

# PACKING DIRECTED CIRCUITS EXACTLY

BERTRAND GUENIN AND ROBIN THOMAS

ABSTRACT. We give an “excluded minor” and a “structural” characterization of digraphs  $D$  that have the property that for every subdigraph  $H$  of  $D$ , the maximum number of disjoint circuits in  $H$  is equal to the minimum cardinality of a set  $T \subseteq V(H)$  such that  $H \setminus T$  is acyclic.

## 1. INTRODUCTION

Graphs and digraphs in this paper may have loops and multiple edges. Paths and circuits have no “repeated” vertices, and in digraphs they are directed. A *transversal* in a digraph  $D$  is a set of vertices  $T$  which intersects every circuit, i.e.  $D \setminus T$  is acyclic. A *packing of circuits* (or *packing* for short) is a collection of pairwise (vertex-)disjoint circuits. The cardinality of a minimum transversal is denoted by  $\tau(D)$  and the cardinality of a maximum packing is denoted by  $\nu(D)$ . Clearly  $\nu(D) \leq \tau(D)$ , and our objective is to study when equality holds. We will show in Section 4 that this is the case for every strongly planar digraph. (A digraph is *strongly planar* if it has a planar drawing such that for every vertex  $v$ , the edges with head  $v$  form an interval in the cyclic ordering of edges incident with  $v$ .) However, in general there is probably no nice characterization of digraphs for which equality holds, and so instead we characterize digraphs such that equality holds for *every subdigraph*. Thus we say that a digraph  $D$  *packs* if  $\tau(D') = \nu(D')$  for every subdigraph  $D'$  of  $D$ .

We will give two characterizations: one in terms of excluded minors, and the other will give a structural description of digraphs that pack. We say that an edge  $e$  of a digraph  $D$  with head  $v$  and tail  $u$  is *special* if either  $e$  is the only edge of  $D$  with head  $v$ , or it is the only edge of  $D$  with tail  $u$ , or both. We say that a digraph  $D$  is a *minor* of a digraph  $D'$  if  $D$  can be obtained from a subdigraph of  $D'$  by repeatedly contracting special edges. It is easy to see that if a digraph packs, then so do all its minors. Thus digraphs that pack can be characterized by a list of minor-minimal digraphs that do not pack. By an *odd double circuit* we mean the digraph obtained from an undirected circuit of odd length at least three by replacing each edge by a pair of directed edges, one in each direction. The digraph  $F_7$  is defined in Figure 1. The following is our excluded minor characterization.

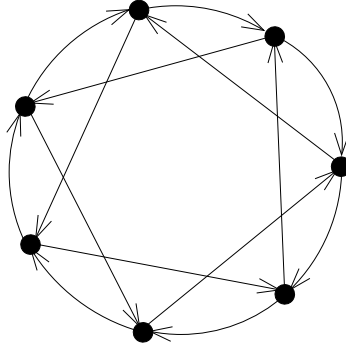
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FIGURE 1. The digraph  $F_7$ .

**Theorem 1.1.** *A digraph packs if and only if it has no minor isomorphic to an odd double circuit or  $F_7$ .*

If  $D$  is an odd double circuit with  $k$  vertices then  $\tau(D) = \lceil k/2 \rceil > \nu(D) = \lfloor k/2 \rfloor$ . Moreover,  $\tau(F_7) = 3 > \nu(F_7) = 2$ . Thus odd double circuits and  $F_7$  do not pack and the content of Theorem 1.1 is to prove the converse.

The structural characterization can be stated directly in terms of digraphs, but it is more convenient to rephrase it in terms of bipartite graphs, and therefore we postpone its statement until Section 5. Roughly, the characterization states that a digraph packs if and only if it can be obtained from strongly planar digraphs by means of certain composition operations.

Our main tool in the proof is a characterization of bipartite graphs that have a Pfaffian orientation, found independently by McCuaig [1] and by Robertson, Seymour and the second author [6]. We present the characterization in Section 5. The rest of the paper is organized as follows. In Section 2 we mention three related results. Section 3 reduces the problem to strongly 2-connected digraphs. It is shown in Section 4 that strongly planar digraphs pack. Sections 6-8 show that the property that digraphs pack is preserved under the composition operations of the characterization theorem, thus completing the proof of Theorem 1.1. Finally, Section 9 offers some closing remarks.

## 2. RELATED RESULTS

In this section we review three related results. The first is a classical theorem of Lucchesi and Younger, of which we only state a corollary [4](Theorem B).

**Theorem 2.1.** *Let  $D$  be a planar digraph and  $\mathcal{F}$  be the family of its directed circuits. Then for any set of weights  $w : E(D) \rightarrow Z_+$  we have,*

$$\begin{aligned} & \min \left\{ \sum_{e \in E(D)} w(e) x_e : \sum_{e \in C} x_e \geq 1, \forall C \in \mathcal{F}, x \in \{0, 1\}^{|E(D)|} \right\} \\ = & \max \left\{ \sum_{C: C \in \mathcal{F}} y_C : \sum_{C: e \in C \in \mathcal{F}} y_C \leq w(e), \forall e \in E(D), y \in Z_+^{|\mathcal{F}|} \right\}. \end{aligned} \quad (2.1)$$

Thus, in particular, in a planar digraph the maximum cardinality of a collection of edge-disjoint circuits is equal to the minimum cardinality of a set of edges whose deletion makes the graph acyclic. This relation does not hold for all digraphs, but there is an upper bound on  $\tau(D)$  as a function of  $\nu(D)$ . (A simple construction — splitting each vertex into a “source” and a “sink,” also used in the proof of Corollary 4.1 — shows that the same function serves as an upper bound for both the edge-disjoint as well as vertex-disjoint version of the problem. Note, however, that this construction does not preserve planarity, but it preserves strong planarity.) McCuaig [1] characterized all digraphs  $D$  with  $\nu(D) \leq 1$ ; the following follows immediately from his characterization (but there does not seem to be a direct proof).

**Theorem 2.2.** *For every digraph  $D$ , if  $\nu(D) \leq 1$ , then  $\tau(D) \leq 3$ .*

In general, Reed, Robertson, Seymour and the second author [5] proved the following.

**Theorem 2.3.** *There is a function  $f$  such that for every digraph  $D$*

$$\tau(D) \leq f(\nu(D)).$$

The function  $f$  from the proof of Theorem 2.3, albeit explicit, grows rather fast. The best lower bound of  $f(k) \geq \Omega(k \log k)$  was obtained by Noga Alon (unpublished). Finally, the undirected analogue of the problem we study is quite easy. It becomes much harder if we only require that the equality  $\nu = \tau$  hold for all *induced* subgraphs. This problem remains open. However, recently Ding and Zang [2] managed to solve the closely related problem of characterizing graphs for which it is required that a weighted version of the relation  $\nu = \tau$  holds. They gave a characterization by means of excluded induced subgraphs, and also gave a structural description of those graphs. We omit the precise statement.

### 3. STRONG 2-CONNECTIVITY

Let  $D$  be a digraph and  $\mathcal{C}$  a packing of circuits. We will say that  $\mathcal{C}$  *uses a vertex  $v$*  if there exists a circuit  $C$  in  $\mathcal{C}$  with  $v \in V(C)$ . Consider a digraph  $D$  that packs. Then some minimum transversal includes  $v$  if and only if  $\tau(D \setminus v) = \tau(D) - 1$ . As  $D$  packs,  $\nu(D \setminus v) = \tau(D \setminus v) = \tau(D) - 1 = \nu(D) - 1$ . But  $\nu(D \setminus v) = \nu(D) - 1$  if and only if every maximum packing uses  $v$ . Thus we have shown the following.

**Remark 3.1.** Let  $D$  be a digraph that packs. There exists a minimum transversal of  $D$  containing  $v$  if and only if every maximum packing of  $D$  uses  $v$ .

A digraph is *strongly connected* if for every pair of vertices  $u$  and  $v$  there is a path from  $u$  to  $v$ . A digraph  $D$  is *strongly  $k$ -connected* if for every  $T \subseteq V(D)$ , where  $|T| \leq k-1$ , the digraph  $D \setminus T$  is strongly connected. If  $D$  is not strongly connected, then  $V(D)$  can be partitioned into non-empty sets  $X_1, X_2$  such that no edge has tail in  $X_1$  and head in  $X_2$ . Let  $D_1 := D \setminus X_2$  and  $D_2 := D \setminus X_1$ . Then  $D$  is said to be a *0-sum* of  $D_1$  and  $D_2$ . Since every circuit of  $D$  is a circuit of precisely one of  $D_1$  and  $D_2$ , the following is straightforward.

**Proposition 3.2.** *Let  $D$  be the 0-sum of  $D_1$  and  $D_2$ . Then  $D_1$  and  $D_2$  pack if and only if so does  $D$ .*

Suppose  $D$  is strongly connected, but not strongly 2-connected; thus there is a vertex  $v$  such that  $D \setminus v$  is not strongly connected. Then there is a partition of  $V(D) - \{v\}$  into non-empty sets  $X_1, X_2$  such that all edges with endpoints in both  $X_1$  and  $X_2$  have tail in  $X_1$  and head in  $X_2$ . Let  $F$  be the set of all these edges. For  $i = 1, 2$  let  $D_i$  be the digraph obtained from  $D$  by deleting all edges with both endpoints in  $X_{3-i} \cup \{v\}$  and identifying all vertices of  $X_{3-i} \cup \{v\}$  into a vertex called  $v$ . Thus edges of  $F$  belong to both  $D_1$  and  $D_2$ ; in  $D_1$  they have head  $v$  and in  $D_2$  they have tail  $v$ . We say that  $D$  is a *1-sum* of  $D_1$  and  $D_2$ .

Let  $D$  be a digraph. We denote by  $D + uv$  the digraph obtained from  $D$  by adding the vertices  $u, v$  (if they are not vertices of  $D$ ) and an edge with tail  $u$  and head  $v$ . Let us stress that we add the edge even if  $D$  already has one or more edges with tail  $u$  and head  $v$ . We use  $D + \{u_1v_1, u_2v_2, \dots, u_kv_k\}$  to denote  $D + u_1v_1 + u_2v_2 + \dots + u_kv_k$ .

**Proposition 3.3.** *Let a strongly connected digraph  $D$  be the 1-sum of  $D_1$  and  $D_2$ . Then  $D_1$  and  $D_2$  pack if and only if so does  $D$ .*

*Proof.* Since  $D$  is strongly connected, the digraphs  $D_1$  and  $D_2$  are minors of  $D$ . So if  $D$  packs, so do  $D_1$  and  $D_2$ . Conversely, assume that  $D_1$  and  $D_2$  pack. Since every subdigraph of  $D$  is either a subdigraph of  $D_1$  or  $D_2$ , or a 0-sum or 1-sum of subdigraphs of  $D_1$  and  $D_2$ , it suffices to show that  $\tau(D) = \nu(D)$ . Let  $v, X_1, X_2$ , and  $F$  be as in the definition of 1-sum. For  $i = 1, 2$  let  $D'_i := D_i \setminus F$  and let  $\mathcal{C}_i$  be a maximum packing of  $D'_i$ . Suppose that, for  $i = 1, 2$ , every maximum packing of  $D'_i$  uses the vertex  $v$ . It follows from Remark 3.1 that there is a minimum transversal  $T_i$  of  $D'_i$  using  $v$ . Let  $\mathcal{C}$  be obtained from the union of  $\mathcal{C}_1, \mathcal{C}_2$  by removing the circuit of  $\mathcal{C}_1$  using  $v$ . Then  $\mathcal{C}$  is a packing of  $D$  and  $T := T_1 \cup T_2$  is a transversal of  $D$ . Moreover, both have cardinality  $\tau(D'_1) + \tau(D'_2) - 1$ , i.e.  $\tau(D) = \nu(D)$ . Thus we can assume one of  $\mathcal{C}_i$  ( $i = 1, 2$ ), say  $\mathcal{C}_1$ , does not use the vertex  $v$ .

For  $i = 1, 2$ , let  $F_i$  be the set of edges  $f$  of  $F$  such that  $\nu(D'_i + f) = \nu(D'_i)$ . Consider first the case where  $F_1 \cup F_2 = F$ . Suppose for a contradiction  $\nu(D'_i + F_i) > \nu(D'_i)$  and let  $\mathcal{F}$  be a corresponding packing. Clearly  $\mathcal{F}$  uses an edge of  $F_i$ . Moreover as all edges  $F$  of  $D_i$  share the endpoint  $v$ ,  $\mathcal{F}$  uses exactly one edge  $f$  of  $F_i$ . Hence  $\nu(D'_i + f) > \nu(D'_i)$ , a contradiction. Since (for  $i = 1, 2$ )  $D'_i + F_i$  packs it has a transversal  $T_i$  of cardinality  $\tau(D'_i)$ . As  $F_1 \cup F_2 = F$  this implies that  $T_1 \cup T_2$  is a transversal of  $D$ . Recall that  $\mathcal{C}_1$  does not use  $v$ ; thus  $\mathcal{C}_1 \cup \mathcal{C}_2$  is a packing of  $D$  and  $|T_1 \cup T_2| = \tau(D'_1) + \tau(D'_2) = |\mathcal{C}_1 \cup \mathcal{C}_2|$ , i.e.  $\tau(D) = \nu(D)$ .

Thus we may assume there exists  $f \in F - F_1 - F_2$ . Let  $\mathcal{C}'_i$  ( $i = 1, 2$ ) be a maximum packing of  $D'_i + f$ . Each  $\mathcal{C}'_i$  contains a circuit  $C_i$  using  $f$ . Define  $\mathcal{C}$  to be the collection of all circuits of  $\mathcal{C}'_1, \mathcal{C}'_2$  distinct from  $C_1$  and  $C_2$  as well as the circuit  $(C_1 \cup C_2) \setminus f$  of  $D$ . Let  $T_i$  ( $i = 1, 2$ ) be a minimum transversal of  $D'_i$ . Then  $T := T_1 \cup T_2 \cup \{v\}$  is a transversal of  $D$  and  $\mathcal{C}$  a packing of  $D$ . Moreover,  $|T| = \tau(D'_1) + \tau(D'_2) + 1 = |\mathcal{C}|$ , i.e.  $\tau(D) = \nu(D)$ , as desired.  $\square$

#### 4. STRONG PLANARITY

Let us recall that a digraph is *strongly planar* if it has a planar drawing such that for all vertices  $v$ , the edges with head  $v$  form an interval in the cyclic ordering of edges incident with  $v$  determined by the drawing.

**Corollary 4.1.** *Every strongly planar digraph packs.*

*Proof.* Let  $D$  be a strongly planar digraph with vertex set  $V$  and edge set  $E$ . We will show that  $D$  packs. Since subdigraphs of strongly planar digraphs are strongly planar it suffices to show  $\tau(D) = \nu(D)$ . Associate to every vertex  $v$  a new vertex  $v'$  and let  $V'$  be the set of all vertices  $v'$ . Associate with every edge  $e \in E(D)$  with tail  $u$  and head  $v$  a new edge  $e'$  with tail  $u'$  and head  $v$ . We define a digraph  $H$  as follows: the vertex-set of  $H$  is  $V \cup V'$ , and the edge-set of  $H$  consists of all the edges  $e'$  for  $e \in E(D)$  and all the edges of the form  $vv'$ , where  $v \in V(D)$ . Define weights  $w : E(H) \rightarrow Z_+$  as follows:  $w(e') = |E(H)|$  for all  $e \in E(D)$  and  $w(vv') = 1$  for all  $v \in V(H)$ . It is easy to see that the drawing associated to the strongly planar digraph  $D$  can be modified to induce a planar drawing of  $H$ . Now equation (2.1) states  $\tau(D) = \nu(D)$ , as desired.  $\square$

#### 5. BRACES

It will be convenient to reformulate our packing problem about digraphs to one about bipartite graphs. Let  $G$  be a bipartite graph with bipartition  $(A, B)$ , and let  $M$  be a perfect matching in  $G$ . We denote by  $D(G, M)$  the digraph obtained from  $G$  by directing every edge of  $G$  from  $A$  to  $B$ , and contracting every edge of  $M$ . When  $G'$  is a subgraph of  $G$  and  $M \cap E(G')$  is a perfect matching of  $G'$  we will abbreviate  $D(G', M \cap E(G'))$  by  $D(G', M)$ . It is clear that every digraph is isomorphic to  $D(G, M)$  for some bipartite graph  $G$  and some perfect matching  $M$ . Moreover, the following is straightforward.

**Remark 5.1.** Let  $G$  be a bipartite graph and let  $M$  be a perfect matching in  $G$ . If  $G$  is planar then  $D(G, M)$  is strongly planar.

A graph  $G$  is  $k$ -extendable, where  $k$  is an integer, if every matching in  $G$  of size at most  $k$  can be extended to a perfect matching. A 2-extendable bipartite graph is called a *brace*. The following straightforward relation between  $k$ -extendability and strong  $k$ -connectivity is very important.

**Proposition 5.2.** *Let  $G$  be a bipartite graph, let  $M$  be a perfect matching in  $G$ , and let  $k \geq 1$  be an integer. Then  $G$  is  $k$ -extendable if and only if  $D(G, M)$  is strongly  $k$ -connected.*

Let  $G$  be a bipartite graph and  $M$  a perfect matching in  $G$  such that  $D(G, M)$  is isomorphic to  $F_7$ . This defines  $G$  uniquely up to isomorphism, and the graph so defined is called the *Heawood graph*.

Let  $G$  be a bipartite graph, and let  $e$  be an edge of  $G$  with ends  $u, v$ . Consider a new graph obtained from  $G$  by replacing  $e$  by a path with an even number of vertices and ends  $u, v$  and otherwise disjoint from  $G$ . Let  $G'$  be obtained from  $G$  by repeating this operation, possibly for different edges of  $G$ . We say that  $G'$  is an *even subdivision* of  $G$ . The graph  $G'$  is clearly bipartite. Now let  $G, H$  be bipartite graphs. We say that  $G$  *contains*  $H$  if  $G$  has a subgraph  $L$  such that  $G \setminus V(L)$  has a perfect matching, and  $L$  is isomorphic to an even subdivision of  $H$ .

A circuit  $C$  in a bipartite graph  $G$  is *central* if  $G \setminus V(C)$  has a perfect matching. Let  $G_0$  be a bipartite graph, let  $C$  be a central circuit of  $G_0$  of length 4, and let  $G_1, G_2$  be subgraphs of  $G_0$  such that  $G_1 \cup G_2 = G_0$ ,  $G_1 \cap G_2 = C$ , and  $V(G_1) - V(G_2) \neq \emptyset \neq V(G_2) - V(G_1)$ . Let  $G$  be obtained from  $G_0$  by deleting all the edges of  $C$ . In this case we say that  $G$  is the *4-sum* of  $G_1$  or  $G_2$  along  $C$ . This is a slight departure from the definition in [6], but the class of simple graphs obtainable according to our definition is the same, because we allow parallel edges.

Let  $G_0$  be a bipartite graph, let  $C$  be a central circuit of  $G_0$  of length 4, and let  $G_1, G_2, G_3$  be three subgraphs of  $G_0$  such that:  $G_1 \cup G_2 \cup G_3 = G_0$  and for distinct integers  $i, j \in \{1, 2, 3\}$   $G_i \cap G_j = C$  and  $V(G_i) - V(G_j) \neq \emptyset$ . Let  $G$  be obtained from  $G_0$  by deleting all the edges of  $C$ . In these circumstances we say that  $G$  is a *trisum* of  $G_1, G_2, G_3$  along  $C$ . We will need the following result.

**Theorem 5.3.** *Let  $G$  be a brace, and let  $M$  be a perfect matching in  $G$ . Then the following conditions are equivalent.*

- (i)  $G$  does not contain  $K_{3,3}$ ,
- (ii) either  $G$  is isomorphic to the Heawood graph, or  $G$  can be obtained from planar braces by repeatedly applying the trisum operation,
- (iii) either  $G$  is isomorphic to the Heawood graph, or  $G$  can be obtained from planar braces by repeatedly applying the 4-sum operation,
- (iv)  $D(G, M)$  has no minor isomorphic to an odd double circuit.

*Proof.* The equivalence of (i), (ii) and (iii) is the main result of [1] and [6]. Condition (iv) is equivalent to the other three by results of Little [3] and Seymour and Thomassen [7]. See also [1].  $\square$

We will need the following small variation of Theorem 5.3.

**Theorem 5.4.** *Let  $G$  be a brace, and let  $M$  be a perfect matching in  $G$ . Then the following conditions are equivalent.*

- (i)  $G$  does not contain  $K_{3,3}$  or the Heawood graph,
- (ii)  $G$  can be obtained from planar braces by repeatedly applying the trisum operation,

- (iii)  $G$  can be obtained from planar braces by repeatedly applying the 4-sum operation,
- (iv)  $D(G, M)$  has no minor isomorphic to an odd double circuit or  $F_7$ .

*Proof.* This follows from Theorem 5.3 and the fact [6, Theorem 6.7] that if  $G$  contains the Heawood graph and is not isomorphic to it, then it contains  $K_{3,3}$ .  $\square$

We deduce the following information about a minimal counterexample to Theorem 1.1.

**Proposition 5.5.** *Let  $G$  be a bipartite graph and  $M$  a perfect matching in  $G$  such that the digraph  $D := D(G, M)$  has no minor isomorphic to an odd double circuit or  $F_7$ , and every digraph  $D'$  with  $|V(D')| + |E(D')| < |V(D)| + |E(D)|$  and no minor isomorphic to an odd double circuit or  $F_7$  packs. If  $\nu(D) < \tau(D)$ , then  $G$  is a brace and there exist braces  $G_1, G_2, G_3$  such that  $G$  is a trisum of  $G_1, G_2, G_3$  along a circuit  $C$ , and each of  $G_1, G_2, G_3$  can be obtained from planar braces by repeatedly applying the trisum operation.*

*Proof.* It follows from Propositions 3.2 and 3.3 that  $D$  is strongly 2-connected. Thus  $G$  is a brace by Proposition 5.2. By Corollary 4.1 the digraph  $D$  is not strongly planar, and hence  $G$  is not planar by Remark 5.1. By Theorem 5.4 the graph  $G$  is obtained from planar braces by trisumming. Since  $G$  itself is not planar, there is at least one trisum operation involved in the construction of  $G$ , and hence  $G_1, G_2, G_3$  and  $C$  exist, as desired.  $\square$

In the next three sections we will prove the following result.

**Proposition 5.6.** *Let  $G, M$ , and  $D$  be as in Proposition 5.5. Then  $\nu(D) = \tau(D)$ .*

*Proof of Theorem 1.1* (assuming Proposition 5.6). We have already established the “only if” part. To prove the “if” part let  $D$  be a digraph with no minor isomorphic to an odd double circuit or  $F_7$  such that every digraph  $D'$  with  $|V(D')| + |E(D')| < |V(D)| + |E(D)|$  and no minor isomorphic to an odd double circuit or  $F_7$  packs. By Proposition 5.6 we have that  $\nu(D) = \tau(D)$ , and hence  $D$  packs, as desired.  $\square$

We now deduce the structural characterization of digraphs that pack.

**Corollary 5.7.** *A digraph packs if and only if it can be obtained from strongly 2-connected digraphs that pack by means of 0- and 1-sums. A strongly 2-connected digraph packs if and only if it is isomorphic to  $D(G, M)$  for some brace  $G$  and some perfect matching  $M$  in  $G$ , where  $G$  is obtained from planar braces by repeatedly applying the trisum operation.*

*Proof.* The first statement follows from Propositions 3.2 and 3.3. For the second statement let  $D$  be a strongly 2-connected digraph. Assume first that  $D$  packs, and let  $G$  be a bipartite graph and  $M$  a perfect matching such that  $D$  is isomorphic to  $D(G, M)$ . By Proposition 5.2 the graph  $G$  is a brace. By Theorem 1.1 the digraph  $D$  has no minor isomorphic to an odd double circuit or  $F_7$ , and so by Theorem 5.4  $G$  is as desired. The converse implication follows along the same lines.  $\square$

As we alluded to in the Introduction, the second part of Corollary 5.7 can be stated purely in terms of “sums” of digraphs. However, three kinds of sums are needed (see [6]), as opposed to just one. Therefore the formulation we chose is clearer, despite the disadvantage that it involves the transition from a digraph to a bipartite graph.

Finally, we deduce a corollary about packing  $M$ -alternating circuits in bipartite graphs. Let  $G$  be a bipartite graph, and let  $M$  be a perfect matching in  $G$ . A circuit  $C$  in  $G$  is  $M$ -alternating if  $2|E(C) \cap M| = |E(C)|$ . Let  $\nu(G, M)$  denote the maximum number of pairwise disjoint  $M$ -alternating circuits, and let  $\tau(G, M)$  denote the minimum number of edges whose deletion leaves no  $M$ -alternating circuit. It is clear that  $\nu(G, M) = \nu(D(G, M))$  and  $\tau(G, M) = \tau(D(G, M))$ . Thus we have the following corollary.

**Corollary 5.8.** *Let  $G$  be a brace, and let  $M$  be a perfect matching in  $G$ . Then the following three conditions are equivalent.*

- (i)  $G$  does not contain  $K_{3,3}$  or the Heawood graph,
- (ii)  $\tau(G', M') = \nu(G', M')$  for every subgraph  $G'$  of  $G$  such that  $M' = M \cap E(G')$  is a perfect matching in  $G'$ , and
- (iii)  $G$  can be obtained from planar braces by repeatedly applying the trisum operation.

In fact, the equivalence of (i) and (ii) holds for all bipartite graphs, not just braces. We conclude this section with a lemma that will be needed later. The lemma follows immediately from [6, Theorem 8.2]. We say that a graph is a *cube* if it is isomorphic to the 1-skeleton of the 3-dimensional cube. Thus every cube has 8 vertices and 12 edges.

**Lemma 5.9.** *Let  $G$  be a trisum of  $G_1, G_2, G_3$  along  $C$ , where the graphs  $G_1, G_2, G_3$  are obtained from planar braces by repeatedly applying the trisum operation. Then for  $i = 1, 2, 3$  we have  $|E(G_i)| \geq 12$  with equality if and only if  $G_i$  is a cube.*

## 6. TRISUM-PART I

Let  $D, G, M, G_1, G_2, G_3, C$  be as in Proposition 5.5. For  $i = 1, 2, 3$  let  $M'_i$  be the set of edges  $M \cap E(G_i)$  with at least one end not in  $V(C)$ , let  $M_0$  be the set of edges of  $C$  that are parallel to a member of  $M$ , and let  $M_i = M'_i \cup M_0$ . We say that  $M_i$  is the *imprint* of  $M$  on  $G_i$ .

**Proposition 6.1.** *Let a bipartite graph  $G$  be a 4-sum of  $G_1$  and  $G_2$  along  $C$ , let  $M$  be a perfect matching in  $G$  such that some two members of  $M$  have both ends in  $V(C)$ , let  $D = D(G, M)$ , and for  $i = 1, 2$  let  $M_i$  be the imprint of  $M$  on  $G_i$ . If both  $D(G_1, M_1)$  and  $D(G_2, M_2)$  pack, then  $\nu(D) = \tau(D)$ .*

*Proof.* For  $i = 1, 2$  let  $D_i = D(G_i, M_i)$ . Then  $|V(D_1) \cap V(D_2)| = 2$ ; let  $V(D_1) \cap V(D_2) = \{u_1, u_2\}$ . Moreover,  $E(D_1) \cap E(D_2) = \{e_1, e_2\}$ , where  $e_1$  has head  $u_2$  and tail  $u_1$ , and  $e_2$  has head  $u_1$  and tail  $u_2$ . For  $i = 1, 2$  let  $D'_i = D_i \setminus \{e_1, e_2\}$ . □

*Claim 1.* For each  $D'_i$  ( $i = 1, 2$ ) one of the following holds:

- (1) There exists a maximum packing not using any of  $u_1$  or  $u_2$ . Every minimum transversal does not contain any of  $u_1$  or  $u_2$ .
- (2) For some  $k \in \{1, 2\}$  the following holds: all maximum packings use  $u_k$ , there exists a maximum packing not using  $u_{3-k}$ , and there exists a minimum transversal which contains  $u_k$  but not  $u_{3-k}$ .
- (3) There exists a maximum packing using both  $u_1$  and  $u_2$ . There exists a minimum transversal using  $u_1$  and a minimum transversal using  $u_2$ . Moreover, either: (a) there is a transversal containing both  $u_1, u_2$ ; or (b) there is a packing of size  $\tau(D'_i) - 1$  not using  $u_1$  or  $u_2$ .

*Proof of Claim:* Observe that for (1)-(3) the statements about transversals (except for the last sentence) follow from the statements about maximum packings and Remark 3.1. Suppose (1) does not hold; then every maximum packing of  $D'_i$  uses one of  $u_1, u_2$ . In particular  $\nu(D_i) = \nu(D'_i)$ . Suppose for a contradiction there exists a maximum packing  $\mathcal{C}'_i$  of  $D'_i$  not using  $u_1$  and a maximum packing  $\mathcal{C}''_i$  of  $D'_i$  not using  $u_2$ . Remark 3.1 implies that no minimum transversal of  $D'_i$  contains  $u_1$  or  $u_2$ . Since  $\{e_1, e_2\}$  is the edge-set of a circuit of  $D_i$  this implies  $\tau(D_i) > \tau(D'_i)$ , a contradiction since  $D_i$  packs. Thus for some  $k \in \{1, 2\}$  every maximum packing of  $D'_i$  uses  $u_k$ . If (2) does not hold, then all maximum packings use  $u_{3-k}$ . If (3)(a) does not hold, no minimum transversal of  $D'_i$  uses both  $u_1$  and  $u_2$ . This implies  $\tau(D'_i \setminus \{u_1, u_2\}) \geq \tau(D'_i) - 1$ . Since  $D_i$  packs (3)(b) must hold.  $\diamond$

*Claim 2.* For  $i = 1, 2$ , let  $T_i$  be a minimum transversal of  $D'_i$  and let  $\mathcal{C}_i$  be a maximum packing of  $D'_i$ . We can assume one of the following holds:

- (a) There exists  $k \in \{1, 2\}$  such that  $\mathcal{C}_1$  and  $\mathcal{C}_2$  use  $u_k$  but  $u_k \notin T_1 \cap T_2$ .
- (b)  $\{u_1, u_2\} \cap (T_1 \cup T_2) = \emptyset$ .

*Proof of Claim:* Let  $T := T_1 \cup T_2$  and let  $\mathcal{C}$  be an inclusion-wise maximal packing in  $\mathcal{C}_1 \cup \mathcal{C}_2$ . If (a) does not hold, then  $|T| \leq |\mathcal{C}|$ . If (b) does not hold, then  $\{u_1, u_2\} \cap T \neq \emptyset$ ; thus  $T$  is a transversal of  $D$ . It follows that  $\tau(D) = \nu(D)$ , as desired. Thus we may assume that (a) or (b) holds.  $\diamond$

We can assume, because of Claim 1 and Claim 2, that  $D_1, D_2$  either both satisfy condition (1) of Claim 1, or they both satisfy condition (3) of Claim 1 and one of  $D'_1, D'_2$ , say  $D'_1$ , satisfies (3)(b). Consider the latter case first. Let  $T_1$  (resp.  $T_2$ ) be a minimum transversal of  $D'_1$  (resp.  $D'_2$ ) using  $u_1$ . Let  $T := T_1 \cup T_2$ . Let  $\mathcal{C}_1$  be a packing of  $D'_1 \setminus \{u_1, u_2\}$  of size  $\tau(D'_1) - 1$  and let  $\mathcal{C}_2$  be a maximum packing of  $D'_2$ . Clearly  $\mathcal{C} := \mathcal{C}_1 \cup \mathcal{C}_2$  is a packing in  $D$ . Since  $|T_1 \cup T_2| = \tau(D'_1) + \tau(D'_2) - 1$  and  $|\mathcal{C}| = \tau(D'_1) - 1 + \tau(D'_2)$ , we have  $\tau(D) = \nu(D)$ .

Thus we may assume that both  $D'_1, D'_2$  satisfy (1). For  $i = 1, 2$ , let  $\mathcal{C}_i$  be a maximum packing of  $D_i$ . Suppose there is  $k \in \{1, 2\}$  such that for  $i = 1, 2$ ,  $\tau(D'_i + u_k u_{3-k}) = \tau(D'_i)$  and let  $T_i$  be the corresponding minimum transversal. Then  $T_i$  intersects all  $u_{3-k} u_k$ -paths of  $D_i$ . Hence  $T := T_1 \cup T_2$  is a transversal of  $D$ . Moreover,  $|T| = \tau(D'_1) + \tau(D'_2) = |\mathcal{C}_1 \cup \mathcal{C}_2|$ , i.e.  $\tau(D) = \nu(D)$ . Thus we can assume there

is for  $k = 1, 2$  an index  $t(k) \in \{1, 2\}$  such that  $\tau(D'_{t(k)} + u_k u_{3-k}) > \tau(D'_{t(k)})$ . Since  $D_1, D_2$  pack  $\nu(D'_{t(k)} + u_k u_{3-k}) > \tau(D'_{t(k)})$ ; let  $\mathcal{F}_{t(k)}$  be the corresponding packing. Some circuit  $C_{t(k)}$  of  $\mathcal{F}_{t(k)}$  is of the form  $P_{t(k)} + u_k u_{3-k}$  where  $P_{t(k)}$  is a  $u_{3-k} u_k$ -path. For  $i = 1, 2$  let  $T_i$  be a minimum transversal of  $D'_i$ . Note that  $T_i$  does not intersect  $P_{t(k)}$ . Observe that we cannot have  $t(1) = t(2) = i \in \{1, 2\}$ , for otherwise there exist both a  $u_1 u_2$ - and  $u_2 u_1$ -paths in  $D'_i$  which are not intersected by  $T_i$  and hence  $T_i$  does not intersect all circuits of  $D'_i$ , a contradiction. Thus we can assume  $t(1) = 1$  and  $t(2) = 2$ . Let  $\mathcal{C} := \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{P_1 \cup P_2\} - \{C_1, C_2\}$ . Then  $\mathcal{C}$  is a packing of  $D$  and  $T := T_1 \cup T_2 \cup \{u_1\}$  is a transversal of  $D$ . Moreover,  $|T| = \tau(D'_1) + \tau(D'_2) + 1 = |\mathcal{C}|$ , i.e.  $\tau(D) = \nu(D)$ , as desired.

**Proposition 6.2.** *Let  $G, M, D$ , where  $\nu(D) < \tau(D)$ , and  $G_1, G_2, G_3, C$  be as in Proposition 5.5. Then at most one member of  $M$  has both ends in  $V(C)$ .*

*Proof.* Suppose for a contradiction that two members of  $M$  have both ends in  $V(C)$ . For  $i = 1, 2, 3$  let  $M_i$  be the imprint of  $M$  on  $G_i$ . The graphs  $G_1$  and  $G_2 \cup G_3$  are obtained from planar braces by repeatedly applying the 4-sum operation, and hence the digraphs  $D_1 = D(G_1, M_1)$  and  $D_2 = D(G_2 \cup G_3, M_2 \cup M_3)$  have no minor isomorphic to an odd double circuit or  $F_7$  by Theorem 5.4. Thus  $D_1$  and  $D_2$  pack, and hence by Proposition 6.1  $\nu(D) = \tau(D)$ , a contradiction.  $\square$

## 7. TRISUM-PART II

**Lemma 7.1.** *Let  $D_1, D_2$  be digraphs with  $V(D_1) \cap V(D_2) = \{u_1, u_2, u_3\}$  and  $E(D_1) \cap E(D_2) = \emptyset$ . Let  $D = D_1 \cup D_2$ ,  $a \notin V(D)$ ,  $E_1 = \{u_1 u_2, u_1 u_3, u_2 u_3\}$ ,  $E_2 = \{u_2 u_1, u_3 u_1, u_3 u_2\}$ ,  $Z_1 = \{a u_2, u_2 a, u_1 a, a u_3\}$ , and  $Z_2 = \{a u_2, u_2 a, a u_1, u_3 a\}$ . Assume that*

- (a) *if, for  $i = 1, 2$ ,  $C_i$  is a circuit of  $D_i$ , then  $V(C_1) \cap V(C_2) \subseteq \{u_2\}$ ,*
- (b) *if  $C$  is a circuit of  $D$  that uses edges of both  $D_1$  and  $D_2$ , then  $C = P_1 \cup P_2$  and there exist integers  $i, j \in \{1, 2, 3\}$  such that  $i < j$  and  $P_1$  is a  $u_j u_i$ -path of  $D_1$  and  $P_2$  is a  $u_i u_j$ -path of  $D_2$ , and*
- (c) *there exist integers  $i, j$  such that  $\{i, j\} = \{1, 2\}$ ,  $D_i + E_i$  packs and is strongly 2-connected, and  $D_j + Z_j$  packs.*

Then  $\tau(D) = \nu(D)$ .

*Proof.* Suppose for a contradiction that  $\nu(D) < \tau(D)$ .

*Claim 1.* The digraph  $D$  has a packing of size  $\nu(D_1) + \nu(D_2) - 1$ .

*Proof of Claim:* Clearly  $\nu(D_2 \setminus u_2) \geq \nu(D_2) - 1$ , and so the union of any maximum packing of  $D_1$  with any packing of  $D_2 \setminus u_2$  of size  $\nu(D_2) - 1$  is as desired by (a). This proves Claim 1.  $\diamond$

*Claim 2.* The digraph  $D$  has a transversal of size at most  $\tau(D_1) + \tau(D_2) + 1$ .

*Proof of Claim:* By (c) we may assume from the symmetry that  $D_1 + E_1$  packs. Clearly  $\nu(D_1 + E_1) \leq \nu(D_1) + 1$ . Thus  $\tau(D_1 + E_1) \leq \tau(D_1) + 1$ . Let  $T_1$  be a transversal of  $D_1 + E_1$  of size at most  $\tau(D_1) + 1$ , and let  $T_2$  be a transversal of  $D_2$  of size  $\tau(D_2)$ . By (b)  $T_1 \cup T_2$  is a transversal of  $D$ , as required. This proves Claim 2.  $\diamond$

For  $i = 1, 2$  let  $F_i$  be the set of all edges  $f \in E_i$  such that  $\nu(D_i + f) = \nu(D_i)$ .

*Claim 3.* For  $i = 1, 2$ ,  $\nu(D_i + F_i) = \nu(D_i)$ .

*Proof of Claim:* If  $\nu(D_i + F_i) > \nu(D_i)$ , then, since every edge of  $E_i$  has both ends in  $\{u_1, u_2, u_3\}$ , we deduce that  $\nu(D_i + f) > \nu(D_i)$  for some  $f \in F_i$ , a contradiction. This proves Claim 3.  $\diamond$

*Claim 4.* Let  $i, j \in \{1, 2, 3\}$  be such that  $i < j$ , and let  $D'$  be a subdigraph of  $D_1$ . If  $\nu(D' + u_i u_j) > \nu(D')$ , then there exist a maximum packing  $\mathcal{C}$  of  $D'$  and a path  $P$  in  $D'$  from  $u_j$  to  $u_i$  such that every member of  $\mathcal{C}$  is disjoint from  $P$ .

*Proof of Claim:* Let  $\mathcal{C}'$  be a maximum packing of  $D' + u_i u_j$ . Since  $\mathcal{C}'$  is not a packing of  $D'$ , some member of  $\mathcal{C}'$ , say  $C$ , uses the edge  $u_i u_j$ . Thus  $\mathcal{C}' - \{C\}$  and  $C \setminus u_i u_j$  satisfy the conclusion of the claim.  $\diamond$

*Claim 5.* If  $D_1 + E_1$  packs and every maximum packing of  $D_1 + u_1 u_3$  uses  $u_2$ , then every maximum packing of  $D_1$  uses  $u_2$ .

*Proof of Claim:* Suppose for a contradiction that every maximum packing of  $D_1 + u_1 u_3$  uses  $u_2$ , but some maximum packing of  $D_1$  does not use  $u_2$ . Then  $\nu(D_1 + u_1 u_3) > \nu(D_1)$ . By Claim 4 applied to  $D' = D_1$  there exist a maximum packing  $\mathcal{C}$  of  $D_1$  and a path  $P$  of  $D_1$  from  $u_3$  to  $u_1$  such that  $P$  is disjoint from every member of  $\mathcal{C}$ . Let  $D'$  be a subdigraph of  $D_1$  such that

- ( $\alpha$ )  $D'$  includes  $P$  and every member of  $\mathcal{C}$ ,
- ( $\beta$ )  $D'$  includes every member of some maximum packing of  $D_1$  that does not use  $u_2$ , and
- ( $\gamma$ ) subject to ( $\alpha$ ) and ( $\beta$ ),  $E(D')$  is minimal.

By ( $\alpha$ )  $\nu(D') = \nu(D_1)$ . We claim that  $\nu(D' + u_1 u_2 + u_2 u_3) > \nu(D')$ . To prove this claim suppose for a contradiction that equality holds. Since  $D_1 + E_1$  packs we deduce that  $\tau(D' + u_1 u_2 + u_2 u_3) = \nu(D')$ . Let  $T$  be a transversal of  $D' + u_1 u_2 + u_2 u_3$  of size  $\nu(D')$ . From ( $\beta$ ) we deduce that  $u_2 \notin T$ , but then it follows that  $T$  is a transversal of  $D' + u_1 u_3$ , contrary to ( $\alpha$ ). This proves that  $\nu(D' + u_1 u_2 + u_2 u_3) > \nu(D')$ . Since  $u_1 u_2, u_2 u_3$  have a common end  $u_2$  and every maximum packing of  $D_1 + u_1 u_3$  uses  $u_2$ , we deduce that either  $\nu(D' + u_1 u_2) > \nu(D')$  or  $\nu(D' + u_2 u_3) > \nu(D')$ , and so we may assume the former. By Claim 4 there exists a maximum packing  $\mathcal{C}'$  of  $D'$  and a path  $P'$  in  $D'$  from  $u_2$  to  $u_1$  disjoint from every member of  $\mathcal{C}'$ . Since the union of  $P$  and all members of  $\mathcal{C}$  does not include a path from  $u_2$  to  $u_1$ , there exists an edge  $e \in E(P')$  that does not belong to  $P$  or any member of  $\mathcal{C}$ . Thus  $D' \setminus e$  satisfies ( $\alpha$ ). But  $D' \setminus e$  includes every member of  $\mathcal{C}'$ , and hence it also satisfies ( $\beta$ ), contrary to ( $\gamma$ ). This proves Claim 5.  $\diamond$

*Claim 6.* Let  $i, j \in \{1, 2, 3\}$  with  $i < j$ . If  $u_i u_j \notin F_1$ , then  $u_j u_i \in F_2$ .

*Proof of Claim:* Suppose for a contradiction that  $u_i u_j \notin F_1$  and  $u_j u_i \notin F_2$ . Let  $\mathcal{C}_1$  be a packing of  $D_1 + u_i u_j$  of size  $\nu(D_1) + 1$ , and let  $\mathcal{C}_2$  be a packing of  $D_2 + u_j u_i$  of size  $\nu(D_2) + 1$ . Then  $\mathcal{C}_1$  includes a circuit  $C_1$  containing  $u_i u_j$ , and  $\mathcal{C}_2$  includes a circuit  $C_2$  containing  $u_j u_i$ . Let  $C$  be the circuit  $(C_1 \setminus u_i u_j) \cup (C_2 \setminus u_j u_i)$ . If one of  $\mathcal{C}_1 - \{C_1\}, \mathcal{C}_2 - \{C_2\}$  does not use  $u_2$ , then  $\mathcal{C} := (\mathcal{C}_1 - \{C_1\}) \cup (\mathcal{C}_2 - \{C_2\}) \cup \{C\}$  is a packing of  $D$  of size  $\nu(D_1) + \nu(D_2) + 1$  by (a). Then because of Claim 2,  $\tau(D) = \nu(D)$  packs, a contradiction. Thus we may assume that both  $\mathcal{C}_1 - \{C_1\}, \mathcal{C}_2 - \{C_2\}$  use  $u_2$  for all choices of  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . Thus  $i = 1$  and  $j = 3$ , and every maximum packing of  $D_1 + u_1 u_3$  or  $D_2 + u_3 u_1$  uses  $u_2$ . By (c) we may assume that  $D_1 + E_1$  packs, and hence by Claim 5 every maximum packing of  $D_1$  uses  $u_2$ . By Remark 3.1  $D_1$  has transversal  $T_1$  of size  $\nu(D_1)$  with  $u_2 \in T_1$ , and  $D_2 + u_3 u_1$  has a transversal  $T_2$  of size  $\nu(D_2) + 1$  with  $u_2 \in T_2$ . By (b)  $T_1 \cup T_2$  is a transversal of  $D$  of size  $\nu(D_1) + \nu(D_2)$ . On the other hand, by deleting one of the circuits of  $\mathcal{C}$  that contain  $u_2$  we obtain a packing of  $D$  of size  $\nu(D_1) + \nu(D_2)$ . Thus  $\nu(D) = \tau(D)$ , a contradiction. This proves Claim 6.  $\diamond$

*Claim 7.* The digraph  $D$  has a packing of size  $\nu(D_1) + \nu(D_2)$ .

*Proof of Claim:* Suppose not. Then for  $i = 1, 2$  every maximum packing of  $D_i$  uses  $u_2$ , for otherwise the union of a maximum packing in  $D_i$  that does not use  $u_2$  with any maximum packing of  $D_{3-i}$  is as desired. By Remark 3.1 the digraph  $D_i$  has a transversal  $T_i$  of size  $\tau(D_i)$  with  $u_2 \in T_i$ . Let us assume first that  $\nu(D_1 + u_1 u_3) > \nu(D_1)$ . Then  $\nu(D_2 + u_3 u_1) = \nu(D_2)$  by Claim 6. The graph  $D_2 + u_3 u_1$  packs (because by (c)  $D_2 + E_2$  or  $D_2 + Z_2$  packs), and so  $\nu(D_2 + u_3 u_1 \setminus u_2) = \tau(D_2 + u_3 u_1 \setminus u_2)$ . If  $\nu(D_2 + u_3 u_1 \setminus u_2) = \nu(D_2)$ , then let  $\mathcal{C}_1$  be a maximum packing in  $D_1 + u_1 u_3$  and let  $\mathcal{C}_2$  be a maximum packing in  $\nu(D_2 + u_3 u_1 \setminus u_2)$ . Then some circuit of  $\mathcal{C}_1$  uses the edge  $u_1 u_3$  (because  $\nu(D_1 + u_1 u_3) > \nu(D_1)$ ), and some circuit of  $\mathcal{C}_2$  uses the edge  $u_3 u_1$  (because every maximum packing of  $D_2$  uses  $u_2$ ). Thus  $\mathcal{C}_1$  and  $\mathcal{C}_2$  can be combined as in the proof of Claim 6 to produce the desired packing of  $D$ . Thus we may assume that  $\nu(D_2 + u_3 u_1 \setminus u_2) < \nu(D_2)$ . Let  $T'_2$  be a transversal in  $D_2 + u_3 u_1 \setminus u_2$  of size  $\nu(D_2) - 1$ ; then  $T_1 \cup T'_2$  is a transversal in  $D$  by (b), and its size is  $\nu(D_1) + \nu(D_2) - 1$ , contrary to Claim 1. This completes the case when  $\nu(D_1 + u_1 u_3) > \nu(D_1)$ .

Thus we may assume that  $\nu(D_1 + u_1 u_3) = \nu(D_1)$  and  $\nu(D_2 + u_3 u_1) = \nu(D_2)$ . From the symmetry and (c) we may assume that  $D_2 + Z_2$  packs. Since every maximum packing of  $D_2$  uses  $u_2$ , and  $\nu(D_2 + u_3 u_1) = \nu(D_2)$ , we see that  $\nu(D_2 + Z_2) = \nu(D_2)$ . Since  $D_2 + Z_2$  packs, there exists a transversal  $T''_2$  of  $D_2 + Z_2$  of size  $\tau(D_2)$ . Since  $T''_2 \cap V(D_2)$  is a transversal of  $D_2$ , we deduce that  $a \notin T''_2$ , and hence  $u_2 \in T''_2$ , because  $T''_2$  intersects the circuit of  $D_2 + Z_2$  with vertex-set  $\{a, u_2\}$ . Thus  $T''_2$  is a transversal of  $D_2 + u_3 u_1$  with  $u_2 \in T''_2$ , and so  $T_1 \cup T''_2$  is a transversal of  $D$  by (b). Moreover,  $|T_1 \cup T''_2| = \tau(D_1) + \tau(D_2) - 1$ , contrary to Claim 1. This completes the proof of Claim 7.  $\diamond$

We are now ready to complete the proof of the lemma. We claim that one of  $D_1 + F_1, D_2 + F_2$  does not pack. Indeed, if both of them pack, then by Claim 3 the digraph  $D_i + F_i$  has a transversal of size  $\nu(D_i)$ , and

the union of those sets is a transversal in  $D$  by (b) of size  $\nu(D_1) + \nu(D_2)$ , contrary to Claim 7. Thus we may assume that  $D_2 + F_2$  does not pack.

By (c) the digraph  $D_1 + E_1$  packs and is strongly 2-connected, and  $D_2 + Z_2$  packs. To motivate the next step, notice that since  $D_2 + Z_2$  packs, but  $D_2 + F_2$  does not, we have  $u_2u_1, u_3u_2 \in F_2$ . Since  $D_1 + E_1$  packs, so does  $D_1 + F_1$ , and hence by Claim 3 there exists a transversal  $T_1$  in  $D_1 + F_1$  of size  $\tau(D_1)$ .

We claim that the set  $T_1$  is a transversal in  $D_1 + F_1 + u_1u_2$  or  $D_1 + F_1 + u_2u_3$ . To prove this claim suppose for a contradiction that this is not the case. We deduce that there exist a  $u_2u_1$ -path  $P_1$  and a  $u_3u_2$ -path  $P_2$  in  $D_1$ , both disjoint from  $T_1$ . Since  $T_1$  intersects every circuit of  $D_1$ , it follows that  $V(P_1) \cap V(P_2) = \{u_2\}$ . Since  $D_1 + E_1$  is strongly 2-connected, there exists a path  $Q$  in  $D_1$  from  $V(P_2) - \{u_2\}$  to  $V(P_1) - \{u_2\}$ ; we may assume that no interior vertex of  $Q$  belongs to  $V(P_1) \cup V(P_2)$ . Let  $H$  be the digraph  $P_1 \cup P_2 \cup Q + E_1$ ; then  $\nu(H) = 1 < 2 = \tau(H)$ , contrary to the fact that  $D_1 + E_1$  packs. This proves our claim that  $T_1$  is a transversal in  $D_1 + F_1 + u_1u_2$  or  $D_1 + F_1 + u_2u_3$ .

From the symmetry we may assume that  $T_1$  is a transversal in  $D_1 + F_1 + u_1u_2$ . Let  $F'_2 = F_2 - \{u_1u_2\}$ . Since  $D_2 + Z_2$  packs, so does its minor  $D_2 + F'_2$ , and so by Claim 3 the digraph  $D_2 + F'_2$  has a transversal  $T_2$  of size  $\tau(D_2)$ . By (b) the set  $T_1 \cup T_2$  is a transversal in  $D$ , and its size is  $\tau(D_1) + \tau(D_2)$ , contrary to Claim 7.  $\square$

**Proposition 7.2.** *Let  $G, M, D$ , where  $\nu(D) < \tau(D)$ , and  $G_1, G_2, G_3, C$  be as in Proposition 5.5. Then either none or exactly two edges of  $M$  have both ends in  $V(C)$ .*

*Proof.* Let  $A, B$  denote a bipartition of  $G$ . Let  $v_1, v'_2, v_2, v_3$  be the vertices of  $C$  (in that order), where  $v_1, v_2 \in A$ . For  $i = 1, 2, 3$  let  $m_i$  be the edge of  $M$  incident with  $v_i$ . Suppose for a contradiction that  $m_2$  is the only member of  $M$  with both ends in  $V(C)$ . We may assume that  $m_2$  is incident with  $v'_2$ . Thus  $m_1, m_3$  are distinct and are incident with vertices not on  $C$ . We may also assume that  $m_1, m_3 \in E(G_1) \cup E(G_2)$ . For  $i = 1, 2, 3$  let  $M_i$  be the imprint of  $M$  on  $G_i$  (see the paragraph prior to Proposition 6.1 for a definition). Let  $J_1 := D(G_1 \cup G_2, M_1 \cup M_2)$ , let  $Q$  be a cube such that  $C$  is a subgraph of  $Q$  and otherwise  $Q$  is disjoint from  $G_3$ , and let  $J_2 := D(G_3 \cup Q, M'_3)$ , where  $M'_3$  is a perfect matching of  $G_3 \cup Q$  with  $M_3 \subseteq M'_3$  that does not use an edge joining  $v_1$  and  $v_3$ . Such a matching is unique, and it has a unique element, say  $m_0$ , not incident with a vertex of  $G_3$ . Let  $a$  denote the vertex of  $J_2$  that results from contracting  $m_0$ , and in both  $J_1, J_2$  let  $u_1, u_2, u_3$  denote the vertices that result from contracting the edges incident with  $v_1, v_2, v_3$ , respectively.

Let  $D_1$  be obtained from  $J_1$  by deleting the edges of  $C$ , and let  $D_2$  be obtained from  $J_2$  by deleting the vertex  $a$  and edges of  $Q \cup C$ . We wish to apply Lemma 7.1 to the digraphs  $D_1$  and  $D_2$ . Since  $u_1$  is a source and  $u_3$  is a sink of  $D_2$ , we see immediately that (a) and (b) of that lemma hold. We will show that  $i = 1$  and  $j = 2$  satisfy (c). Since  $G_1$  and  $G_2$  are braces, so is  $G_1 \cup G_2$ , and thus  $J_1$  is strongly 2-connected by Proposition 5.2. To show that  $D_1 + E_1$  packs we first notice that  $D_1 + E_1$  is isomorphic to  $J_1$ . But  $G_1 \cup G_2$  is obtained from planar braces by repeatedly applying the trisum operation, and hence  $J_1$  has no odd double circuit or  $F_7$

minor by Theorem 5.4. Moreover,  $|V(J_1)| + |E(J_1)| = |E(G_1 \cup G_2)| < |E(G)| = |V(D)| + |E(D)|$  by Lemma 5.9, and hence  $J_1$  (and thus  $D_1 + E_1$ ) pack by the hypothesis of Proposition 5.5. Finally,  $D_2 + Z_2$  is a subdigraph of  $J_2$ , and hence it packs, by the argument of this paragraph. Thus  $\nu(D) = \tau(D)$  by Proposition 7.1, a contradiction.  $\square$

## 8. TRISUM-PART III

Let  $D_1, D_2$  be edge-disjoint subdigraphs of a digraph  $D$ , let  $X \subseteq V(D_1) \cap V(D_2)$ , and let  $C$  be a circuit of  $D$ . We say that  $C$  passes from  $D_1$  to  $D_2$  through  $X$  if there is no vertex  $v \in V(D) - X$  such that the edge of  $C$  with head  $v$  belongs to  $D_1$  and the edge of  $C$  with tail  $v$  belongs to  $D_2$ .

**Lemma 8.1.** *Let  $D_1$  and  $D_2$  be digraphs with  $V(D_1) \cap V(D_2) = \{u_1, u_2, u_3, u_4\}$  and  $E(D_1) \cap E(D_2) = \emptyset$ . Let  $D = D_1 \cup D_2$ , let  $E_1 = \{u_1u_2, u_3u_2, u_3u_4, u_1u_4\}$ , and let  $E_2 = \{u_2u_1, u_2u_3, u_4u_3, u_4u_1\}$ . Assume that*

- (1) for  $i = 1, 2$ ,  $D_i + E_i$  packs,
- (2) every circuit of  $D_1$  is disjoint from every circuit of  $D_2$ ,
- (3) every circuit of  $D$  passes from  $D_1$  to  $D_2$  through  $\{u_1, u_3\}$ , and it passes from  $D_2$  to  $D_1$  through  $\{u_2, u_4\}$ .

Moreover, assume that for every pair  $e_1, e_2 \in E_i$  of independent edges one of the following holds:

- (a)  $\nu(D_i + e_1 + e_2) \geq \nu(D_i) + 2$ ,
- (b)  $\tau(D_i + e_1) = \tau(D_i)$ , or
- (c)  $\tau(D_i + e_2) = \tau(D_i)$ .

Then  $\tau(D) = \nu(D)$ .

*Proof.* Suppose for a contradiction that  $\nu(D) < \tau(D)$ .  $\square$

*Claim 1.* Let  $i = 1$  or  $i = 2$ , and let  $F \subseteq E_i$ . Then one of the following holds:

- (i) There is an edge  $e \in F$  such that  $\nu(D_i + e) > \nu(D_i)$ ,
- (ii)  $\tau(D_i + F) = \tau(D_i)$ , or
- (iii) there exist independent edges  $e_1, e_2 \in F$  such that

$$\nu(D_i) = \nu(D_i + e_1) = \nu(D_i + e_2) < \nu(D_i + e_1 + e_2).$$

*Proof of Claim:* Suppose (ii) does not hold, i.e.  $\tau(D_i + F) > \tau(D_i)$ . As  $D_i + E_i$  packs,  $\nu(D_i + F) > \nu(D_i)$ . Now if (i) does not hold then (iii) must hold since if two edges  $e_1, e_2 \in F$  appear in the same circuit then  $e_1, e_2$  are independent.  $\diamond$

*Claim 2.*  $D$  has a transversal of size  $\nu(D_1) + \nu(D_2) + 1$ .

*Proof of Claim:* If  $\tau(D_i + E_i) \leq \tau(D_i) + 1$  for some  $i \in \{1, 2\}$ , then take the corresponding transversal, and union it with any transversal of  $D_{3-i}$  of size  $\tau(D_{3-i})$ . The resulting set is as required by (3). Thus we may assume that  $\tau(D_i + E_i) \geq \tau(D_i) + 2$  for  $i = 1, 2$ . Since  $\nu(D_i + E_i) = \tau(D_i + E_i)$  we may assume that there is a packing of size  $\nu(D_1)$  in  $D_1$  and two disjoint paths disjoint from the packing joining  $u_2$  to  $u_3$  and  $u_4$  to  $u_1$ , respectively. Likewise, we may assume that a similar situation occurs in  $D_2$ , but with paths joining  $u_3$  to  $u_4$  and  $u_1$  to  $u_2$ . (If the paths join the other pairs we get a packing of size  $\nu(D_1) + \nu(D_2) + 2$ , a contradiction, because the union of  $\{u_1, u_3\}$ , any transversal of  $D_1$  and any transversal of  $D_2$  is a transversal of  $D$  of the same size.) Now we use the fact that  $D_2$  satisfies (a), (b) or (c) for the edges  $u_2u_3$  and  $u_4u_1$ . If (a) holds, then we have a packing in  $D$  of size  $\nu(D_1) + \nu(D_2) + 2$ , and so we may assume from the symmetry that (b) holds, where  $e_1 = u_2u_3$ . Let  $T_2$  be the corresponding transversal. We may also assume that  $\nu(D_1 + u_3u_4 + u_1u_2) \leq \nu(D_1) + 1$ , for otherwise we produce a packing of  $D$  of size  $\nu(D_1) + \nu(D_2) + 2$ . It follows that  $\nu(D_1 + u_3u_4 + u_1u_2 + u_1u_4) \leq \nu(D_1) + 1$ , because a packing of  $D_1 + u_3u_4 + u_1u_2 + u_1u_4$  that uses  $u_1u_4$  cannot use  $u_3u_4$  or  $u_1u_2$ . Hence  $\tau(D_1 + u_3u_4 + u_1u_2 + u_1u_4) = \nu(D_1 + u_3u_4 + u_1u_2 + u_1u_4) \leq \tau(D_1) + 1$ . Let  $T_1$  be a corresponding transversal. Then  $T_1 \cup T_2$  is a transversal in  $D$  of size  $\nu(D_1) + \nu(D_2) + 1$  by (3), as desired.  $\diamond$

Let  $F_i$  be the set of all edges  $e \in E_i$  such that  $\tau(D_i + e) > \tau(D_i)$ .

*Claim 3.* The reversal of no edge in  $F_1$  belongs to  $F_2$ .

*Proof of Claim:* Otherwise we can construct a packing in  $D$  of size  $\nu(D_1) + \nu(D_2) + 1$ , contrary to Claim 2.  $\diamond$

*Claim 4.* The digraph  $D$  has a packing of size  $\nu(D_1) + \nu(D_2)$ .

*Proof of Claim:* The union of any maximum packing of  $D_1$  with any maximum packing of  $D_2$  is as desired by (2).  $\diamond$

*Claim 5.* For some  $i \in \{1, 2\}$ ,  $F_i$  includes two independent edges.

*Proof of Claim:* Suppose for a contradiction that no  $F_i$  includes two independent edges. It follows from Claim 3 that there exist adjacent edges  $e_1, e_2 \in E_1 - F_1$  and adjacent edges  $e_3, e_4 \in E_2 - F_2$  such that  $e_3, e_4$  are the reverses of the edges in  $E_1 - \{e_1, e_2\}$ . Since  $e_1, e_2 \notin F_1$  we deduce from Claim 1 that  $\tau(D_1 + e_1 + e_2) = \nu(D_1 + e_1 + e_2) = \nu(D_1)$  and similarly  $\tau(D_2 + e_3 + e_4) = \nu(D_2)$ . But the union of the corresponding transversals is a transversal in  $D$  of size  $\nu(D_1) + \nu(D_2)$ , contrary to Claim 4.  $\diamond$

*Claim 6.* At most one of  $F_1, F_2$  includes two independent edges.

*Proof of Claim:* If both of them do, then (a) holds for those pairs, and we get a packing in  $D$  of size at least  $\nu(D_1) + \nu(D_2) + 1$ , contradicting Claim 2.  $\diamond$

By Claim 5 we may assume that  $F_2$  includes two independent edges. We wish to define a set  $F \subseteq E_1 - F_1$ . If  $E_2 = F_2$ , then  $F_1 = \emptyset$  by Claim 3, and we put  $F = E_1$ . Otherwise we proceed as follows. If  $F_1 \neq \emptyset$ , then it includes a unique edge by Claim 3, Claim 6 and the fact that  $F_2$  includes two independent edges. Let  $e$  be the unique member of  $F_1$ . If  $F_1 = \emptyset$ , then we select  $e \in E_1$  such that its reverse does not belong to  $F_2$ . In either case the reverse of  $e$  does not belong to  $F_2$ . We put  $F = E_1 - \{e\}$ . This completes the definition of  $F$ . We apply Claim 1 to  $D_1$  and  $F$ . Then (i) does not hold, because  $F \cap F_1 = \emptyset$ . If (ii) holds, then let  $T_1$  be the corresponding transversal, and let  $T_2$  be a transversal of size  $\tau(D_2)$  in  $D_2$  if  $e$  does not exist, and in  $D_2$  with the reverse of  $e$  added otherwise. Then  $T_1 \cup T_2$  is a transversal in  $D$  by (3) of size  $\nu(D_1) + \nu(D_2)$ , contrary to Claim 4. Thus (iii) holds. That is, there exist independent edges  $e_1, e_2 \in F$  such that  $\nu(D_1 + e_1 + e_2) > \nu(D_1)$ . Let  $e_3, e_4 \in E_2$  be the reverses of  $e_1, e_2$ . Since  $F_2$  includes two independent edges we deduce from the choice of  $F$  that  $e_3, e_4 \in F_2$ . Thus  $\nu(D_2 + e_1 + e_2) \geq \nu(D_2) + 2$  by (a). By combining the resulting packings we get a packing in  $D$  of size at least  $\nu(D_1) + \nu(D_2) + 1$ , contrary to Claim 2.

**Proposition 8.2.** *Let  $G, M, D$ , where  $\nu(D) < \tau(G)$ , and  $G_1, G_2, G_3, C$  be as in Proposition 5.5. Then at least one edge of  $M$  has both ends in  $V(C)$ .*

*Proof.* Let  $A, B$  denote a bipartition of  $G$ . Let  $u_1, u_2, u_3, u_4$  be the vertices of  $C$  (in that order), where  $u_1, u_3 \in A$  and  $u_2, u_4 \in B$ . Suppose for a contradiction that no edge of  $M$  has both ends in  $V(C)$ , and let the edges of  $M$  incident to vertices of  $C$  be  $m_1 = u_1u'_1, m_2 = u_2u'_2, m_3 = u_3u'_3, m_4 = u_4u'_4$ . For  $i = 1, 2, 3, 4$  we will use  $u_i$  to also denote the vertex of  $D$  that results from contracting  $m_i$ . Let  $Q$  be a cube such that  $C$  is a subgraph of  $Q$ , and  $Q$  is otherwise disjoint from  $G_1 \cup G_2 \cup G_3$ . Since  $G$  is a brace, we may assume (by renumbering  $G_1, G_2, G_3$ ) that  $\{m_1, m_2, m_3, m_4\} \subseteq E(G_1)$ , or  $\{m_3, m_4\} \subseteq E(G_1)$  and  $\{m_1, m_2\} \subseteq E(G_2)$ . In the former case we may also assume that  $|E(G_2)| \leq |E(G_3)|$ . If  $\{m_1, m_2, m_3, m_4\} \subseteq E(G_1)$  and  $|E(G_1)| > 12$ , then let  $H_1 = G_1$  and  $H_2 = G_2 \cup G_3$ ; otherwise let  $H_1 = G_1 \cup G_2$  and  $H_2 = G_3$ . Thus  $|E(H_1)| > 12$  by Lemma 5.9. Then both  $H_1$  and  $H_2$  are obtained from planar braces by repeatedly applying the trisum operation. Let  $J_1 = D(H_1, M)$ , and let  $D_1 = J_1 \setminus E(C)$ . Let  $J_2$  be obtained from  $H_2$  by directing every edge from  $A \cap V(H_2)$  to  $B \cap V(H_2)$ , and then contracting every edge of  $M \cap E(H_2)$ , and let  $D_2 = J_2 \setminus E(C)$ . Let us notice that  $u_1, u_3$  are sources, and  $u_2, u_4$  are sinks of  $D_2$ . Thus conditions (2) and (3) of Lemma 8.1 hold.

We now prove that condition (1) holds. The graph  $H_1$  is obtained from planar braces by repeatedly applying the 4-sum operation. By Theorem 5.4 the digraph  $J_1$  has no minor isomorphic to an odd double circuit or  $F_7$ . Moreover  $|V(J_1)| + |E(J_1)| < |V(D)| + |E(D)|$  by Lemma 5.9, and so  $J_1$  packs by the hypothesis of Proposition 5.5. But  $J_1$  is isomorphic to  $D_1 + E_1$ , and hence  $D_1 + E_1$  packs. To prove that  $D_2 + E_2$  packs we first notice that  $D_2 + E_2$  is a subdigraph of  $D(H_2 \cup Q, M_2)$ , where  $M_2$  is a perfect matching

of  $H_2 \cup Q$  that includes  $E(H_2) \cap M$  and no edge with both ends in  $V(C)$ . But  $D(H_2 \cup Q, M_2)$  packs by the hypothesis of Proposition 5.5 and the fact that  $|E(H_1)| > 12$ . Thus conditions (1)–(3) of Lemma 8.1 hold.

Next we show that for  $i = 1, 2$ , and for every pair  $e_1, e_2 \in E_i$  of independent edges one of (a), (b), (c) holds. We first do so for  $i = 2$ . It suffices to argue for  $e_1 = u_2u_1$  and  $e_2 = u_4u_3$ . Since  $D(H_2 \cup Q, M_2)$  packs by the previous paragraph, we see that  $D'_2 = D_2 + \{u_2u_1, u_3u_2, u_4u_3, u_1u_4\}$  also packs. But clearly  $\tau(D'_2) > \tau(D_2)$ , because  $u_1, u_3$  are sources and  $u_1, u_4$  are sinks of  $D_2$ , and  $\{u_1, u_2, u_3, u_4\}$  is the vertex-set of a circuit of  $D'_2$ . If  $\nu(D'_2) \geq \nu(D_2) + 2$ , then (a) holds. Thus we may assume that  $\tau(D'_2) = \tau(D_2) + 1$ . Let  $T$  be a corresponding transversal. Since  $\{u_1, u_2, u_3, u_4\}$  is the vertex-set of a circuit of  $D'_2$ , and  $|T| = \nu(D_2) + 1$ , we see that  $|\{u_1, u_2, u_3, u_4\} \cap T| = 1$ . Let  $T' = T - \{u_1, u_2, u_3, u_4\}$ . If  $u_1 \in T$  or  $u_2 \in T$ , then  $T'$  shows that (c) holds and if  $u_3 \in T$  or  $u_4 \in T$ , then  $T'$  shows that (b) holds, as desired. This proves that one of (a), (b), (c) holds for  $i = 2$ .

It remains to show that one of (a), (b), (c) holds for  $i = 1$ . Let  $M_1$  be the union of  $M \cap E(H_1)$  and two edges of  $Q$ : one with ends adjacent to  $u_1$  and  $u_2$ , and the other with ends adjacent to  $u_3$  and  $u_4$ . Let us consider the digraph  $D'_1 := D(H_1 \cup Q \setminus E(C), M_1)$ . Then  $D'_1$  is isomorphic to the graph  $D_1 + \{u_1a, au_2, ab, ba, u_3b, bu_4\}$ . If  $D'_1$  packs, then one of (a), (b), (c) holds: clearly  $\tau(D'_1) > \tau(D_1)$  because  $D'_1$  has a circuit disjoint from  $D_1$ . If  $\nu(D'_1) \geq \nu(D_1) + 2$ , then (a) holds; if  $\tau(D'_1) = \tau(D_1) + 1$ , then let  $T$  be a corresponding transversal. The set  $T \cap V(D_1)$  proves that (b) or (c) holds. Thus we may assume that  $D'_1$  does not pack, and so by the hypothesis of Proposition 5.5 we see that  $|E(H_2)| \leq |E(Q)|$ . Thus  $H_2$  is a cube by Lemma 5.9. In particular,  $H_2 = G_3$  and  $H_1 = G_1 \cup G_2$ . The definition of  $H_1$  and  $H_2$  implies that  $\{m_1, m_2, m_3, m_4\} \not\subseteq E(G_1)$  or  $|E(G_1)| = 12$ .

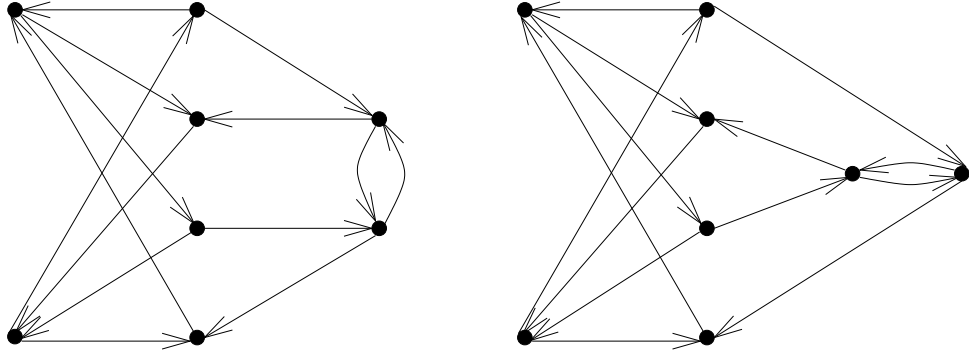


FIGURE 2. Two digraphs.

Let us first assume that  $\{m_1, m_2, m_3, m_4\} \subseteq E(G_1)$ . Then  $|E(G_1)| = 12$ , and so  $G_1$  is a cube. Since  $|E(G_2)| \leq |E(G_3)|$  and  $G_3 = H_2$  is a cube, we deduce that  $G_1, G_2, G_3$  are all cubes. It follows that  $D$  is isomorphic to one of the digraphs depicted in Figure 2, but both those digraphs satisfy  $\nu(D) = 2 = \tau(D)$ ,

a contradiction. Thus we may assume that  $\{m_1, m_2, m_3, m_4\} \not\subseteq E(G_1)$ , and so  $\{m_3, m_4\} \subseteq E(G_1)$  and  $\{m_1, m_2\} \subseteq E(G_2)$ . Moreover,  $H_1 = G_1 \cup G_2$ . For  $i = 1, 2$  let  $L_i = D(G_i, M)$ . Then

(\*)  $u_1$  is a source and  $u_2$  is a sink of  $L_1$ , and  $u_3$  is a source and  $u_4$  is a sink of  $L_2$ .

*Claim 1.* For  $i = 1, 2$ , the digraph  $L_i$  does not have two disjoint paths, one joining  $u_1$  and  $u_3$  and the other joining  $u_2$  and  $u_4$ .

*Proof of Claim:* We may assume that  $i = 1$ , and suppose for a contradiction that the paths exist. By (\*) they join  $u_4$  to  $u_2$  and  $u_1$  to  $u_3$ . Let  $L$  denote the union of the two paths. The digraph  $D(G_1 \cup Q, M)$  packs by Theorem 5.4, Lemma 5.9 and the hypothesis of Proposition 5.5. Let  $D'$  be obtained from  $L$  by adding a new vertex  $w$  and edges  $u_2u_1, u_1u_2, u_3w, wu_4, wu_1$ , and  $u_2w$ . Then  $D'$  is isomorphic to a subgraph of  $D(G_1 \cup Q, M)$ , and hence packs. But that is a contradiction, because  $D'$  has an odd double circuit minor (contract all but one edge of each path comprising  $L$ ).  $\diamond$

We now show that one of (a), (b), (c) holds for the pair of edges  $u_1u_4$  and  $u_3u_2$ . Indeed, suppose that none of (a), (b), (c) hold. Then  $D_1$  has a path  $P_1$  from  $u_2$  to  $u_3$  disjoint from a maximum packing, and a path  $P_2$  from  $u_4$  to  $u_1$  disjoint from a maximum packing. Since  $P_1$  and  $P_2$  are disjoint from any minimum transversal, we deduce that their union is acyclic. By (\*) we may assume that, say in  $G_2$ , a subpath  $P'_1$  of  $P_1$  goes from  $u_2$  to  $u_4$ , and that a subpath  $P'_2$  of  $P_2$  goes from  $u_3$  to  $u_1$ . We see that  $P'_1$  and  $P'_2$  are disjoint, because  $P_1 \cup P_2$  is acyclic. The existence of  $P'_1$  and  $P'_2$  contradicts Claim 1.

It remains to show that one of (a), (b), (c) holds for the pair of edges  $u_1u_2$  and  $u_3u_4$ . Suppose it does not. Thus  $D_1 + u_3u_4$  has a packing of size  $\nu(D_1) + 1$ . This packing includes a circuit containing the edge  $u_3u_4$ , and hence  $D_1$  has a packing  $\mathcal{C}$  of size  $\nu(D_1)$ , and a path  $P$  from  $u_4$  to  $u_3$  disjoint from every member of  $\mathcal{C}$ . It follows from (\*) and Claim 1 that  $P$  is a subgraph of  $L_1$ . Since  $\mathcal{C}$  does not use  $u_3$  or  $u_4$  (because every member of  $\mathcal{C}$  is disjoint from  $P$ ) we deduce that at most one circuit of  $\mathcal{C}$  intersects both  $E(L_1)$  and  $E(L_2)$ . Thus either (letting  $\nu = \nu(D_1)$  and using (\*))

- (A)  $\nu(L_1 + u_3u_4) + \nu(L_2) \geq \nu + 1$ , or
- (B)  $\nu(L_1 + u_2u_1 + u_3u_4) + \nu(L_2 + u_1u_2) \geq \nu + 2$ .

Similarly, either

- (C)  $\nu(L_2 + u_1u_2) + \nu(L_1) \geq \nu + 1$ , or
- (D)  $\nu(L_2 + u_1u_2 + u_4u_3) + \nu(L_1 + u_3u_4) \geq \nu + 2$ .

By (\*)  $\nu(L_1) + \nu(L_2) \leq \nu$ . Thus if (A) and (C) hold we deduce that

$$\nu(D_1 + u_1u_2 + u_3u_4) \geq \nu(L_1 + u_3u_4) + \nu(L_2 + u_1u_2) = 2\nu + 2 - \nu(L_1) - \nu(L_2) \geq \nu + 2,$$

where the first inequality follows from (\*). It follows that (a) holds, a contradiction. Assume now that (B) and (D) hold. Clearly  $\nu(L_2 + u_1u_2 + u_4u_3) \geq \nu(L_2 + u_1u_2)$ ,  $\nu(L_1 + u_2u_1 + u_3u_4) \geq \nu(L_1 + u_3u_4)$  and

$\nu(L_1 + u_2u_1 + u_3u_4) + \nu(L_2 + u_1u_2 + u_4u_3) \leq \nu + 2$ . Therefore

$$4\nu + 4 \geq \nu(L_1 + u_2u_1 + u_3u_4) + \nu(L_2 + u_1u_2) + \nu(L_2 + u_1u_2 + u_4u_3) + \nu(L_1 + u_3u_4) \geq 4\nu + 4.$$

Thus equality holds throughout, and, in particular,  $\nu(L_1 + u_2u_1 + u_3u_4) = \nu(L_1 + u_3u_4)$ . Since  $\nu(L_2) \geq \nu(L_2 + u_1u_2) - 1$  we have

$$\nu(L_1 + u_3u_4) + \nu(L_2) \geq \nu(L_1 + u_2u_1 + u_3u_4) + \nu(L_2 + u_1u_2) - 1 \geq \nu + 1,$$

and so (A) holds. Thus we have shown that if (B) and (D) hold, then (A) holds as well.

To complete the proof we may assume that (A) and (D) hold, but (C) does not. We need two claims.

$$(E) \quad \nu(L_2 + u_1u_2) \leq \nu(L_2)$$

To prove (E) we subtract the negation of (C) from (A), and use the fact that  $\nu(L_1 + u_3u_4) \leq \nu(L_1) + 1$ . We find that  $\nu(L_2 + u_1u_2) \leq \nu(L_2)$ , which is (E).

$$(F) \quad \nu(L_2 + u_4u_3) \leq \nu(L_2)$$

To prove (F) we use the fact that  $\nu(L_1 + u_3u_4) + \nu(L_2 + u_4u_3) \leq \nu + 1$ . (Otherwise those packings could be combined to produce a packing in  $D_1$  of size  $\nu + 1$ .) By subtracting this inequality from (A) we obtain (F).

Let  $L'_2 = L_2 + \{u_1a, au_2, u_3b, bu_4, ab, ba, u_4u_3\}$ . Let  $Q'$  be obtained from  $Q$  by adding a three-edge path  $P'$  joining  $u_3$  and  $u_4$ , and otherwise disjoint from  $G \cup Q$ . Let  $M'_2$  be a perfect matching of  $G_2 \cup Q'$  that includes  $M \cap E(G_2)$ , two edges of  $P'$ , and two edges of  $Q \setminus V(C)$ : one with ends adjacent to  $u_1$  and  $u_2$ , and the other with ends adjacent to  $u_3$  and  $u_4$ . This  $L'_2$  is isomorphic to  $D(G_2 \cup Q' \setminus E(C), M'_2)$ . The graph  $G_2 \cup Q'$  is a subgraph of a brace  $H$  in such a way that  $H \setminus V(G_2 \cup Q')$  has a perfect matching and  $H$  is obtained from planar braces by trisumming. By Theorem 5.4 the digraph  $L'_2$  has no minor isomorphic to an odd double circuit or  $F_7$ . By Lemma 5.9 the digraph  $L'_2$  satisfies  $|V(L'_2)| + |E(L'_2)| < |V(D)| + |E(D)|$ , and hence  $L'_2$  packs by the hypothesis of Proposition 5.5. We will show that  $\tau(L'_2) \geq \nu(L_2) + 2$  and  $\nu(L'_2) \leq \nu(L_2) + 1$ . This is a contradiction that will prove the proposition.

We first show that  $\tau(L'_2) \geq \nu(L_2) + 2$ . Indeed, suppose for a contradiction that  $L'_2$  has a transversal  $T$  of size at most  $\nu(L_2) + 1$ . Since  $\{b, u_3, u_4\}$  is the vertex-set of a circuit of  $L'_2$ , one of those vertices belongs to  $T$ . If  $b \in T$ , then  $T - \{b\}$  is a transversal of  $L_2 + u_1u_2 + u_4u_3$  of size  $\nu(L_2)$ . Thus  $\nu(L_2 + u_1u_2 + u_4u_3) + \nu(L_1 + u_3u_4) \leq \nu(L_2) + \nu(L_1 + u_3u_4) \leq \nu + 1$ , contrary to (D). If  $b \notin T$ , then  $u_3 \in T$  or  $u_4 \in T$ , and  $a \in T$ , because  $\{a, b\}$  is the vertex-set of a circuit of  $L'_2$ . Then  $T - \{u_3, u_4, a\}$  is a transversal of  $L_2$  by (\*) of size  $\nu(L_2) - 1$ , a contradiction. This proves that  $\tau(L'_2) \geq \nu(L_2) + 2$ .

Finally, it remains to prove that  $\nu(L'_2) \leq \nu(L_2) + 1$ . To this end suppose for a contradiction that  $\mathcal{C}$  is a packing in  $L'_2$  of size  $\nu(L_2) + 2$ . Choose a circuit  $C \in \mathcal{C}$  such that  $b \in V(C)$ . If such a choice is not possible choose  $C$  with  $a \in V(C)$ , and if that is not possible choose  $C$  arbitrarily. It follows that the packing  $\mathcal{C} - \{C\}$  uses at most one of  $a$  and  $u_4$ , and hence the packing  $\mathcal{C} - \{C\}$  proves that either  $\nu(L_2 + u_4u_3) > \nu(L_2)$ , or

$\nu(L_2 + u_1u_2) > \nu(L_2)$ , contrary to (E) and (F). This proves that  $\nu(L'_2) \leq \nu(L_2) + 1$ , and hence completes the proof of the proposition.  $\square$

### 9. CONCLUDING REMARKS

Consider a digraph  $D$  with weight function  $w : V(D) \rightarrow Z_+$ . The weight of a subset  $T \subseteq V(D)$  is defined as  $\sum_{v \in T} w(v)$ . The value of the minimum weight transversal is written  $\tau(D, w)$ . The cardinality of the largest family  $\mathcal{C}$  of circuits with the property that for every  $v \in V(D)$  at most  $w(v)$  circuits of  $\mathcal{C}$  use  $v$ , is denoted  $\nu(D, w)$ . Let  $e : V(D) \rightarrow Z_+$  where  $e(v) = 1, \forall v \in V(D)$ . Then  $\tau(D) = \tau(D, e)$

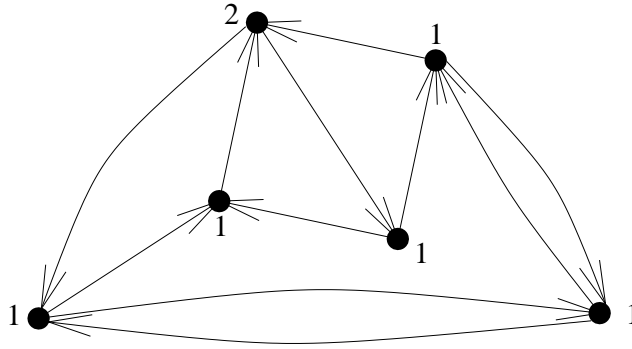


FIGURE 3. Digraph  $D$  with  $\tau(D, w) > \nu(D, w)$ .

and  $\nu(D) = \nu(D, e)$ . Observe that for every digraph  $D$  and for all positive weight functions  $w$  we have  $\tau(D, w) \geq \nu(D, w)$ . A natural extension of Theorem 1.1 would be to characterize which are the digraphs  $D$  for which  $\tau(H, w) = \nu(H, w)$ , for every subdigraph  $H$  of  $D$  and for every weights  $w : V(D) \rightarrow Z_+$ . This class of digraphs is closed under taking minors, and thus does not contain  $F_7$  or odd double circuits. However, there are other obstructions as is illustrated by the digraph  $D$  of Figure 3. Next to each vertex  $v$  we indicate the weight  $w(v)$ . Here we have  $3 = \tau(D, w) > \nu(D, w) = 2$ , and  $D$  does not contain  $F_7$  or an odd double circuit as a minor. In fact many other obstructions can be obtained by a similar construction. A related problem is to study the class of digraphs for which  $\tau(D, w) = \nu(D, w)$  for all  $w : V(D) \rightarrow Z_+$  but without requiring that the same property hold for every subdigraph. This can be formulated as a hypergraph matching problem where the vertices of the hypergraph are the vertices of the digraph and the edges are the vertex set of circuits of  $D$ . There is a long list of obstructions to this property. However the problem has been solved for the special case when  $D$  is a tournament [8] or a bipartite tournament [9].

### REFERENCES

- [1] W. McCuaig, Pólya's permanent problem, manuscript, June 1997.
- [2] G. Ding and W. Zang, Packing cycles in graphs, manuscript, April 2000.
- [3] C. H. C. Little, A characterization of convertible  $(0, 1)$ -matrices, *J. Combin. Theory Ser. B* **18** (1975), 187–208.

- [4] C. L. Lucchesi and D. H. Younger, A minimax relation for directed graphs, *J. London Math. Soc.* **17** (1978), 369–374.
- [5] B. Reed, N. Robertson, P. D. Seymour and R. Thomas, Packing directed circuits, *Combinatorica* **16** (1996), 535–554.
- [6] N. Robertson, P. D. Seymour and R. Thomas, Permanents, Pfaffian orientations, and even directed circuits, *Ann. Math.* **150** (1999), 929–975.
- [7] P. D. Seymour and C. Thomassen, Characterization of even directed graphs, *J. Combin. Theory Ser. B* **42** (1987), 36–45.
- [8] W. Zang, M. Cai and X. Deng, A tdi system and its application to approximation algorithm, In *Proc. 39th IEEE Symposium on Foundations of Computer Science*, 1998.
- [9] W. Zang, M. Cai and X. Deng, A min-max theorem on feedback vertex sets, In *Integer Programming and Combinatorial Optimization*, Lecture Notes in Computer Science, 7th IPCO Conference, Graz, Austria, 1999.

BERTRAND GUENIN

DEPARTMENT OF COMBINATORICS AND OPTIMIZATION

FACULTY OF MATHEMATICS

UNIVERSITY OF WATERLOO

WATERLOO, ON N2L 3G1, CANADA

ROBIN THOMAS

SCHOOL OF MATHEMATICS

GEORGIA INSTITUTE OF TECHNOLOGY

ATLANTA, GEORGIA 30332, USA