$.

Lemma 4.1. *The relation "to be smaller than" is a well–ordering on the set of sizes of tree–decompositions of G.*

Proof. Since this ordering is clearly linear, it is enough to show that it is well–founded. Suppose for a contradiction that $\{(a^{(i)}_{n,r} : n \geq 0, r \in \mathcal{R})\}_{i=1}^{\infty}$ is a strictly decreasing sequence of sizes, and for $i = 1, 2, \ldots$, let n_i, r_i be such that $a^{(i)}_{n_i,r_i} > a^{(i+1)}_{n_i,r_i}$ and $a^{(i)}_{n,r} = a^{(i+1)}_{n,r}$ for (n, r) such that either $n > n_i$, or $n = n_i$ and $r > r_i$. Since $a^{(1)}_{n,r} = 0$ for all $r \in \mathcal{R}$ and all $n > |V(G)|$, we may assume (by taking a suitable subsequence) that $n_1 = n_2 = \cdots$, and that $r_1 \leq r_2 \leq r_3 \leq \cdots$. Since clearly $a^{(i)}_{n,r} \geq a^{(i)}_{n,r'}$ for all $n \geq 0$, all $r \leq r'$ and all $i = 1, 2, \ldots$, we have

$$a^{(1)}_{n_1,r_1} > a^{(2)}_{n_1,r_1} \geq a^{(2)}_{n_2,r_2} > a^{(3)}_{n_2,r_2} \geq a^{(3)}_{n_3,r_3} > \cdots ,$$

a contradiction. □

We say that a tree-decomposition (T, W) of a graph G is *minimal* if there is no tree-decomposition of G of smaller size.

Lemma 4.2. *Let w be an integer, and let G be a graph of tree-width at most w. Then a minimal tree-decomposition of G exists, and every minimal tree-decomposition of G has width at most w.*

Proof. The existence of a minimal tree-decomposition follows from Lemma 4.1. If G has a tree-decomposition of width at most w, then every minimal tree-decomposition has width at most w, as desired. □

Theorem 4.3. *Let (T, W) be a minimal tree-decomposition of a graph G. Then (T, W) satisfies* (W1)–(W6)*.*

Proof. That (T, W) satisfies (W3) is shown in [10], and that it satisfies (W4), (W5) and (W6) is shown in [7]. Let us remark that [7] and [10] use a slightly different definition of minimality, but the proofs are adequate, because a minimal tree-decomposition in our sense is minimal in the sense of [7] and [10] as well. □

Lemma 4.4. *Let (T,W) be a minimal tree-decomposition of a graph G. Then for every edge $tt' \in E(T)$ either $W_t \subseteq W_{t'}$ or $W_{t'} \subseteq W_t$.*

Proof. Assume for a contradiction that there exists an edge $tt' \in E(T)$ such that $W_t \not\subseteq W_{t'}$ and $W_{t'} \not\subseteq W_t$. Let R be obtained from T by subdividing the edge tt' and let t'' be the new vertex. Let $Y_{t''} = W_t \cap W_{t'}$ and $Y_r = W_r$ for all $r \in V(T)$, and let $Y = (Y_r : r \in V(R))$. Then (R,Y) is a tree-decomposition of G of smaller size than (T,W), contrary to the minimality of (T,W). □

Lemma 4.5. *Let (T,W) be a minimal tree-decomposition of a graph G, let $t \in V(T)$, let $X \subseteq W_t$, let B be a component of $T\backslash t$, let t' be the neighbor of t in B, let $Y = \bigcup_{r\in V(B)} W_r$, and let H be the subgraph of G induced by $Y \cup W_t$. If $H\backslash X = H_1 \cup H_2$, where $V(H_1) \cap V(H_2) = \emptyset$ and both of $V(H_1), V(H_2)$ intersect W_t, then either $W_{t'} - X \subseteq W_t \cap V(H_1)$ or $W_{t'} - X \subseteq W_t \cap V(H_2)$.*

Proof. We first prove the following claim.

Claim 4.5.1. *Either $W_t \cap W_{t'} - X \subseteq V(H_1)$ or $W_t \cap W_{t'} - X \subseteq V(H_2)$.*

To prove the claim suppose for a contradiction that there exist vertices $v_1 \in W_t \cap W_{t'} \cap V(H_1)$ and $v_2 \in W_t \cap W_{t'} \cap V(H_2)$. Thus both v_1 and v_2 are B-tied, and so by (W6), which (T,W) satisfies by Theorem 4.3, there exists a B-confined path Q with ends v_1 and v_2. Since Q is B-confined, it is a subgraph of $H\backslash X$, contrary to the fact that $V(H_1) \cap V(H_2) = \emptyset$ and $H_1 \cup H_2 = H\backslash X$. This proves Claim 4.5.1.

Since both of $V(H_1), V(H_2)$ intersect W_t, Claim 4.5.1 implies that $W_t \not\subseteq W_{t'}$, and hence $W_{t'} \subseteq W_t$ by Lemma 4.4. By another application of Claim 4.5.1 we deduce that either $W_{t'} - X \subseteq W_t \cap V(H_1)$ or $W_{t'} - X \subseteq W_t \cap V(H_2)$, as desired. □

Lemma 4.6. *Let $k \geq 1$ be an integer, let (T,W) be a minimal tree-decomposition of a graph G, let $t_1, t_2 \in V(T)$, let $X = W_{t_1} \cap W_{t_2}$, let H be the subgraph of G induced by $\bigcup W_t$, the union taken over all vertices $t \in V(T)$ such that either $t \in \{t_1, t_2\}$, or for $i = 1, 2$ the vertex t belongs to the component of $T\backslash t_i$ containing t_{3-i}, let $H\backslash X = H_1 \cup H_2$, where $V(H_1) \cap V(H_2) = \emptyset$, and assume that $|W_{t_i} \cap V(H_j)| = k$ and $|W_t \cap V(H_i)| \geq k$ for all $i, j \in \{1,2\}$ and all $t \in t_1Tt_2$. Let t, t' be two adjacent vertices on the path of T between t_1 and t_2. Then there exists an integer $i \in \{1,2\}$ such that $W_t \cap V(H_i) = W_{t'} \cap V(H_i)$ and this set has cardinality k.*

Proof. We begin with the following claim.

Claim 4.6.1. *For every $t \in t_1Tt_2$ either $|W_t \cap V(H_1)| = k$ or $|W_t \cap V(H_2)| = k$.*

To prove the claim let R be the subtree of T induced by vertices $r \in V(T)$ such that either $r \in \{t_1, t_2\}$ or r belongs to the component of $T\backslash\{t_1, t_2\}$ that contains neighbors of both t_1 and t_2, let R_1, R_2 be two isomorphic copies of R, and for $r \in V(R)$ let r_1 and r_2 denote the copies of r in R_1 and R_2, respectively. Assume for a contradiction that there is $t_0 \in t_1Tt_2$ such that $|W_{t_0} \cap V(H_i)| > k$ for all $i \in \{1,2\}$, and choose such a vertex with $t_0 \in V(R)$ and $|W_{t_0}|$ maximum. We construct a new tree-decomposition (T', W') as follows. The tree T' is obtained from the disjoint union of $T\backslash(V(R)-\{t_1,t_2\})$, R_1 and R_2 by identifying t_1 with $(t_1)_1$, $(t_2)_1$ with $(t_1)_2$ and $(t_2)_2$ with t_2 (here $(t_1)_2$ denotes the copy of t_1 in R_2 and similarly for the other three quantities). The family $W' = (W'_t : t \in V(T'))$ is defined as follows:

$$W'_t = \begin{cases} W_t & \text{if } t \in V(T) - V(R) \\ (W_r \cap V(H_1)) \cup (W_{t_1} \cap V(H_2) \cup X & \text{if } t = r_1 \text{ for } r \in t_1Tt_2 \\ (W_r \cap V(H_2)) \cup (W_{t_2} \cap V(H_1) \cup X & \text{if } t = r_2 \text{ for } r \in t_1Tt_2 \\ W_r \cap V(H_1) & \text{if } t = r_1 \text{ for } r \in V(R) - t_1Tt_2 \\ W_r \cap V(H_2) & \text{if } t = r_2 \text{ for } r \in V(R) - t_1Tt_2 \end{cases}$$

Please note that the value of W'_t is the same for $t = (t_2)_1$ and $t = (t_1)_2$, and hence W' is well-defined. Since no edge of G has one end in $V(H_1)$ and the other end in $V(H_2)$, it follows that (T', W') is a tree-decomposition of G.

We claim that the size of (T', W') is smaller than the size of (T, W). Indeed, let $n_0 = |W_{t_0}|$, and let $Z = \{t \in V(T') : |W'_t| \geq n_0\}$. Then $n_0 > 2k + |X|$. We define a mapping $f : Z \to V(T)$ by $f(t) = t$ for $t \in Z - V(R_1) - V(R_2)$, $f(r_1) = r$ for $r \in V(R)$ such that $r_1 \in Z$ and $f(r_2) = r$ for $r \in V(R)$ such that $r_2 \in Z$. We remark that the vertex obtained by identifying $(t_2)_1$ with $(t_1)_2$ does not belong to Z, and hence there is no ambiguity. Then Z and f have the following properties:

- $|W_{f(t)}| \geq |W'_t|$ for every $t \in Z$,
- for $r \in V(R)$, at most one of r_1, r_2 belongs to Z, and
- $(t_0)_1, (t_0)_2 \notin Z$

These properties follow from the assumptions that $|W_{t_i} \cap V(H_j)| = k$ and $|W_t \cap V(H_i)| \geq k$ for all $i, j \in \{1, 2\}$ and all $t \in t_1Tt_2$. (To see the second property assume for a contradiction that for some $r \in V(R)$ both r_1 and r_2 belong to Z. Then $n_0 = |W_{t_0}| \geq |W_{f(r_i)}| \geq |W_{r_i}| \geq n_0$, by the maximality of $|W_{t_0}|$ and the first property, and so equality holds throughout, contrary to the construction.) It follows from the first two properties that f maps injectively (n, r)-cells in (T', W') to (n, r)-cells in (T, W) for all $n \geq n_0$ and all ranks r. On the other hand, the third property implies that, letting r_1 denote the rank of one-vertex trees, no (n_0, r_1)-cell in (T', W') is mapped onto the (n_0, r_1)-cell in (T, W) with vertex-set $\{t_0\}$. Thus the size of (T', W') is smaller than the size of (T, W), contrary to the minimality of (T, W). This proves Claim 4.6.1.

Now let $t, t' \in t_1Tt_2$ be adjacent. By Lemma 4.4 we may assume that $W_t \subseteq W_{t'}$. Then $W_t \cap V(H_1) \subseteq W_{t'} \cap V(H_1)$ and $W_t \cap V(H_2) \subseteq W_{t'} \cap V(H_2)$. By Claim 4.6.1 we may assume that $|W_{t'} \cap V(H_1)| = k$. Given that $|W_t \cap V(H_1)| \geq k$ we have $W_t \cap V(H_1) = W_{t'} \cap V(H_1)$ and this set has cardinality k, as desired. □

Lemma 4.7. *Let (T, W) be a minimal tree-decomposition of a graph G, let t_1, t_2, t_3 be a W-separable triad in T with center t_0, and let X, H, H_1, H_2 and H_3 be as in the definition of W-separable triad. Let $k = |W_{t_1} - X|/2$ and for $i = 1, 2, 3$ let t'_i denote the neighbor of t_0*

in the component of $T\backslash t_0$ *containing* t_i. *Then for all distinct* $i, j \in \{1,2,3\}$, $V(H_i) \cap W_{t'_j} = V(H_i) \cap W_{t_0}$, *and this set has cardinality* k.

Proof. Let $X_3 = \bigcup W_t$, the union taken over all $t \in V(T)$ that do not belong to the component of $T\backslash t_3$ containing t_0. Since $|W_{t_0} \cap V(H_1)| \geq k$ and $|W_{t_0} \cap V(H_2)| \geq k$ by the definition of W-separable triad, by Lemma 4.6 applied to t_1, t_2, H_3 and the subgraph of G induced by $V(H_1) \cup V(H_2) \cup X_3$ we deduce that $V(H_3) \cap W_{t_0} = V(H_3) \cap W_{t'_1} = V(H_3) \cap W_{t'_2}$, and this set has cardinality k. Similarly we deduce that $V(H_2) \cap W_{t_0} = V(H_2) \cap W_{t'_1} = V(H_2) \cap W_{t'_3}$ and $V(H_1) \cap W_{t_0} = V(H_1) \cap W_{t'_2} = V(H_1) \cap W_{t'_3}$, and that the latter two sets also have cardinality k. □

We are finally ready to prove Theorem 2.4, which, by Lemma 4.2 is implied by the following theorem.

Theorem 4.8. *Let* (T, W) *be a minimal tree-decomposition of a graph* G. *Then* (T, W) *satisfies* (W1)–(W7).

Proof. That (T, W) satisfies (W1)–(W6) follows from Theorem 4.3. Thus it remains to show that (T, W) satisfies (W7). Suppose for a contradiction that (T, W) does not satisfy (W7), and let t_1, t_2, t_2 be a W-separable triad in T with center t_0 such that $W_{t_i} \cap W_{t_0} \subseteq X$ for every $i = 1, 2, 3$, where $X = W_{t_1} \cap W_{t_2} \cap W_{t_3}$. Let H, H_1, H_2 and H_3 be as in the definition of W-separable triad, and for $i \in \{1,2,3\}$ let t'_i denote the neighbor of t_0 in the component of $T\backslash t_0$ containing t_i.

Let $n := |W_{t_1}|$, let $k := |W_{t_1} - X|/2$, let r_1 denote the rank of 1-vertex trees, and let T_0 denote the (n, r_1)-cell containing t_0. By the definition of W-separable triad we have $|W_{t'_i}| \geq n$ for all $i \in \{1,2,3\}$, and hence the degree of t_0 in T_0 is at least three and by Lemmas 4.7 and 4.5 it is at most three.

Let $(T_0, P_0, T_1, P_1, \ldots, T_l, P_l)$ be a spine-decomposition of T_0 relative to t_0 with $t_0 \in V(P_l)$. Since P_l is a maximal path in T_l we may assume that $t'_1, t'_2 \in V(P_l)$ and $t'_3 \notin V(P_l)$.

It follows from Lemma 4.7 that $W_{t_3} \cap W_{t'_3} = X$. By Lemma 4.6 applied to t_3 and t'_3 and t'_3 and its neighbor in $t_3Tt'_3$ we deduce that there exists a vertex $r_3 \in t_3Tt'_3 - \{t'_3\}$ such that either $V(H_1) \cap W_{t'_3} = V(H_1) \cap W_r$ for every $r \in r_3Tt'_3$, or $V(H_2) \cap W_{t'_3} = V(H_2) \cap W_r$ for every $r \in r_3Tt'_3$. Without loss of generality we may assume the latter. We may choose r_3 to be as close to t_3 as possible. The fact that $W_{t_3} \cap W_{t'_3} = X$ implies that $r_3 \neq t_3$. By another application of Lemma 4.6, this time to t_3, t'_3, r_3 and the neighbor of r_3 in r_3Tt_3, we deduce that $|V(H_1) \cap W_{r_3}| = |V(H_2) \cap W_{r_3}| = k$.

Let r'_3 be the neighbor of r_3 in r_3Tt_0 and let the tree T'' be defined as follows: for every component B of $T\backslash t_0Tr'_3$ not containing t_1, t_2 or t_3 let $r(B)r'(B)$ denote the edge connecting B to $t_0Tr'_3$, where $r(B) \in V(B)$ and $r'(B) \in t_0Tr'_3$. By Lemma 4.5 there exists an integer $i \in \{1,2,3\}$ such that $W_{r(B)} \subseteq W_{r'(B)} \cap V(H_i)$. Let us mention in passing that this, the choice of r_2 and Lemma 4.7 imply that for every such component B, every (n, r_1)-cell is either a subgraph of B or is disjoint from B. The tree T'' is obtained from T by, for every such component B for which either $i = 2$, or $i = 3$ and $r'(B) = t_0$, deleting the edge $r(B)r'(B)$ and adding the edge $t'_1r(B)$; and for every such component B for which $i = 1$ and $r'(B) = t_0$ deleting the edge $r(B)r'(B)$ and adding the edge $t'_2r(B)$. Since $W_{r'(B)} \cap (V(H_2) \cup V(H_3)) \subseteq W_{t'_1}$ by the choice of r_3 and Lemma 4.7; and $W_{r'(B)} \cap V(H_1) \subseteq W_{t'_2}$ by Lemma 4.7 it follows that (T'', W) is a tree-decomposition of G.

Let T' be defined as in Lemma 3.5, starting from the tree T'', let t'_0 be the vertex that resulted from contracting the edges $t_0t'_2$ and $t_0t'_3$, and let $W' = (W'_t \mid t \in V(T'))$ be defined by

$$W'_t = \begin{cases} W_t & \text{if } t \in V(T') - r'''_3T't'_0 \\ W_{r_3} \cup (V(H_3) \cap W_{t_0}) & \text{if } t = r'''_3 \\ (W_{r_3} - V(H_2)) \cup (V(H_3) \cap W_{t_0}) & \text{if } t = r''_3 \\ W_{t'_2} & \text{if } t = t'_0 \\ (W_t - V(H_2)) \cup (V(H_3) \cap W_{t_0}) & \text{if } t \in r'_3T't'_0 - \{t'_0\} \end{cases}$$

We claim that (T', W') is a tree decomposition of G. Indeed, since $V(H_2) \cap W_r \subseteq W_{t_0}$ for all $r \in r'_3Tt_0$ it follows that (T', W') satisfies (W1).

To show that (T', W') satisfies (W2) let $v \in V(G)$, let $Z = \{t \in V(T) : v \in W_t\}$. and let $Z' = \{t \in V(T') : v \in W'_t\}$. It suffices to show that Z' induces a connected subset of T', for this is easily seen to be equivalent to (W2). To that end assume first that $v \notin W_{t'_1} = W'_{t'_1} = W_{t_0} \cap (V(H_2) \cup V(H_3))$. It follows that, since Z induces a subtree of T, that Z' induces a subtree of T'. We assume next that $v \in W_{t_0} \cap V(H_2)$. The choice of T'' and the definition of W' imply that no vertex in the component of $T' \backslash r'''_3$ containing t'_0 belongs to Z'. Again, it follows that Z' induces a subtree of T'. Finally, let $v \in W_{t_0} \cap V(H_3)$. Then $t'_1T't'_0 \subseteq Z'$, and it again follows that Z' induces a subtree of T'. This proves our claim that (T', W') is a tree-decomposition.

We claim that the size of (T', W') is smaller than the size of (T, W). Let r denote the rank of T_0, and let T'_0 denote the (n, r_1)-cell in (T', W') containing t'_0. First, by the passing remark made a few paragraphs ago, for every integer $m \geq n$ and every rank s, to every (m, s)-cell in (T', W') other than T'_0 there corresponds a unique (m, s)-cell in (T, W). (To the $(n+1, r_1)$-cell in (T', W') with vertex-set $\{r'''_3\}$ there corresponds the $(n+1, r_1)$-cell in (T, W) with vertex-set $\{t_0\}$.) Second, by Lemma 3.5 the rank of T_0 is strictly larger than the rank of T'_0. Thus no (n, r)-cell in (T', W') corresponds to T_0. It follows that (T', W') is a tree-decomposition of G of smaller size, contrary to the minimality of (T, W). □

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