Three-coloring triangle-free graphs on surfaces V. Coloring planar graphs with distant anomalies*

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Abstract

We settle a problem of Havel by showing that there exists an absolute constant d such that if G is a planar graph in which every two distinct triangles are at distance at least d, then G is 3-colorable. In fact, we prove a more general theorem. Let G be a planar graph, and let $\mathcal{H}$ be a set of connected subgraphs of G, each of bounded size, such that every two distinct members of $\mathcal{H}$ are at least a specified distance apart and all triangles of G are contained in $\bigcup \mathcal{H}$. We give a sufficient condition for the existence of a 3-coloring ϕ of G such that for every $H \in \mathcal{H}$ the restriction of ϕ to H is constrained in a specified way.

1 Introduction

This paper is a part of a series aimed at studying the 3-colorability of graphs on a fixed surface that are either triangle-free, or have their triangles restricted in some way. Here, we are concerned with 3-coloring planar graphs. All *graphs* in this paper are finite and simple; that is, have no loops or multiple edges. All *colorings* that we consider are proper, assigning different colors to adjacent vertices. The following is a classical theorem of Grötzsch [17].

Theorem 1.1. *Every triangle-free planar graph is* 3*-colorable.*

There is a long history of generalizations that extend the theorem to classes of graphs that include triangles. An easy modification of Grötzsch' proof shows

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that every planar graph with at most one triangle is 3-colorable. Even more is true—every planar graph with at most three triangles is 3-colorable. This was first claimed by Grünbaum [18], however his proof contains an error. This error was fixed by Aksionov [1] and later Borodin [5] gave another proof. There are infinitely many 4-critical planar graphs with four triangles, but they were recently completely characterized by Borodin et al. [6].

As another direction of research, Grünbaum [18] conjectured that every planar graph with no intersecting triangles is 3-colorable. This was disproved by Havel [19], who formulated a more cautious question whether there exists a constant d such that every planar graph such that the distance between every two triangles is at least d is 3-colorable. In [20], Havel shows that if such a constant d exists, then $d \geq 3$, and Aksionov and Mel'nikov [2] improved this bound to $d \geq 4$. Borodin [4] constructed a family of graphs that suggests that it may not be possible to obtain a positive answer to Havel's question using local reductions only.

The answer to Havel's question is known to be positive under various additional conditions (e.g., no 5-cycles [8], no 5-cycles adjacent to triangles [7], a distance constraint on 4-cycles [9]), see the on-line survey of Montassier [21] for a more complete list. The purpose of this paper is to describe a solution to Havel's problem.

Theorem 1.2. *There exists an absolute constant d such that if G is a planar graph and every two distinct triangles in G are at distance at least d, then G is 3-colorable.*

Let us remark that our proof gives an explicit upper bound on the constant d of Theorem 1.2, which however is not very good ($d \leq 10^{100}$, say) compared with the known lower bounds.

A natural extension of Havel's question is whether instead of triangles, we could allow other kinds of distant anomalies, such as 3-colorable subgraphs containing several triangles (the simplest one being a diamond, that is, K_4 without an edge) or even more strongly, prescribing specific colorings of some distant subgraphs. Similar questions have been studied for other graph classes. For example, Albertson [3] proved that if S is a set of vertices in a planar graph G that are precolored with colors $1, \ldots, 5$ and are at distance at least 4 from each other, then the precoloring of S can be extended to a 5-coloring of G. Furthermore, using the results of the third paper of this series [12], it is easy to see that any precoloring of sufficiently distant vertices of a planar graph G of girth at least 5 can be extended to a 3-coloring of G. We can even precolor larger connected subgraphs, as long as these precolorings can be extended locally to the vertices of G at some bounded distance from the precolored subgraphs. Both for 5-coloring planar graphs and 3-coloring planar graphs of girth at least five this follows from the fact that the corresponding critical graphs satisfy a certain isoperimetric inequality [22].

The situation is somewhat more complicated for graphs of girth four. Firstly, as we will discuss in Section 4, there is a global constraint on 3-colorings of plane graphs based on winding number, which implies that in graphs with almost all faces of length four, precoloring a subgraph may give restrictions on possible colorings of distant parts of the graph. For example, if we prescribed specific colorings of the triangles in Theorem 1.2, the resulting claim would be false, even though such precolorings extend locally. Secondly, non-facial (separating) 4-cycles are problematic as well and they need to be treated with care in many of the results of this series, see e.g. Theorem 2.2 below. Specifically, we cannot replace triangles in Theorem 1.2 by diamonds, even though this seems viable when considering only the winding number argument, as shown by the class of graphs (with many separating 4-cycles) constructed by Thomas and Walls [23].

Thus, in our second result, we only deal with graphs without separating 4-cycles, and we need to allow certain flexibility in the prescribed colorings of distant subgraphs. The exact formulation of the result (Theorem 5.1) is somewhat technical, and we postpone it till Section 5. Here, let us give just a special case covering several interesting kinds of anomalies. The *pattern* of a 3-coloring ψ is the set $\{\psi^{-1}(1), \psi^{-1}(2), \psi^{-1}(3)\}$. That is, two 3-colorings have the same pattern if they only differ by a permutation of colors.

Theorem 1.3. *There exists an absolute constant $d \geq 2$ with the following property. Let G be a plane graph without separating 4-cycles. Let S_1 be a set of vertices of G. Let S_2 be a set of (≤ 5)-cycles of G. Let S_3 be a set of vertices of G of degree at most 4. For each $v \in S_1 \cup S_3$, let $c_v \in \{1, 2, 3\}$ be a color. For each $K \in S_2$, let ψ_K be a 3-coloring of K. Suppose that the distance between any two vertices or subgraphs belonging to $S_1 \cup S_2 \cup S_3$ is at least d. If all triangles in G belong to S_2, then G has a 3-coloring φ such that*

- *$\varphi(v) = c_v$ for every $v \in S_1$,*
- *φ has the same pattern on K as ψ_K for every $K \in S_2$, and*
- *$\varphi(u) = c_v$ for every neighbor u of a vertex $v \in S_3$.*

Let us remark that forbidding separating 4-cycles is necessary when the anomalies S_2 (except for triangles) and S_3 are considered, as shown by simple variations of the construction of Thomas and Walls [23]. On the other hand, there does not appear to be any principal reason to exclude 4-cycles when only precolored single vertices are allowed.

Conjecture 1.4. *There exists an absolute constant $d \geq 2$ with the following property. Let G be a plane triangle-free graph, let S be a set of vertices of G and let $\psi : S \to \{1, 2, 3\}$ be an arbitrary function. If the distance between every two vertices of S is at least d, then ψ extends to a 3-coloring of G.*

In Theorem 5.1, we show that Conjecture 1.4 is implied by the following seemingly simpler statement.

Conjecture 1.5. *There exists an absolute constant $d \geq 2$ with the following property. Let G be a plane triangle-free graph, let C be a 4-cycle bounding a face of G and let v be a vertex of G. Let ψ be a 3-coloring of $C + v$. If the distance between C and v is at least d, then ψ extends to a 3-coloring of G.*

Every planar triangle-free graph G on n vertices contains $\Omega(n)$ vertices of degree at most 4, and if the graph has bounded maximum degree, then we can select a subset S_3 of such vertices of size $\Omega(n)$ such that the distance between any two of its vertices is at least d. If G does not contain separating 4-cycles, then by Theorem 1.3, we can 3-color G so that all vertices of S_3 have a monochromatic neighborhood. By independently altering the colors of vertices in S_3, we obtain exponentially many 3-colorings of G. This solves a special case of a conjecture of Thomassen [24] that all triangle-free planar graphs have exponentially many 3-colorings.

Corollary 1.6. *For every $k \geq 0$, there exists $c > 1$ such that every planar triangle-free graph G of maximum degree at most k and without separating 4-cycles has at least $c^{|V(G)|}$ 3-colorings.*

The rest of the paper is structured as follows. In the next section, we state several previous results which we need in the proofs. In Section 3, we study the structure of graphs where no 4-faces can be collapsed without decreasing distances between anomalies, showing that they contain long cylindrical quadrangulated subgraphs. In Section 4, we study the colorings of such cylindrical subgraphs. Finally, in Section 5, we prove a statement generalizing Theorems 1.2 and 1.3.

2 Previous results

We use the following lemma of Aksionov [1].

Lemma 2.1. *Let G be a plane graph with at most one triangle, and let C be either the null graph or a facial cycle of G of length at most five. Assume that if C has length five and G has a triangle T, then C and T are edge-disjoint. Then every 3-coloring of C extends to a 3-coloring of G.*

We also need several results from previous papers of this series. Let G be a graph and C its subgraph. We say that G is *C-critical* if $G \neq C$ and for every proper subgraph G' of G that includes C, there exists a 3-coloring of C that extends to a 3-coloring of G', but does not extend to a 3-coloring of G. The following claim is a special case of the general form of the main result of [13] (Theorem 4.1).

Theorem 2.2. *There exists an absolute constant η with the following property. Let G be a plane graph and Z a (not necessarily connected) subgraph of G such that all triangles and all separating 4-cycles in G are contained in Z. If G is Z-critical, then $\sum |f| \leq \eta |V(Z)|$, where the summation is over all faces f of G of length at least five.*

The following is a simple corollary of Lemma 5.3 of [13].

Lemma 2.3. *Let G be a triangle-free plane graph with outer face f_0 bounded by a cycle and with another face f bounded by a cycle of length at least $|f_0| - 1$.*

If every cycle separating f_0 from f in G has length at least $|f_0|-1$, then every 3-coloring of the cycle bounding f_0 extends to a 3-coloring of G.

Finally, let us state a basic property of critical graphs.

Proposition 2.4. *Let G be a graph and C its subgraph such that G is C-critical. If $G = G_1 \cup G_2$, $C \subseteq G_1$ and $G_2 \not\subseteq G_1$, then G_2 is $(G_1 \cap G_2)$-critical.*

3 Structure of graphs without collapsible 4-faces

Essentially all papers dealing with 3-colorability of triangle-free planar graphs first eliminate 4-faces by identifying their opposite vertices, thus reducing the problem to graphs of girth 5. However, this reduction might decrease distances in the resulting graph, which constrains its applicability for the problems we consider. In this section, we give a structural result on graphs in that no 4-face can be reduced.

Let C be a cycle in a graph G, and let $S \subseteq V(G)$. We say that the cycle C is *S-tight* if C has length four and the vertices of C can be numbered v_1, v_2, v_3, v_4 in order such that for some integer $t \geq 0$ the vertices v_1, v_2 are at distance exactly t from S, and the vertices v_3, v_4 are at distance exactly $t+1$ from S. We say that a face is S-tight if it is bounded by an S-tight cycle.

Lemma 3.1. *Let $d \geq 1$ be an integer, let G be a graph, and let $\mathcal{S}$ be a family of subsets of $V(G)$ such that the distance between every two distinct sets of $\mathcal{S}$ is at least $2d$. Let C be a cycle in G of length four and assume that for each pair u, v of diagonally opposite vertices of C, two distinct sets in $\mathcal{S}$ are at distance at most $2d-1$ in the graph obtained from G by identifying u and v. Then there exists a unique set $S_0 \in \mathcal{S}$ at distance at most $d-1$ from C. Furthermore, C is S_0-tight.*

Proof. Let the vertices of C be v_1, v_2, v_3, v_4 in order. By hypothesis there exist sets $S_1, S_2, S_3, S_4 \in \mathcal{S}$, where S_i is at distance d_i from v_i, such that $S_1 \neq S_3$, $S_2 \neq S_4$, $d_1 + d_3 \leq 2d-1$, and $d_2 + d_4 \leq 2d-1$. From the symmetry we may assume that $d_1 \leq d-1$ and $d_2 \leq d-1$. The distance between S_1 and S_2 is at most $d_1 + d_2 + 1 \leq 2d-1$, and thus $S_1 = S_2$. Let us set $S_0 = S_1$. If any $S \in \mathcal{S}$ is at distance at most $d-1$ from C, then the distance between S and S_0 is at most $2(d-1)+1 < 2d$, and thus $S = S_0$. It follows that S_0 is the unique element of $\mathcal{S}$ at distance at most $d-1$ from C.

Note that $S_4 \neq S_2 = S_1$, and hence $d_1 + d_4 + 1 \geq 2d$, because S_1 and S_4 are at distance at least $2d$. This and the inequality $d_2 + d_4 \leq 2d-1$ imply that $d_1 \geq d_2$. But there is a symmetry between d_1 and d_2, and hence an analogous argument shows that $d_1 \leq d_2$. Thus for $t := d_1 = d_2$ the vertices v_1, v_2 are both at distance t from $S_0 = S_1 = S_2$. If v_4 was at distance t or less from S_0, then S_0 and S_4 would be at distance at most $t + d_4 = d_2 + d_4 \leq 2d-1$, a contradiction. The same holds for v_3 by symmetry, and hence v_3 and v_4 are at distance $t+1$ from S_0, as desired. □

Let G be a graph, let $S \subseteq V(G)$ and let K be a cycle in G. We say that K is *equidistant* from S if for some integer $t \geq 0$ every vertex of K is at distance exactly t from S. We will also say that K is equidistant from S at distance t.

We say that a plane graph H is a *cylindrical quadrangulation with boundary faces f_1 and f_2* if the distinct faces f_1 and f_2 of H are bounded by cycles and all other faces of H have length four. The union of the cycles bounding f_1 and f_2 is called the *boundary* of H. The cylindrical quadrangulation H is a *joint* if $|f_1| = |f_2|$, every cycle of H separating f_1 from f_2 has length at least $|f_1|$ and the distance between f_1 and f_2 in H is at least $4|f_1|$. If H appears as a subgraph of another plane graph G, we say that the appearance is *clean* if every face of H except for f_1 and f_2 is also a face of G. An $r \times s$ *cylindrical grid* is the Cartesian product of a path with r vertices and a cycle of length s.

Lemma 3.2. *Let G be a plane graph and let $S \subseteq V(G)$ induce a connected subgraph of G. Let C_0 be an equidistant cycle at some distance $d_0 \geq 1$ from S. Let $s = |C_0|$ and $d_1 = d_0 + 2(s-2)(s+3)$. Let V_0 denote the set of vertices of G at distance at least d_0 and at most d_1 from S that are separated from S by C_0, including $V(C_0)$. Let F_0 denote the set of faces of G which are separated from S by C_0 and are incident with at least one vertex of V_0. Assume that every face in F_0 is S-tight and every vertex in V_0 has degree at least three. Then G contains a clean joint H such that $V(H) \subseteq V_0$.*

Proof. For an integer j, let $d(j) = d_0 + 2(s-j)(s+j+1)$. Note that $d(j) + 4j = d(j-1)$ for every j, $d_0 = d(s)$ and $d_1 = d(2)$. Choose the smallest integer $j \geq 3$ for that there exists an equidistant j-cycle C at distance t from S such that $d_0 \leq t \leq d(j)$ and $V(C) \subset V_0$. Such an integer j exists, since C_0 satisfies the requirements for $j = s$. Let $p \leq 4j$ be the maximum integer such that G contains a clean $(p+1) \times |C|$ cylindrical grid H with boundary faces f_1 and f_2 as a subgraph such that $V(H) \subset V_0$, f_1 is bounded by C and f_2 is bounded by an equidistant cycle K at distance $t + p$ from S. Such an integer p exists, since C (treated as a $1 \times |C|$ cylindrical grid) satisfies the requirements for $p = 0$.

We claim that $p = 4j$, and thus H satisfies the conclusion of the theorem. Suppose that $p \leq 4j - 1$. Removing K splits the plane to two open sets, let Δ be the one that does not contain S. Observe that K has no chord contained in Δ, as otherwise there exists an equidistant cycle of length less than j at distance $t + p \leq t + 4j - 1 < d(j-1)$ from S, contrary to the minimality of j. Consider an edge $uv \in E(K)$ and let f be the face of G incident with uv contained in Δ. Then f is at distance $t+p < d(j-1) \leq d(2) = d_1$ from S, and thus f is S-tight, i.e., the vertices on the boundary on f may be denoted by u, v, v', u' in order and u' and v' are at distance $t+p+1$ from S. Now let w be the other neighbor of v on K, and let us repeat the same argument to the edge vw, obtaining a face boundary v, w, w', v''. If $v' \neq v''$, then there exists a face f' incident with v and contained in Δ that is incident neither with uv nor with vw. However, since K has no chord contained in Δ, both neighbors of v in f' are at distance at least $t + p + 1$ from S, contrary to the assumption that f' is S-tight. This proves that $v' = v''$.

Hence, every $v \in V(K)$ has a unique neighbor v' contained in Δ. Let Z be the subgraph of G induced by $\{v' : v \in V(K)\}$. If Z contained a vertex v' whose degree in Z is one, then v' would have degree two in G, contrary to the assumptions of this lemma. Hence, the minimum degree of Z is at least two, and thus Z contains a cycle Z'. Note that Z' is equidistant at distance $t+p+1 \leq t+4j \leq d(j-1)$ from S, and by the minimality of j, it follows that $|Z'| = j$, and thus $|V(Z)| = j = |V(K)|$. Therefore, $v'_1 \neq v'_2$ for distinct vertices $v_1, v_2 \in V(K)$. We conclude that we can extend H to a clean $(p+2) \times |C|$ cylindrical grid by adding the vertices and edges of the faces of G incident with $V(K)$ and contained in Δ, contrary to the maximality of p. This finishes the proof. □

Next, we consider the case that some of the faces of the set F_0 are not tight, but instead are contained in a short separating cycle. A 4-face f is *attached* to a cycle C if the boundary cycle of f and C intersect in a path of length two.

Lemma 3.3. *Let $d_2 \geq 4$ and $s \geq 3$ be integers, and let $d_3 = d_2 + 34(s-2)(s+3) + 483$. Let G be a connected plane graph and let $S_1, S_2 \subseteq V(G)$ induce connected subgraphs of G at distance at least $d_3 + 4$ from each other. Suppose that every equidistant cycle in G at distance at least d_2 and at most d_3 from S_1 has length at most s and that all vertices of G at distance at least d_2 and at most d_3 from S_1 have degree at least three. Assume furthermore that every face of G at distance at least d_2 and at most d_3 from S_1 has length four, and it is either S_1-tight or attached to a cycle of length at most 6 that separates S_1 from S_2. Then G contains a clean joint H such that every vertex of H is at distance at least d_2 and at most d_3 from S_1.*

Proof. Let R be the set of all (≤ 6)-cycles in G that separate S_1 from S_2 and contain a vertex at distance at least d_2 and at most d_3 from S_1. For an integer t such that $d_2 \leq t \leq d_3$, let G_t denote the subgraph of G induced by vertices at distance exactly t from S_1.

If $d_2 + 2 \leq t \leq d_3$ and Q is a connected component of G_t whose distance from every element of R in G is at least 2, then Q has minimum degree at least two. (1)

Proof. Consider a vertex $v \in V(Q)$ and any face f incident with v. The distance between f and S_1 is at least $t-2 \geq d_2$ and at most $t \leq d_3$, and thus f is a 4-face. Since the distance between v and every element of R is at least two, f is not attached to a (≤ 6)-cycle separating S_1 from S_2; hence, f is S_1-tight. Therefore, the boundary of f contains an edge of Q incident with v. Since the same claim holds for each face incident with v and since v has degree at least three in G, we conclude that v has degree at least two in Q. □

Consider a cycle $K \in R$. Removing K splits the plane to two open sets, denote by Δ_K the one that does not contain S_1. For $K_1, K_2 \in R$, we write $K_1 \prec K_2$ if K_2 is contained in the closure of Δ_{K_1}, and we write G_{K_1,K_2} for the subgraph of G drawn in the closure of $\Delta_{K_1} \setminus \Delta_{K_2}$.

Consider cycles $K_1, K_2 \in R$ of the same length r such that $K_1 \prec K_2$ and no $K \in R$ with $|K| < r$ satisfies $K_1 \prec K \prec K_2$. For $i \in \{1,2\}$, let k_i denote the distance between S_1 and K_i. If $k_1 \geq d_2$ and $k_1 + 4r + 3 \leq k_2 \leq d_3 - 2(s-2)(s+3) - 5$, then G contains a clean joint H such that every vertex of H is at distance at least d_2 and at most d_3 from S_1.

(2)

Proof. Note that by the assumptions of the claim, no cycle in G_{K_1,K_2} that separates K_1 from K_2 has length less than r and the distance between K_1 and K_2 is at least $4r$. If each vertex of G_{K_1,K_2} is at distance at most d_3 from S_1, then by the assumptions of this lemma, every face of G_{K_1,K_2} has length four, and thus we can set $H = G_{K_1,K_2}$.

Therefore, assume that G_{K_1,K_2} contains a vertex at distance more than d_3 from S_1. Let $t = k_2 + 5$ and let Q be a connected component of G_t contained in G_{K_1,K_2}. Observe that every element of R which intersects G_{K_1,K_2} is at distance at most k_2 from S_1, and thus its distance from Q is at least two. By (1), we conclude that Q contains a cycle C. Note that C is equidistant from S_1 at distance t, and by the assumptions of this lemma, C has length at most s.

Consider any face f at distance at most $t + 2(s-2)(s+3)$ from S_1 that is separated from S_1 by C. Note that f is a 4-face and its distance from every element of R is at least two, and thus it is not attached to any cycle of R. We conclude that each such face f is S_1-tight. By Lemma 3.2, G contains a clean joint H as required. $\square$

Let $b_2 = d_2 - 1$ and $e_2 = d_3 - 2(s-2)(s+3) - 4$. For $3 \leq r \leq 6$, let b_r and e_r be chosen so that $b_{r-1} \leq b_r \leq e_r \leq e_{r-1}$, every cycle in R of length r is at distance either at most b_r or at least e_r from S_1, and subject to these conditions, $e_r - b_r$ is as large as possible.

Consider a fixed $r \in \{3,4,5,6\}$. If no cycle in R has length r and is at distance more than b_{r-1} and less than e_{r-1} from S_1, then we have $b_r = b_{r-1}$ and $e_r = e_{r-1}$. Otherwise, let $K_1 \in R$ be a cycle of length r whose distance k_1 from S_1 satisfies $b_{r-1} < k_1 < e_{r-1}$ and subject to that, k_1 is as small as possible; and, let $K_2 \in R$ be a cycle of length r whose distance k_2 from S_1 satisfies $b_{r-1} < k_2 < e_{r-1}$ and subject to that, k_2 is as large as possible. If $k_2 \geq k_1 + 4r + 3$, then (2) implies that the conclusion of this lemma holds, and thus we can assume that $k_2 \leq k_1 + 4r + 2$. Note that the distance of every cycle in R of length r from S_1 is at most b_r, or between k_1 and k_2 (inclusive), or at least e_r. Furthermore, $(k_1 - b_{r-1}) + (e_{r-1} - k_2) = (e_{r-1} - b_{r-1}) - (k_2 - k_1) \geq (e_{r-1} - b_{r-1}) - 4r - 2$, and thus $e_r - b_r \geq \max(k_1 - b_{r-1}, e_{r-1} - k_2) \geq \frac{e_{r-1} - b_{r-1}}{2} - 2r - 1$.

It follows that $e_6 - b_6 \geq \frac{e_2 - b_2}{16} - 22 = \frac{d_3 - d_2 - 2(s-2)(s+3) - 355}{16} = 2(s-2)(s+3) + 8$. Let $t = b_6 + 5$ and let Q be a connected component of G_t. Note that the distance between Q and every element of R is at least two, and thus by (1), Q contains a cycle C. Since C is equidistant at distance t from S_1, the assumptions of this lemma imply that C has length at most s. Consider a face f of G at distance at least t and at most $t + 2(s-2)(s+3)$ from S_1. Note that f is a 4-face and observe that the distance between f and any element of R is at least one. Therefore, f is S_1-tight by the assumptions of this lemma. By Lemma 3.2, G contains a clean joint H such that every vertex of H is at distance at least d_2 and at most d_3 from S_1, as required. $\square$

Let G be a plane graph, let B be an odd cycle in G, let Δ be one of the two connected open subsets of the plane bounded by B, let uv be an edge of B, let w be the vertex of B that is farthest (as measured in B) from uv and let z be a vertex of G such that either $z = w$, or z does not belong to the closure of Δ. Let P_u and P_v be the paths in $B - uv$ joining u and v, respectively, with w. We say that Δ is a *z-petal with top uv* if there exists a path Q in G between w and z such that $Q \cup P_u$ and $Q \cup P_v$ are shortest paths from z to u and v in G, respectively.

Let S be a connected subgraph of G and consider a cycle K which is equidistant at some distance $t \geq 1$ from S. The removal of K splits the plane into two open sets, let Δ be the one containing S. For each $v \in V(K)$, choose a path P_v of length t joining v to S. We can choose the paths so that for every $u, v \in V(K)$, the paths P_u and P_v are either disjoint or intersect in a path ending in S. Removing $G[S]$ and the paths P_v for $v \in V(K)$ splits Δ to several parts; for each $e \in E(K)$, let Δ_e be the one whose boundary contains e. Clearly, Δ_e and $\Delta_{e'}$ are disjoint for distinct $e, e' \in E(K)$. Note that if $e = uv$ is an edge of K, the path P_u intersects P_v and z is the common endpoint of P_u and P_v in S, then Δ_e is a z-petal with top uv. We call the collection $\{\Delta_e : e \in E(K)\}$ a *flower of K with respect to S*.

Lemma 3.4. *Let $D_1 \geq 0$ and $D_2 \geq D_1 + 3$ be integers, let G be a connected plane graph and let S be a subset of $V(G)$ inducing a connected subgraph. Let C be either the null graph, or a cycle in G bounding a face of length at most 5. Suppose that every 4-face of G not bounded by C at distance at least D_1 and at most $D_2 - 1$ from S is S-tight, and all vertices belonging to $V(G) \setminus V(C)$ at distance exactly $D_2 - 1$ from S have degree at least three. Let Δ be a z-petal with top uv for some $z \in S$ and some edge uv of G, such that $S \setminus \{z\}$ is disjoint from the closure of Δ and the distance between S and uv is at most $D_2 - 1$. Then, there exists a face f of G at distance at most $D_2 - 1$ from S which is contained in Δ, such that either f has length other than 4 or it is bounded by C.*

Proof. We can assume that Δ is minimal, i.e., there is no $\Delta' \subsetneq \Delta$ such that Δ' is a z-petal satisfying the assumptions of the lemma. Since Δ is bounded by an odd cycle, there exists an odd face f contained in Δ. It suffices to consider the case that the distance between f and S is at least D_2. Let Q be the subgraph of G induced by vertices at distance exactly $D_2 - 1$ from S that are contained in the closure of Δ. Note that Q may intersect the boundary of Δ only in the edge uv.

If $Q = uv$, then $\{u, v\}$ forms a cut in G that separates the rest of the boundary of Δ from the vertices incident with f. Observe that this implies that there exists a face f' contained in Δ and incident with both u and v which either does not have length 4 or is bounded by C (let us remark that f' cannot have length 4 unless it is bounded by C, since it would not be S-tight). Hence, the conclusion of this lemma is satisfied.

Therefore, we can assume that $Q \neq uv$. Consider a vertex $w \in V(Q) \setminus \{u, v\}$. If any face incident with w has length other than 4 or is bounded by C, then the conclusion of this lemma is satisfied. Hence, assume that this is not the case, and thus all such faces are S-tight. Furthermore, w has degree at least three in G. As in (1) in the proof of Lemma 3.3, we conclude that w has degree at least two in Q. Hence, Q contains a cycle K, which is equidistant at distance $D_2 - 1$ from S. Let $F = \{\Delta_e : e \in E(K)\}$ be a flower of K with respect to S. We can choose F so that Δ_e is a z-petal for every $e \in E(K)$. There exists exactly one element $\Delta_0 \in F$ such that uv is contained in the closure of Δ_0. Every $\Delta' \in F$ distinct from Δ_0 is a subset of Δ. Since $|F| = |K| \geq 3$, it follows that each such z-petal Δ' is a proper subset of Δ. This contradicts the minimality of Δ. $\square$

Consider a face f in a plane graph G, bounded by a closed walk $v_1 v_2 \ldots v_m$ going clockwise around f. A pair $(v_{i-1} v_i v_{i+1}, f)$ for $1 \leq i \leq m$ (where $v_0 = v_m$ and $v_{m+1} = v_1$) is called an *angle* in G, and v_i is its *tip*.

Lemma 3.5. *For all integers $D_1, p \geq 1$, there exists an integer $D_2 > D_1$ with the following property. Let G be a connected plane graph. Let $\mathcal{S}$ be a set of subsets of $V(G)$ such that each $S \in \mathcal{S}$ induces a connected subgraph of G with at most p vertices and the distance between every two elements of $\mathcal{S}$ is at least $2D_2$. Let C be either the null graph or a cycle of length at most 5 bounding a face of G. Suppose that every 4-face of G not bounded by C at distance at least D_1 and at most $D_2 - 1$ from any element $S \in \mathcal{S}$ is S-tight. Let $Z_0 = G\left[\bigcup_{S \in \mathcal{S}} S\right]$, and assume that every triangle of G is contained in Z_0. Furthermore, assume that for every separating 4-cycle F of G, if one of the open regions of the plane bounded by F is disjoint from C and contains a vertex of exactly one $S \in \mathcal{S}$, then the distance between F and S is less than $D_1 - 2$. Let $Z = C \cup Z_0$. If G is Z-critical and $|\mathcal{S}| \geq 2$, then G contains a clean joint H whose vertices are at distance at least D_1 and at most $D_2 - 1$ from some element of $\mathcal{S}$. Furthermore, H is vertex-disjoint from C.*

Proof. Let $\mu = 5(p+1)\eta + 6p + 30$, where η is the constant from Theorem 2.2, and let $s = \mu + 8p$. Let $D_2 = \max(D_1, 4) + (\mu + 1)(34(s-2)(s+3) + 484)$.

Without loss of generality, we can assume that if C is non-null, then it bounds the outer face of G. If there exists $Q \subset G$ such that either

- Q is a separating 4-cycle such that for at least two distinct $S_1, S_2 \in \mathcal{S}$, the open disk Δ bounded by Q contains at least one vertex of both S_1 and S_2, or
- for some distinct $S_1, S_2 \in \mathcal{S}$, Q forms the boundary of a face Δ of $G[S_1]$ which is not outer and which contains S_2,

then choose Q so that Δ is inclusionwise-minimal and let G_0 be the subgraph of G drawn in the closure of Δ. Otherwise, let $G_0 = G$ and $Q = C$. Let $\mathcal{S}_0$ consist of the sets $S \in \mathcal{S}$ such that $G[S] \subseteq G_0$. If there exists $S \in \mathcal{S} \setminus \mathcal{S}_0$ (necessarily unique by the assumptions of this lemma) such that $G[S]$ is not disjoint from G_0, then let $Q_0 = Q \cup (G[S] \cap G_0)$, otherwise let $Q_0 = Q$.

For each $S \in \mathcal{S}_0$, let $H_S \subset G_0$ consist of $G_0[S]$ and of all separating 4-cycles $F \subset G_0$ such that the open disk bounded by F contains at least one vertex of S.

For each $S \in \mathcal{S}_0$, the boundary B_S of the outer face of H_S has only one component and $|V(B_S)| \leq 4p$. The distance in G_0 between S and every vertex of B_S is less than D_1. Furthermore, for any $S' \in \mathcal{S}_0$ distinct from S, the subgraph $B_{S'}$ is drawn in the outer face of H_S. (3)

Proof. Let $F_1, \ldots, F_k$ be all separating 4-cycles in G_0 such that for $1 \leq i \leq k$, there exists a vertex $s_i \in S$ contained in the open disk Λ_i bounded by F_i, no other separating 4-cycle of G_0 bounds an open disk containing Λ_i, and $F_i \not\subseteq G_0[S]$. Consider distinct indices i and j such that $1 \leq i, j \leq k$. By the choice of G_0, no set in $\mathcal{S}$ distinct from S has vertices in the disk bounded by F_i, and by the assumptions of this lemma, the distance between F_i and S is less than $D_1 - 2$. It follows that the distance between S and each vertex of B_S is less than D_1. If say F_i had a chord, then it would be contained in a union of two triangles and we would have $F_i \subseteq G_0[S]$ by the assumptions of this lemma. Thus, neither F_i nor F_j has a chord, and we conclude that if the open disks bounded by F_1 and F_2 intersected, then their union would contain a disk bounded by a 4-cycle contradicting the maximality of F_i or F_j. Therefore, the open disks bounded by F_i and F_j are disjoint. If a vertex $s \in S$ is contained in the open disk bounded by F_i, then $s \notin V(B_S)$, and thus each vertex of S contributes at most 4 vertices to B_S.

Since $G[S]$ is connected, either $k = 1$ and $B_S = F_1$, or each of the cycles $F_1, \ldots, F_k$ contains a vertex of S. We conclude that B_S is connected. Finally, consider a set $S' \in \mathcal{S}_0$ distinct from S. Since the distance between each vertex of B_S and S (or $B_{S'}$ and S') is less than D_1 and the distance between S and S' is at least $2D_2 > 2D_1$, it follows that B_S and B'_S are vertex-disjoint. If $B_{S'}$ were not drawn in the outer face of H_S, then $G[S']$ would be contained either inside a face of $G[S]$ or inside one of the 4-cycles $F_1, \ldots, F_k$, which contradicts the choice of G_0. □

Let Λ be the intersection of the outer faces of the graphs H_S for $S \in \mathcal{S}_0$, and let G_1 be the subgraph of G contained in the closure of Λ. Observe that Q_0 is a subgraph of G_1. Let $Z_1 = Q_0 \cup \bigcup_{S \in \mathcal{S}_0} B_S$ and note that all triangles and separating 4-cycles in G_1 are contained in Z_1. By Proposition 2.4, G_1 is Z_1-critical.

A face f of G_1 is *poisonous* if either it has length at least 5 or its boundary is contained in Q_0. Since $|V(B_S)| \leq 4p$ for each $S \in \mathcal{S}_0$, $|\mathcal{S}_0| \geq 1$ and $|V(Q_0)| \leq p + 5$, the choice of μ and Theorem 2.2 imply that

$\sum |f| \leq \mu|\mathcal{S}_0|$, *where the summation is over all poisonous faces of G_1.* (4)

Consider $S \in \mathcal{S}_0$. We say that an angle (xyz, f) in G_1 is *S-contaminated* if f is poisonous and the distance between S and y in G is at most $D_2 - 1$. Since every S-contaminated angle contributes at least one toward the sum in (4), we deduce that there exists $S \in \mathcal{S}_0$ such that there are at most μ angles that are S-contaminated. Let us fix such S.

We say that an integer i such that $D_1 \leq i \leq D_2 - 1$ is *S-contaminated* if there exists an S-contaminated angle in G_1 whose tip is at distance exactly i from S in G. By the choice of D_2 and S, we conclude that there exist integers i_0 and i_1 such that

$\max(4, D_1) \leq i_0$ *and* $i_0 + 34(s-2)(s+3) + 483 \leq i_1 \leq D_2 - 1$ *and no integer* i *such that* $i_0 \leq i \leq i_1$ *is* S*-contaminated.* (5)

Our next claim bounds the length of equidistant cycles. Note that G_1 contains some vertices whose distance from S in G is at least D_2. By the choice of G_1 and the assumptions that $|\mathcal{S}| \geq 2$, the distance between elements of $\mathcal{S}$ is at least $2D_2$ and G_1 is connected.

If K *is a cycle in* G_1 *that is equidistant at distance* i *from* S *in* G*, where* $i_0 \leq i \leq i_1$*, then* $|K| \leq s$*.* (6)

Proof. Let $\{\Delta_e : e \in E(K)\}$ be a flower of K with respect to S. Since $|V(B_S)| \leq 4p$, it follows that the outer face of B_S has length at most $8p$, and thus at most $8p$ elements of the flower contain an edge of B_S in their closure.

Let us recall that at most $\mu = s - 8p$ angles are S-contaminated. If $|K| > s$, then there exists $e \in E(K)$ such that Δ_e contains no S-contaminated angle and the closure of Δ_e does not contain any edge of B_S. Observe that Δ_e is a z-petal for some $z \in S$. By Lemma 3.4, there exists a face f of G contained in Δ_e which is odd or bounded by C, and the distance of f from S is at most $D_2 - 1$. Since Δ_e does not contain a contaminated angle, it follows that f is not a face of G_1.

Let $f_1 \subseteq \Delta_e$ be the face of G_1 that contains f. Note that the distance between S and f_1 is at most $D_2 - 1$, and thus the boundary of f_1 is not contained in $G[S']$ for any $S' \in \mathcal{S}$ distinct from S. Furthermore, by the choice of Δ_e, the boundary of f_1 intersects B_S in at most one vertex. Since no angle in Δ_e is S-contaminated, we conclude that f_1 is not a face of Q_0 and it is a 4-face whose boundary F forms a separating 4-cycle in G. By the construction of G_1, there exists $S' \in \mathcal{S}_0$ such that $F \subseteq H_{S'}$. By the choice of Δ_e, we have $S' \neq S$. However, the distance between S' and F in G is less than $D_1 - 2$ by (3), and thus the distance between S and S' in G is less than $(D_2 - 1) + 2 + (D_1 - 2) < 2D_2$. This is a contradiction. □

Consider a vertex $v \in V(G_1)$ whose distance from S in G is i, where $i_0 \leq i \leq i_1$. Since i is not S-contaminated by (5), all faces incident with v are S-tight. Furthermore, $v \notin V(Z_1)$, and since G_1 is Z_1-critical, it follows that v has degree at least three in G_1. Let M be the set of vertices of G whose distance from S is at most D_1 and that are not drawn in the outer face of H_S. Let $G_2 = G_1 \cup G[M]$. Note that a vertex $v \in V(G_2)$ is at distance i from S in G_2, where $i_0 \leq i \leq i_1$, if and only if $v \in V(G_1)$ and the distance between v and S in G is i. The conclusion of this lemma then follows by Lemma 3.3 applied to G_2 with $S_1 = S$ and S_2 equal to an arbitrary vertex of G_2 at distance at least D_2 from S. $\square$

We also need a variant of Lemma 3.5 dealing with the case that $|\mathcal{S}| = 1$.

Lemma 3.6. *For all integers $p \geq 1$ and $r \geq 0$, there exists an integer $D_0 > r$ with the following property. Let G be a connected plane graph, S a subset of at most p vertices of G inducing a connected subgraph and C a cycle of length at most 5 bounding a face of G, such that the distance between S and C is at least $2D_0$. Suppose that every 4-face of G at distance at least $r + 4$ and at most $D_0 - 1$ from S is either S-tight or attached to a (≤ 6)-cycle separating S from C. Let $Z = C \cup G[S]$ and assume that every triangle in G is contained in Z and that the distance of every separating 4-cycle of G from S is at most r. If G is Z-critical, then G contains a clean joint vertex-disjoint from C.*

Proof. Let $\mu = (4p + 5)\eta + 4$, where η is the constant from Theorem 2.2, and let $s = \mu + 8p$. Let $D_0 = \max(r + 2, 4) + (\mu + 1)(34(s - 2)(s + 3) + 484)$.

Without loss of generality, assume that C bounds the outer face of G. By Lemma 2.1, each separating 4-cycle in G bounds an open disk containing at least one vertex of S, since G is Z-critical. Let B_S be the boundary of the outer face of the subgraph of G consisting of $G[S]$ and all separating 4-cycles. As in (3), B_S is connected and $|V(B_S)| \leq 4p$. Let G_1 be the subgraph of G drawn in the closure of the outer face of B_S, let $Z_1 = B_S \cup C$ and note that G_1 is Z_1-critical (by Proposition 2.4) and contains no separating 4-cycles. Let us define a poisonous face and S-contamination as in the proof of Lemma 3.5.

Since $|V(B_S)| \leq 4p$ and $|V(C)| \leq 5$, the choice of μ and Theorem 2.2 implies that $\sum |f| \leq \mu$, where the summation is over all poisonous faces of G. Consequently, there exist integers $i_0 \geq \max(r + 2, 4)$ and i_1 such that $i_0 + 34(s - 2)(s + 3) + 483 \leq i_1 \leq D_0 - 1$ and no integer i such that $i_0 \leq i \leq i_1$ is S-contaminated.

Next, we need to bound the lengths of equidistant cycles. Consider a cycle $K \subset G_1$ which is equidistant at distance i from S in G, where $i_0 \leq i \leq i_1$. Let $\{\Delta_e : e \in E(K)\}$ be a flower of K with respect to S. As in the proof of (6), we argue that the flower contains at least $|K| - 8p$ elements which are z-petals for some $z \in V(S)$ and their closure contains no edge of B_S, and thus the subgraph of G contained in their closure is also contained in G_1. Since each petal is bounded by an odd cycle, each of them contains an odd face. Since all triangles in G are contained in Z, such an odd face has length at least 5, and thus it is poisonous. Since G contains at most μ poisonous faces, we conclude that $|K| \leq \mu + 8p = s$.

Let M be the set of vertices at distance at most $r + 2$ from S that are not contained in the outer face of B_S. Note that $V(B_S) \subseteq M$, since every separating 4-cycle is at distance at most r from S in G. Let $G_2 = G[M] \cup G_1$.

The conclusion of this lemma then follows by Lemma 3.3 applied to G_2 with $S_1 = S$ and $S_2 = V(C)$. □

4 Colorings of quadrangulations of a cylinder

In this section, we give a lemma on extending a precoloring of boundaries of a quadrangulated cylinder. This is a special case of a more general theory which we develop in the following paper of the series [14].

Let C be a cycle drawn in plane, let $v_1, v_2, \ldots, v_k$ be the vertices of C listed in the clockwise order of their appearance on C, and let $\varphi : V(C) \to \{1, 2, 3\}$ be a 3-coloring of C. We can view φ as a mapping of $V(C)$ to the vertices of a triangle, and speak of the *winding number of φ on C*, defined as the number of indices $i \in \{1, 2, \ldots, k\}$ such that $\varphi(v_i) = 1$ and $\varphi(v_{i+1}) = 2$ minus the number of indices i such that $\varphi(v_i) = 2$ and $\varphi(v_{i+1}) = 1$, where v_{k+1} means v_1. We denote the winding number of φ on C by $W_\varphi(C)$.

Consider a plane graph G and its 3-coloring φ. For a face f of G bounded by a cycle C, we define the *winding number of φ on f* (denoted by $w_\varphi(f)$) as $-W_\varphi(C)$ if f is the outer face of G and as $W_\varphi(C)$ otherwise. The following two propositions are easy to prove.

Proposition 4.1. *Let G be a plane graph such that every face of G is bounded by a cycle, and let $\varphi : V(G) \to \{1, 2, 3\}$ be a 3-coloring of G. Then the sum of the winding numbers of all the faces of G is zero.*

Proposition 4.2. *The winding number of every 3-coloring on a cycle of length four is zero.*

Let G be a cylindrical quadrangulation with boundary faces f_1 and f_2. We say that the cylindrical quadrangulation is *boundary-linked* if every cycle K in G separating f_1 from f_2 and not bounding either of these faces has length at least $\max(|f_1|, |f_2|)$, and if $|K| = |f_i| = \max(|f_1|, |f_2|)$ for some $i \in \{1, 2\}$, then $V(K) \cap V(f_{3-i}) \neq \emptyset$. The cylindrical quadrangulation is *long* if the distance between f_1 and f_2 is at least $|f_1| + |f_2|$.

Lemma 4.3. *Let G be a long boundary-linked cylindrical quadrangulation with boundary faces f_1 and f_2 and let ψ be a 3-coloring of the boundary of G. Suppose that $|f_1| \geq \max(5, |f_2|)$ and let $v_1v_2v_3$ be a subpath of the cycle bounding f_1, where $\psi(v_1) = \psi(v_3)$. Then, there exists a long boundary-linked cylindrical quadrangulation G' with boundary faces f'_1 and f'_2 such that $|f'_1| = |f_1| - 2$ and $|f'_2| = |f_2|$ together with a 3-coloring ψ' of the boundary of G' such that if ψ' extends to a 3-coloring of G', then ψ extends to a 3-coloring of G.*

Proof. Note that since $\max(|f_1|, |f_2|) \geq 5$ and G is boundary-linked, it follows that G contains no triangle other than possibly the cycle bounding f_2, and thus the neighbors of v_2 form an independent set in G_2. Furthermore, f_1 is an induced cycle. Let G' be the cylindrical quadrangulation obtained from $G - v_2$ by contracting all neighbors of v_2 (including v_1 and v_3) to a single vertex w and by suppressing the arising 2-faces. Let f'_1 and f'_2 be the faces of G' corresponding to f_1 and f_2, respectively. Clearly, G' is long.

Let ψ' be the coloring of the boundary of G' such that $\psi'(w) = \psi(v_1)$ and $\psi'(z) = \psi(z)$ for $z \neq w$. If ψ' extends to a 3-coloring φ of G', then we can turn φ to a 3-coloring of G extending ψ by setting $\varphi(z) = \psi(v_1)$ for every neighbor z of v_2 and $\varphi(v_2) = \psi(v_2)$.

Consider a cycle K' separating f'_1 from f'_2 in G' and not bounding either of these faces. Let K be the corresponding cycle in G (equal to K', or obtained from K' by replacing w by a neighbor of v_2, or obtained from K' by replacing w by a path xv_2y for some neighbors x and y of v_2).

Let us first consider the case that $|f_1| > |f_2|$. Note that $|f_1|$ and $|f_2|$ have the same parity, and thus $|f_1| \geq |f_2|+2$ and $|f'_1| \geq |f_1|-2 \geq |f_2|$. Consequently, $|K'| \geq |K|-2 \geq |f_1|-2 = \max(|f'_1|, |f'_2|)$. Furthermore, the equality only holds if $v_2 \in V(K)$ and $|K| = |f_1|$. Since G is boundary-linked, the latter implies that K also contains a vertex incident with f_2. However, this contradicts the assumption that G is long. Therefore, we have $|K'| > \max(|f'_1|, |f'_2|)$.

Next, we consider the case that $|f_1| = |f_2|$, and thus $\max(|f'_1|, |f'_2|) = |f_2| > |f'_1|$. If $|K| = |f_2|$, then since G is boundary-linked, it would follow that K intersects both f_1 and f_2, contrary to the assumption that G is long. Therefore, $|K| > |f_2|$, and by parity, $|K| \geq |f_2| + 2$. Consequently, $|K'| \geq |K| - 2 \geq |f_2|$. The equality can only hold when K contains v_2, and thus K' contains the vertex w incident with f'_1. We conclude that G' is boundary-linked. □

Lemma 4.4. *Let G be a long cylindrical quadrangulation with boundary faces f_1 and f_2 and let ψ be a 3-coloring of the boundary of G. If $|f_1| = |f_2| = 4$, then ψ extends to a 3-coloring of G.*

Proof. Let $v_1v_2v_3v_4$ be the cycle bounding f_1, where $\psi(v_1) = \psi(v_3)$. Note that G is bipartite, and thus the vertices at distance exactly three from $\{v_2, v_4\}$ form an independent set. Let G' be the quadrangulation of the plane obtained from G by removing all vertices at distance at most two from $\{v_2, v_4\}$, identifying all vertices at distance exactly three from $\{v_2, v_4\}$ to a single (non-boundary) vertex w and by suppressing the arising 2-faces.

Let ψ' be a restriction of ψ to the 4-cycle bounding the face of G' corresponding to f_2. By Lemma 2.3, ψ' extends to a 3-coloring φ of G'. We can extend φ to a 3-coloring of G as follows. Give all vertices at distance exactly 1 from $\{v_2, v_4\}$ the color $\psi(v_1) = \psi(v_3)$, all vertices at distance exactly 3 from $\{v_2, v_4\}$ the color $\varphi(w)$ and all vertices at distance exactly 2 from $\{v_2, v_4\}$ an arbitrary color different from $\psi(v_1)$ and $\varphi(w)$. The resulting assignment is a 3-coloring of G extending ψ. □

Lemma 4.5. *Let G be a long boundary-linked cylindrical quadrangulation with boundary faces f_1 and f_2 and let ψ be a 3-coloring of the boundary of G. The coloring ψ extends to a 3-coloring of G if and only if $w_\psi(f_1) + w_\psi(f_2) = 0$.*

Proof. If ψ extends to a 3-coloring of G, then $w_\psi(f_1) + w_\psi(f_2) = 0$ by Propositions 4.1 and 4.2.

Let us now show the converse implication. We proceed by induction and assume that the claim holds for all graphs whose boundary has less than $|f_1| + |f_2|$ vertices. By symmetry, we can assume that $|f_1| \geq |f_2|$. If $|f_1| = 4$, then since $|f_1|$ and $|f_2|$ have the same parity, we have $|f_2| = 4$, and ψ extends to a 3-coloring of G by Lemma 4.4. Thus, assume $|f_1| \geq 5$. If the cycle bounding f_1 contains a path $v_1v_2v_3$ with $\psi(v_1) = \psi(v_3)$, then ψ extends to a 3-coloring of G by Lemma 4.3 and the induction hypothesis.

Therefore, we can assume that the boundary cycle of f_1 contains no such path, and thus $|f_1|$ is a multiple of 3 and $|w_\psi(f_1)| = |f_1|/3$. Since $w_\psi(f_1) + w_\psi(f_2) = 0$, we have $|w_\psi(f_2)| = |f_1|/3$, and since $|f_2| \leq |f_1|$ and $|w_\psi(f_2)| \leq |f_2|/3$, we conclude that $|f_2| = |f_1|$. Since G is long and boundary-linked, every cycle in G that separates f_1 from f_2 and does not bound either of the faces has length at least $|f_1| + 2$.

Let $G^\star$ be the dual of G. Let K_i be the edge-cut in G consisting of the edges incident with $V(f_i)$ that do not belong to $E(f_i)$. Note that the dual $K_i^\star$ of K_i is a cycle in $G^\star$. Let $H = G^\star - (E(K_1^\star) \cup E(K_2^\star))$. Let $f_1^\star$ and $f_2^\star$ be the vertices of the dual corresponding to f_1 and f_2, respectively. Suppose that H contains an edge-cut of size less than $|f_1|$ separating $f_1^\star$ from $f_2^\star$. Then, there exists a cycle K in G separating f_1 from f_2 and not bounding either of the faces such that $|E(K) \setminus (E(K_1) \cup E(K_2))| < |f_1|$. Since G is long, K does not intersect both K_1 and K_2. As we observed before, $|K| \geq |f_1| + 2$, and thus we can by symmetry assume that K intersects K_1 in at least three edges. Let us choose such a cycle K that shares as many edges with the cycle bounding f_1 as possible. Let P be a subpath of K with both endpoints incident with f_1, but no other vertex or edge incident with f_1. Let Q_1 and Q_2 be the two subpaths of the cycle bounding f_1 joining the endpoints of P labelled so that $P \cup Q_2$ is a cycle separating f_1 from f_2. Consider the cycle $K' = (K - P) \cup Q_1$. Since K intersects K_1 in at least three edges, K' is not the cycle bounding f_1. Since K' shares more edges with the cycle bounding f_1 than K, the choice of K implies that $|E(K') \setminus (E(K_1) \cup E(K_2))| \geq |f_1|$, and since $|Q_1 \cap (E(K_1) \cup E(K_2))| = 0$ and $|P \cap (E(K_1) \cup E(K_2))| = 2$, we conclude that $|Q_1| > |P| - 2$. However, then the cycle $P \cup Q_2$ has length less than $|f_1| + 2$, contrary to the assumption that G is boundary-linked.

Therefore, there is no such edge-cut in H, and by Menger's theorem, H contains pairwise edge-disjoint paths $P_1, \ldots, P_{|f_1|}$ joining $f_1^\star$ with $f_2^\star$. Note that all vertices of $H' = H - E(P_1 \cup P_2 \cup \ldots \cup P_{|f_1|})$ have even degree, and thus H' is a union of pairwise edge-disjoint cycles $C_1, \ldots, C_m$. For $1 \leq i \leq m$, direct the edges of C_i so that all vertices of C_i have outdegree 1. For $1 \leq i \leq |f_1|$, direct the edges of P_i so that all its vertices except for $f_1^\star$ have outdegree 1. This gives an orientation $\vec{H}$ of H such that the indegree of every vertex of $V(H) \setminus \{f_1^\star, f_2^\star\}$ equals its outdegree, $f_1^\star$ has outdegree 0 and $f_2^\star$ has indegree 0. Let $\vec{G}_1^\star$ be the orientation of $G^\star$ obtained from $\vec{H}$ by orienting all edges of $K_1^\star$ and $K_2^\star$ in the clockwise direction along the cycles. Let $\vec{G}_2^\star$ be the orientation of $G^\star$ obtained from $\vec{G}_1^\star$ by reversing the orientation of the edges of $K_1^\star$, and let $\vec{G}_3^\star$ be the orientation of $G^\star$ obtained from $\vec{G}_2^\star$ by reversing the orientation of the edges of $K_2^\star$.

Since $|f_1| = |f_2|$ is a multiple of 3, it follows that the orientations $\vec{G}_1^\star$, $\vec{G}_2^\star$ and $\vec{G}_3^\star$ define nowhere-zero Z_3-flows in $G^\star$. Let φ_1, φ_2 and φ_3 be the corresponding 3-colorings of G; since $|w_\psi(f_1)| = |f_1|/3$, we can choose the colorings so that their restrictions to the cycle bounding f_1 match ψ. Since $|w_\psi(f_2)| = |f_2|/3$ and $w_\psi(f_1) + w_\psi(f_2) = 0$, the restrictions of φ_1, φ_2 and φ_3 to the boundary of f_2 differ from ψ only by a cyclic permutation of colors. Observe that the colors $\varphi_1(v)$, $\varphi_2(v)$ and $\varphi_3(v)$ are pairwise distinct for every $v \in V(f_2)$, and thus there exists $i \in \{1, 2, 3\}$ such that φ_i is a 3-coloring of G extending ψ. □

The inspection of the proofs of Lemmas 4.3, 4.4, and 4.5 shows that they are constructive and can be implemented as linear-time algorithms to find the described 3-colorings (Lemma 2.3 is only used in the proof of Lemma 4.4 to extend the precoloring of a 4-cycle, and a linear-time algorithm for this special case appears in [10]). Hence, we obtain the following corollary which we use in the next paper of the series [14].

Corollary 4.6. *For all positive integers d_1 and d_2, there exists a linear-time algorithm as follows. Let G be a cylindrical quadrangulation with boundary faces f_1 and f_2 and let ψ be a 3-coloring of the boundary of G such that $w_\psi(f_1) + w_\psi(f_2) = 0$. Suppose that $|f_1| = d_1$, $|f_2| = d_2$, every cycle in G separating f_1 from f_2 and not bounding either of these faces has length greater than $\max(d_1, d_2)$, and the distance between f_1 and f_2 is at least $d_1 + d_2$. Then the algorithm returns a 3-coloring of G that extends ψ.*

We also need another result similar to Lemma 4.5.

Corollary 4.7. *Let G be a joint with boundary faces f_1 and f_2 and let ψ be a 3-coloring of the boundary of G such that $w_\psi(f_1) + w_\psi(f_2) = 0$. If $|w_\psi(f_1)| < |f_1|/3$, then ψ extends to a 3-coloring of G.*

Proof. Since $|w_\psi(f_1)| < |f_1|/3$, we have $|f_1| \neq 3$. If $|f_1| = 4$, then ψ extends to a 3-coloring of G by Lemma 4.4. Therefore, assume $|f_1| \geq 5$. Since $|w_\psi(f_1)| < |f_1|/3$ and $|w_\psi(f_2)| < |f_2|/3$, there exist paths $u_1u_2u_3$ and $v_1v_2v_3$ in the cycles bounding f_1 and f_2, respectively, such that $\psi(u_1) = \psi(u_3)$ and $\psi(v_1) = \psi(v_3)$. Let G' be the cylindrical quadrangulation obtained from $G - u_2 - v_2$ by identifying all neighbors of u_2 to a single vertex w_1 and all neighbors of v_2 to a single vertex w_2. Let ψ' be the coloring of the boundary of G' such that $\psi'(w_1) = \psi(u_1)$, $\psi'(w_2) = \psi(v_1)$ and $\psi'(z) = \psi(z)$ for any other boundary vertex of G'. Clearly, it suffices to show that ψ' extends to a 3-coloring of G'.

Let f_1' and f_2' be the boundary faces of G' corresponding to f_1 and f_2, respectively. Note that every cycle in G' separating f_1' from f_2' has length at least $|f_1'|$, and each such cycle of length $|f_1'|$ contains either w_1 or w_2. We can assume that G' is drawn so that f_1' is its outer face. Let A be a subset of the plane homeomorphic to the closed annulus such that the boundary of A is formed by cycles in G' of length $|f_1'|$ separating f_1' from f_2', one of them containing w_1, the other one containing w_2, such that no other cycle separating

f_1' from f_2' is contained in A. Let G_0 be the subgraph of G' drawn in A. Removing A splits the plane into two connected open sets B_1 and B_2, where $f_1' \subset B_1$. For $i \in \{1,2\}$, let G_i be the subgraph of G' drawn B_i. Note that G_0 is a long boundary-linked cylindrical quadrangulation. By Lemma 2.3, ψ' extends to a 3-coloring of $G_1 \cup G_2$, and by Lemma 4.5, the resulting coloring of the boundary of G_0 extends to a 3-coloring of G_0. This gives a 3-coloring of G' extending ψ'. □

An *s-cap* is a cylindrical quadrangulation G with boundary faces f_1 and f_2, such that G does not contain separating 4-cycles, $|f_1| = s$, $|f_2| = 4 + (s \bmod 2)$ and for every $u, v \in V(f_1)$, the distance between u and v in G is the same as their distance in the cycle bounding f_1. We call f_2 the *special face* of the s-cap.

Lemma 4.8. *For every $s \geq 4$, there exists an s-cap G that has fewer vertices than every joint with boundary faces of length s.*

Proof. Let G be an s-cap obtained from the $s \times s$ cylindrical quadrangulation by adding chords to one of its boundary faces. We have $|V(G)| = s^2$.

Consider any joint H with boundary faces f_1 and f_2 of length s. For $1 \leq i \leq 4s-1$, let V_i denote the set of vertices of H at distance exactly i from f_1. Observe that since all faces of H other than f_1 and f_2 have length 4, $H[V_i \cup V_{i+1}]$ contains a cycle separating f_1 from f_2 for $1 \leq i \leq 4s-2$, and thus $|V_i| + |V_{i+1}| \geq s$. Therefore, $|V(H)| \geq |f_1| + |f_2| + (2s-1)s = (2s+1)s > |V(G)|$. □

5 3-coloring with distant anomalies

An *anomaly* is a triple $T = (H_T, B_T, \Phi_T)$, where H_T is a connected plane graph, $B_T \subseteq V(H_T)$ and Φ_T is a set of 3-colorings of H_T such that for every $\psi \in \Phi_T$, there exist distinct colors a and b such that the 3-coloring obtained from ψ by swapping the colors a and b also belongs to Φ_T. An anomaly T *appears* in a plane graph G if H is an induced subgraph of G (where the plane embedding of H is induced by the embedding of G) and every $v \in B_T$ satisfies $\deg_G(v) = \deg_{H_T}(v)$. Given a 3-coloring φ of a plane graph G and an anomaly T appearing in H, we say that φ is *compatible with* T if $\varphi \restriction V(H_T) \in \Phi_T$.

An anomaly T is *locally extendable* if for every plane graph G, if T appears in G and all triangles in G are contained in H_T, then there exists a 3-coloring of G compatible with T. For an integer $r \geq 0$, an anomaly T is *strongly locally extendable with margin r* if for all plane graphs G in that T appears so that all triangles of G are contained in H_T, and for all 4-faces C of G at distance at least r from H_T, every 3-coloring ψ of C extends to a 3-coloring of G compatible with T.

The following anomalies are of interest for Theorems 1.2 and 1.3. Recall that the pattern of a 3-coloring ψ is the set $\{\psi^{-1}(1), \psi^{-1}(2), \psi^{-1}(3)\}$.

- A single precolored vertex (H_T is a single vertex, B_T is empty and Φ_T consists of a coloring assigning to the vertex of H_T the prescribed color).

This anomaly is locally extendable by Grötzsch' theorem. We believe it is also strongly locally extendable with some margin, see Conjecture 1.5.

- A cycle of length at most 5 with a prescribed pattern of coloring (H_T is a (≤ 5)-cycle, B_T is empty and Φ_T consists of all 3-colorings of H_T with the prescribed pattern). This anomaly is locally extendable by Lemma 2.1. Furthermore, the same lemma implies that if the cycle has length 3, then the anomaly is strongly locally extendable with margin 0.

- A vertex of degree at most 4 with neighborhood precolored by one color (H_T is a star with at most 4 rays, B_T contains the center of the star and Φ_T consists of all 3-colorings of H_T which assign the prescribed color to the rays). This anomaly is locally extendable by the results of Gimbel and Thomassen [16] for degree at most 3 and Dvořák and Lidický [15] for degree 4 (given a vertex v of degree $k \leq 4$ with precolored neighborhood, split v into k vertices of degree two colored arbitrarily and extend the coloring of the resulting $2k$-cycle).

Thus, both Theorem 1.2 and Theorem 1.3 are implied by the following general statement (which also shows that Conjecture 1.5 implies Conjecture 1.4), by letting C be the null graph, $p = 5$ and $r = 0$.

Theorem 5.1. *For all integers $p \geq 1$ and $r \geq 0$, there exist constants $0 < d_0 < d_1$ with the following property. Let G be a plane graph and let $\mathcal{T} = \{T_i : 1 \leq i \leq n\}$ be a set of locally extendable anomalies appearing in G, such that $|V(H_{T_i})| \leq p$ for $1 \leq i \leq n$. Suppose that*

- *for $1 \leq i < j \leq n$, the distance between H_{T_i} and H_{T_j} in G is at least $2d_1$,*

- *every triangle in G is contained in H_T for some $T \in \mathcal{T}$, and*

- *if a separating 4-cycle K is at distance less than $2d_0$ from H_T for some $T \in \mathcal{T}$, then either K is contained in H_T, or T is strongly locally extendable with margin r.*

Let C be either the null graph or a facial cycle of G of length at most five, at distance at least $2d_0$ from H_T for each $T \in \mathcal{T}$. Then, every 3-coloring of C extends to a 3-coloring of G compatible with all elements of $\mathcal{T}$.

Proof. Let d_0 be equal to D_0 from Lemma 3.6 with the given p and r. Let d_1 be equal to D_2 from Lemma 3.5 applied with $D_1 = 2d_0 + 3$ and the given p. We will prove by induction on $|V(G)|$ that d_0 and d_1 satisfy the conclusion of the theorem.

Let G be as stated, let ψ be a 3-coloring of C, and assume for a contradiction that ψ does not extend to a 3-coloring of G compatible with all elements of $\mathcal{T}$. Let $\mathcal{S} = \{V(H_T) : T \in \mathcal{T}\}$, $Z_0 = \bigcup_{T \in \mathcal{T}} H_T$ and $Z = C \cup Z_0$. For a set $X \subseteq V(G)$, let $\mathcal{T}[X] = \{T \in \mathcal{T} : V(H_T) \subseteq X\}$.

We may assume, by taking a subgraph of G, that ψ extends to a 3-coloring compatible with all elements of $\mathcal{T}$ for every proper subgraph of G that includes

Z. Note that $G \neq Z$, as otherwise we can color each component of Z_0 separately by the local extendability of the anomalies. Consequently, G is Z-critical. Note that G is connected, as otherwise we can color each component of G separately by induction.

If K is a separating (≤ 5)-cycle and Δ is one of the connected open regions of the plane bounded by K, then at least one vertex or edge of Z is drawn in Δ, since G is Z-critical and every 3-coloring of a (≤ 5)-cycle extends to a 3-coloring of a triangle-free planar graph by Lemma 2.1. We claim that

if K is a separating cycle of length at most five in G and $K \not\subseteq Z_0$, then the distance between K and Z_0 is less than $2d_0$. Furthermore, if $|K| = 4$ and one of the connected open regions of the plane bounded by K is disjoint from C and contains a vertex of exactly one $S \in \mathcal{S}$, then the distance between K and S is at most r.

(7)

Proof. Let K be a separating cycle of length at most five in G. Removing K splits the plane into two open sets Δ_1 and Δ_2, labelled so that the face bounded by C (if any) is contained in Δ_1.

Suppose that either the distance between K and Z_0 is at least $2d_0$, or that $|K| = 4$ and Δ_2 contains a vertex of exactly one $S \in \mathcal{S}$ and the distance between K and S is greater than r (and in particular, $G[S]$ is contained in Δ_2). For $i \in \{1, 2\}$, let G_i be the subgraph of G drawn in the closure of Δ_i. Note that $|V(G_i)| < |V(G)|$. By the induction hypothesis, the precoloring ψ extends to a 3-coloring φ_1 of G_1 compatible with all elements of $\mathcal{T}[V(G_1)]$. Similarly, the restriction of φ_1 to K extends to a 3-coloring of G_2 compatible with all elements of $\mathcal{T}[V(G_2)]$ (either by the induction hypothesis if the distance between K and $Z_0 \cap G_2$ is at least $2d_0$, or by the strong local extendability of the anomaly corresponding to S otherwise). The union of these 3-colorings is a 3-coloring of G compatible with all elements of $\mathcal{T}$. This is a contradiction. □

Suppose first that $|\mathcal{S}| \geq 2$. We consider 4-faces of G.

Let f be a 4-face of G at distance at least $2d_0 + 3$ from Z_0. If f is not bounded by C, then f is S-tight for a unique $S \in \mathcal{S}$ at distance at most $d_1 - 1$ from f.

(8)

Proof. Let the vertices of f be numbered u_1, u_2, u_3, u_4 in order. By (7), no vertex of f is contained in a separating 4-cycle. Furthermore, C does not share an edge with a triangle, and thus the intersection of the boundary of f with C is a path of length at most two.

If say $u_1, u_2, u_3 \in V(C)$, then note that u_2 has degree two. Consider the graph $G - u_2$ and color u_4 by $\psi(u_2)$. By the minimality of G, this coloring extends to a 3-coloring of $G - u_2$ compatible with all elements of $\mathcal{T}$, which also gives a 3-coloring of G extending ψ and compatible with all elements of $\mathcal{T}$, a contradiction.

Therefore, we can assume that $u_3, u_4 \notin V(C)$. Note that $u_1u_2u_3$ and $u_1u_4u_3$ are the only paths of length at most three joining u_1 with u_3, since f is at distance at least $2d_0+3$ from Z_0, u_1 and u_3 are not contained in a separating (≤ 5)-cycle by (7), and u_4 has degree at least three. Let G_{13} be the graph obtained from G by identifying u_1 and u_3 and suppressing parallel edges, and observe that G_{13} contains no new triangles and C is edge-disjoint from all triangles in G_{13}. Furthermore, every new separating 4-cycle in G_{13} is at distance at least $2d_0$ from Z_0. Let G_{24} be defined analogously.

If G_{13} or G_{24} satisfies the assumptions of Theorem 5.1, then it has a 3-coloring extending ψ and compatible with all elements of $\mathcal{T}$ by induction, which would give such a 3-coloring of G. Otherwise, both G_{13} and G_{24} contain a pair of anomalies at distance at most $2d_1 - 1$ from each other, and thus f is S-tight for a unique $S \in \mathcal{S}$ at distance at most $d_1 - 1$ from f by Lemma 3.1. $\square$

Therefore, we can apply Lemma 3.5 and conclude that G contains a clean joint H vertex-disjoint from C, at distance at least $2d_0$ from Z_0. Let f_1 and f_2 be the boundary faces of H, labelled so that the face of G bounded by C (if any) is contained in f_1. For $i \in \{1, 2\}$, let G'_i be the subgraph of G drawn in the closure of f_i. Let H_i be an $|f_i|$-cap with its non-special boundary cycle equal to the boundary of f_i, but otherwise disjoint from G'_i, such that $|V(H_i)| < |V(H)|$, which exists by Lemma 4.8. Let h_i be the special face of H_i. Let $G_i = G'_i + H_i$. Note that the distance between any two elements of $\mathcal{S} \cup \{C\}$ in G_i is the same as the distance between them in G'_i, which is greater or equal to their distance in G. By induction, ψ extends to a 3-coloring φ_1 of G_1 compatible with all the elements of $\mathcal{T}[V(G'_1)]$. Consider the restriction of φ_1 to H_1. Propositions 4.1 and 4.2 imply that $w_{\varphi_1}(f_1) + w_{\varphi_1}(h_1) = 0$. Furthermore, since h_1 has length at most 5, we have $w_{\varphi_1}(h_1) = 0$ if $|h_1| = 4$ (f_1 has even length) and $|w_{\varphi_1}(h_1)| = 1$ if $|h_1| = 5$ (f_1 has odd length).

Let ψ_2 be an arbitrary 3-coloring of the boundary of h_2 with winding number equal to $-w_{\varphi_1}(h_1)$. By induction, ψ_2 extends to a 3-coloring φ_2 of G_2 compatible with all elements of $\mathcal{T}[V(G'_2)]$. By Propositions 4.1 and 4.2 for H_2, we have $w_{\varphi_2}(f_2) = -w_{\varphi_2}(h_2) = w_{\varphi_1}(h_1) = -w_{\varphi_1}(f_1)$. By Corollary 4.7, the restriction of $\varphi_1 \cup \varphi_2$ to the boundary cycles of f_1 and f_2 extends to a 3-coloring φ_3 of H. Consequently, the restriction of φ_1 to G'_1, the restriction of φ_2 to G'_2, and φ_3 together give a 3-coloring of G extending ψ and compatible with all the elements of $\mathcal{T}$. This contradiction finishes the proof in the case that $|\mathcal{S}| \ge 2$.

If $\mathcal{S} = \emptyset$, then ψ extends to a 3-coloring of G by Lemma 2.1. Therefore, we can assume that $\mathcal{S} = \{S\}$ and $\mathcal{T} = \{T\}$. If C is the null graph, then G has a 3-coloring compatible with T, since T is locally extendable. Hence, suppose that C bounds a (≤ 5)-face.

By (7) and the assumptions of this theorem, if T is strongly locally extendable with margin r, then every separating 4-cycle is at distance at most r from S in G, and otherwise G contains no separating 4-cycles.

Let f be a 4-face of G at distance at least $r+4$ and at most $d_0 - 1$ from S. If f is not S-tight, then f is attached to a (≤ 6)-cycle separating S from C.

(9)

Proof. Let the vertices of f be numbered u_1, u_2, u_3, u_4 in order. If neither u_1 and u_3, nor u_2 and u_4 are joined by a path of length at most 4 not contained in the boundary of f, then we proceed as in the proof of (8), since neither G_{13} nor G_{24} contains separating 4-cycles. Hence, suppose that say u_1 and u_3 are joined by a path of length at most 4 not containing u_2 or u_4. Let K be the cycle formed by this path together with $u_1u_2u_3$.

Suppose that K does not separate S from C. If $|K| \leq 5$, then by the minimality of G and Lemma 2.1, we conclude that K bounds a face of G, and thus u_4 has degree two, which is a contradiction since G is Z-critical. If $|K| = 6$, then we can still proceed as in the proof of (8)—G_{13} contains a separating 4-cycle K', however K' does not separate S from C, and thus we can split G_{13} on K', color the part containing C and S by induction, and extend the coloring to the other part by Lemma 2.1.

Therefore, K separates S from C, and f is attached to K. □

We can now apply Lemma 3.6 and obtain a clean joint H in G, with boundary faces f_1 and f_2. Let G'_i, H_i, h_i and G_i be defined as in the case $|\mathcal{S}| \geq 2$. If both C and S are contained in G_1, then we proceed in the same way as in the case $|\mathcal{S}| \geq 2$. Hence, suppose that S is contained in G_2, while $C \subset G_1$. We extend ψ to a 3-coloring φ_1 of G_1 by Lemma 2.1. We find a 3-coloring φ'_2 of G_2 compatible with T by the local extendability of T. Let a and b be distinct colors such that the 3-coloring φ''_2 obtained from φ' by swapping the colors a and b is also compatible with T. If $w_{\varphi_1}(h_1) = -w_{\varphi_2}(h_2)$, then we set $\varphi_2 = \varphi'_2$, otherwise we set $\varphi_2 = \varphi''_2$. Note that $w_{\varphi_1}(h_1) + w_{\varphi_2}(h_2) = 0$, and thus we can extend these colorings to a 3-coloring of G compatible with T as in the case $|\mathcal{S}| \geq 2$. This contradiction finishes the proof. □

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