Three-coloring triangle-free graphs on surfaces
III. Graphs of girth five

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Abstract
We show that the size of a 4-critical graph of girth at least five is bounded by a linear function of its genus. This strengthens the previous bound on the size of such graphs given by Thomassen. It also serves as the basic case for the description of the structure of 4-critical triangle-free graphs embedded in a fixed surface, presented in a future paper of this series.

1 Introduction
This paper is a part of a series aimed at studying the 3-colorability of graphs on a fixed surface that are either triangle-free, or have their triangles restricted in some way. Historically the first result in this direction is the following classical theorem of Grötzsch [10].

Theorem 1.1. Every triangle-free planar graph is 3-colorable.

Thomassen [13, 14, 16] found three reasonably simple proofs of this claim. Recently, two of us, in joint work with Kawarabayashi [4] were able to design a linear-time algorithm to 3-color triangle-free planar graphs, and as a by-product found perhaps a yet simpler proof of Theorem 1.1. The statement of Theorem 1.1 cannot be directly extended to any surface other than the sphere. In fact, for every non-planar surface Σ there are infinitely many 4-critical graphs that can be embedded in Σ. For instance, the graphs obtained from an odd cycle of length five or more by applying Mycielski’s construction [3, Section 8.5] have that property. Thus an algorithm for testing 3-colorability of triangle-free graphs on a fixed surface will have to involve more than just testing the presence of finitely many obstructions.

The situation is different for graphs of girth at least five by another deep theorem of Thomassen [15], the following.

Theorem 1.2. For every surface Σ, there are only finitely many 4-critical graphs of girth at least five that can be embedded in Σ.

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Thus the 3-colorability problem on a fixed surface has a linear-time algorithm for graphs of girth at least five, but the presence of cycles of length four complicates matters. Let us remark that there are no 4-critical graphs of girth at least five on the projective plane and the torus [13] and on the Klein bottle [12].

In his proof of Theorem 1.2, Thomassen does not give a specific bound on the size of a 4-critical graph of girth at least five embedded in Σ. It appears that if one was to extract a bound from the argument, that bound would be at least doubly-exponential in the genus of Σ. In this paper, we give a different proof of the result, which gives a linear bound.

**Theorem 1.3.** There exists a constant \( C \) with the following property. If \( G \) is a 4-critical graph of Euler genus \( g \) and girth at least 5, then \(|V(G)| \leq Cg|\).

Let us now outline the relationship of this result to the structure of triangle-free 4-critical graphs. The only non-planar surface for which the 3-colorability problem for triangle-free graphs is fully characterized is the projective plane. Building on earlier work of Youngs [17], Gimbel and Thomassen [9] obtained the following elegant characterization. A graph embedded in a surface is a quadrangulation if every face is bounded by a cycle of length four.

**Theorem 1.4.** A triangle-free graph embedded in the projective plane is 3-colorable if and only if it has no subgraph isomorphic to a non-bipartite quadrangulation of the projective plane.

For other surfaces there does not seem to be a similarly nice characterization, but in a later paper of this series we will present a polynomial-time algorithm to decide whether a triangle-free graph in a fixed surface is 3-colorable. The algorithm naturally breaks into two steps. The first is when the graph is a quadrangulation, except perhaps for a bounded number of larger faces of bounded size, which will be allowed to be precolored. In this case there is a simple topological obstruction to the existence of a coloring extension based on the so-called “winding number” of the precoloring. Conversely, if the obstruction is not present and the graph is highly “locally planar”, then we can show that the precoloring can be extended to a 3-coloring of the entire graph. This can be exploited to design a polynomial-time algorithm. With additional effort the algorithm can be made to run in linear time.

The second step covers the remaining case, when the graph has either many faces of size at least five, or one large face, and the same holds for every subgraph. In that case, we reduce the problem to Theorem 1.3 and show that the graph is 3-colorable. More precisely, in a future paper of this series, we use Theorem 1.3 to derive the following cornerstone result.

**Theorem 1.5.** There exists an absolute constant \( K \) with the following property. Let \( G \) be a graph embedded in a surface \( \Sigma \) of Euler genus \( \gamma \) with no separating cycles of length at most four, and let \( t \) be the number triangles in \( G \). If \( G \) is 4-critical, then \( \sum |f| \leq K(t + \gamma) \), where the summation is over all faces \( f \) of \( G \) of length at least five.

The fact that the bound in Theorems 1.3 and 1.5 is linear is needed in our solution [6] of a problem of Havel [11], as follows.

**Theorem 1.6.** There exists an absolute constant \( d \) such that if \( G \) is a planar graph and every two distinct triangles in \( G \) are at distance at least \( d \), then \( G \) is 3-colorable.
Our technique to prove Theorem 1.3 is a refinement of the standard method of reducible configurations. We show that every sufficiently generic graph $G$ (i.e., a graph that is large enough and cannot be decomposed to smaller pieces along cuts simplifying the problem) embedded in a surface contains one of a fixed list of configurations. Each such configuration enables us to obtain a smaller 4-critical graph $G'$ with the property that every 3-coloring of $G'$ corresponds to a 3-coloring of $G$. Furthermore, we perform the reduction in such a way that a properly defined weight of $G'$ is greater or equal to the weight of $G$.

A standard inductive argument then shows that the weight of every 4-critical graph is bounded, which also restricts its size. This brief exposition however hides a large number of technical details that were mostly dealt with in the previous paper in the series [7]. There, we introduced this basic technique and used it to prove the following special case of Theorem 1.5.

Theorem 1.7. Let $G$ be a graph of girth at least 5 embedded in the plane and let $C$ be a cycle in $G$. Suppose that there exists a precoloring $\phi$ of $C$ by three colors that does not extend to a proper 3-coloring of $G$. Then there exists a subgraph $H \subseteq G$ such that $C \subseteq H$, $|V(H)| \leq 1715|C|$ and $H$ has no proper 3-coloring extending $\phi$.

Further results of [7] needed in this paper are summarized in Section 3.

2 Definitions

In this section, we give a few basic definitions All graphs in this paper are simple, with no loops or parallel edges.

A surface is a compact connected 2-manifold with (possibly null) boundary. Each component of the boundary is homeomorphic to the circle, and we call it a cuff. For non-negative integers $a$, $b$ and $c$, let $\Sigma(a, b, c)$ denote the surface obtained from the sphere by adding $a$ handles, $b$ crosscaps and removing interiors of $c$ pairwise disjoint closed discs. A standard result in topology shows that every surface is homeomorphic to $\Sigma(a, b, c)$ for some choice of $a$, $b$ and $c$. Note that $\Sigma(0, 0, 0)$ is a sphere, $\Sigma(0, 0, 1)$ is a closed disk, $\Sigma(0, 0, 2)$ is a cylinder, $\Sigma(1, 0, 0)$ is a torus, $\Sigma(0, 1, 0)$ is a projective plane and $\Sigma(0, 2, 0)$ is a Klein bottle. The Euler genus $g(\Sigma)$ of the surface $\Sigma = \Sigma(a, b, c)$ is defined as $2a + b$. For a cuff $C$ of $\Sigma$, let $\hat{C}$ denote an open disk with boundary $C$ such that $\hat{C}$ is disjoint from $\Sigma$, and let $\Sigma + \hat{C}$ be the surface obtained by gluing $\Sigma$ and $\hat{C}$ together, that is, by closing $C$ with a patch. Let $\Sigma = \Sigma + \hat{C}_1 + \ldots + \hat{C}_c$, where $C_1, \ldots, C_c$ are the cuffs of $\Sigma$, be the surface without boundary obtained by patching all the cuffs.

Consider a graph $G$ embedded in the surface $\Sigma$; when useful, we identify $G$ with the topological space consisting of the points corresponding to the vertices of $G$ and the simple curves corresponding to the edges of $G$. A face $f$ of $G$ is a maximal connected subset of $\Sigma - G$ (let us remark that this definition is somewhat non-standard, and in particular the face is not necessarily a subset of the surface). The boundary of a face is equal to a union of closed walks of $G$, which we call the boundary walks of $f$ (in case that $G$ contains an isolated vertex, this vertex forms a boundary walk by itself).

An embedding of $G$ in $\Sigma$ is normal if each cuff $C$ that intersects $G$ either does so in exactly one vertex $v$ or is equal to a cycle $B$ in $G$. In the former case, we call $v$ a vertex ring and the face of $G$ that contains $C$ the cuff face of $v$. In
the latter case, note that $B$ is the boundary walk of the face $C$ of $G$; we say that $B$ is a facial ring. A face of $G$ is a ring face if it is equal $C$ for some ring $C$, and internal otherwise (in particular, the cuff face of a vertex ring is internal). We write $F(G)$ for the set of internal faces of $G$. A vertex $v$ of $G$ is a ring vertex if $v$ is incident with a ring (i.e., it is drawn in the boundary of $\Sigma$), and internal otherwise. A cycle $K$ in $G$ is separating the surface if $\Sigma - K$ has two components, and $K$ is non-separating otherwise. A cycle $K$ is contractible if there exists a closed disk $\Delta \subseteq \Sigma$ with boundary equal to $K$. A cycle $K$ surrounds the cuff $C$ if $K$ is not contractible in $\Sigma$, but it is contractible in $\Sigma + C$. We say that $K$ surrounds a ring $R$ if $K$ surrounds the cuff incident with $R$.

Let $G$ be a graph embedded in a surface $\Sigma$, let the embedding be normal, and let $R$ be the set of vertex rings and facial rings of this embedding. In those circumstances we say that $G$ is a graph in $\Sigma$ with rings $R$. Furthermore, some vertex rings are designated as weak vertex rings. Unless explicitly specified otherwise, we assume that every cuff of $\Sigma$ is incident with a ring in $R$.

The length $|R|$ of a facial ring is the number of edges of $R$. For a vertex ring $R$, we define $|R| = 0$ if $R$ is weak and $|R| = 1$ otherwise. For an internal face $f$, by $|f|$ we mean the sum of the lengths of the boundary walks of $f$ (in particular, if an edge appears twice in the boundary walks, it contributes 2 to $|f|$); if a boundary walk consists just of a vertex ring $R$, it contributes $|R|$ to $|f|$. For a set of rings $R$, let us define $\ell(R) = \sum_{R \subseteq R} |R|$.

Let $G$ be a graph with rings $R$. A precoloring of $R$ is a proper 3-coloring of the graph $H = \bigcup R$. Note that $H$ is a (not necessarily induced) subgraph of $G$. A precoloring $\phi$ of $R$ extends to a 3-coloring of $G$ if there exists a proper 3-coloring $\psi$ of $G$ such that $\phi(v) \neq \psi(v)$ for every weak vertex ring $v$ and $\phi(v) = \psi(v)$ for every other vertex $v$ incident with one of the rings. The graph $G$ is $R$-critical if $G \neq H$ and for every proper subgraph $G'$ of $G$ that contains $R$, there exists a precoloring of $R$ that extends to a 3-coloring of $G'$, but not to a 3-coloring of $G$. If $R$ consists of a single ring $R$, then we abbreviate $\{R\}$-critical to $R$-critical. For a precoloring $\phi$ of the rings, the graph $G$ is $\phi$-critical if $G \neq H$ and $\phi$ does not extend to a 3-coloring of $G$, but it extends to a 3-coloring of every proper subgraph of $G$ that contains $R$.

Let us remark that if $G$ is $\phi$-critical for some $\phi$, then it is $R$-critical, but the converse is not true (for example, consider a graph consisting of a single facial ring with two chords). On the other hand, if $\phi$ is a precoloring of the rings of $G$ that does not extend to a 3-coloring of $G$, then $G$ contains a (not necessarily unique) $\phi$-critical subgraph.

Let $G$ be a graph embedded in a disk with one ring $R$ of length $l \geq 5$. We say that $G$ is exceptional if it satisfies one of the conditions below (see Figure 1):

- (E0) $G = R$,
- (E1) $l \geq 8$ and $E(G) - E(R) = 1$,
- (E2) $l \geq 9$, $V(G) - V(R)$ has exactly one vertex of degree three, and the internal faces of $G$ have lengths $5, 5, l - 4$,
- (E3) $l \geq 11$, $V(G) - V(R)$ has exactly one vertex of degree three, and the internal faces of $G$ have lengths $5, 6, l - 5$,
- (E4) $l \geq 10$, $V(G) - V(R)$ consists of two adjacent degree three vertices, and the internal faces of $G$ have lengths $5, 5, 5, l - 5$,
- (E5) $l \geq 10$, $V(G) - V(R)$ consists of five degree three vertices forming a facial cycle of length five, and the internal faces of $G$ have lengths $5, 5, 5, 5, l - 5$.

We say that $G$ is very exceptional if it satisfies (E1), (E2) or (E3).
Figure 1: Exceptional graphs.
3 Definitions and results from [7]

Let $G$ be a graph in a surface $\Sigma$ with rings $\mathcal{R}$. A face is open 2-cell if it is homeomorphic to an open disk. A face is closed 2-cell if it is open 2-cell and bounded by a cycle. A face $f$ is omnipresent if it is not open 2-cell and each of its boundary walks is either a vertex ring or a cycle bounding a closed disk $\Delta \subseteq \hat{\Sigma} \setminus f$ containing exactly one ring. We say that $G$ has an internal 2-cut if there exist sets $A, B \subseteq V(G)$ such that $A \cup B = V(G)$, $|A \cap B| = 2$, $A - B \neq \emptyset \neq B - A$, $A$ includes all vertices of $\mathcal{R}$, and no edge of $G$ has one end in $A - B$ and the other in $B - A$.

We wish to consider the following conditions that the triple $G, \Sigma, \mathcal{R}$ may or may not satisfy:

(I0) every internal vertex of $G$ has degree at least three,

(I1) $G$ has no even cycle consisting of internal vertices of degree three,

(I2) $G$ has no cycle $C$ consisting of internal vertices of degree three, together with two distinct adjacent vertices $u, v \in V(G) - V(C)$ such that both $u$ and $v$ have a neighbor in $C$,

(I3) every internal face of $G$ is closed 2-cell and has length at least 5,

(I4) if a path of length at most two has both ends in $\mathcal{R}$, then it is a subgraph of $\mathcal{R}$,

(I5) no two vertices of degree two in $G$ are adjacent.

(I6) if $\Sigma$ is the sphere and $|\mathcal{R}| = 1$, or if $G$ has an omnipresent face, then $G$ does not contain an internal 2-cut,

(I7) the distance between every two distinct members of $\mathcal{R}$ is at least four,

(I8) every cycle in $G$ that does not separate the surface has length at least seven,

(I9) if a cycle $C$ of length at most 9 in $G$ bounds an open disk $\Delta$ in $\hat{\Sigma}$ such that $\Delta$ is disjoint from all rings, then $\Delta$ is a face, a union of a 5-face and a $(|C| - 5)$-face, or $C$ is a 9-cycle and $\Delta$ consists of three 5-faces intersecting in a vertex of degree three.

Some of these properties are automatically satisfied by critical graphs; see [7] for the proofs of the following observations.

Lemma 3.1. Let $G$ be a graph in a surface $\Sigma$ with rings $\mathcal{R}$. If $G$ is $\mathcal{R}$-critical, then it satisfies (I0), (I1) and (I2).

Lemma 3.2. Let $G$ be a graph in a surface $\Sigma$ with rings $\mathcal{R}$. Suppose that each component of $G$ is a planar graph containing exactly one of the rings. If $G$ is $\mathcal{R}$-critical and contains no non-ring triangle, then each component of $G$ is 2-connected and $G$ satisfies (I6).

Let $G$ be a graph in a surface $\Sigma$ with rings $\mathcal{R}$, and let $P$ be a path of length at least one and at most four with ends $u, v \in V(\mathcal{R})$ and otherwise disjoint from $\mathcal{R}$. We say that $P$ is allowable if

- $u, v$ belong to the same ring of $\mathcal{R}$, say $R$,
- $P$ has length at least three,
- there exists a subpath $Q$ of $R$ with ends $u, v$ such that $P \cup Q$ is a cycle of length at most eight that bounds an open disk $\Delta \subset \Sigma$,
- if $P$ has length three, then $P \cup Q$ has length five and $\Delta$ is a face of $G$, and
• if \( P \) has length four, then \( \Delta \) includes at most one edge of \( G \), and if it includes one, then that edge joins the middle vertex of \( P \) to the middle vertex of \( Q \) (which also has length four).

We say that \( G \) is well-behaved if every path \( P \) of length at least one and at most four with ends \( u, v \in V(\mathcal{R}) \) and otherwise disjoint from \( \mathcal{R} \) is allowable.

Let \( M \) be a subgraph of \( G \). A subgraph \( M \subseteq G \) captures \((\leq 4)-cycles\) if \( M \) contains all cycles of \( G \) of length at most 4 and furthermore, \( M \) is either null or has minimum degree at least two.

Throughout the rest of the paper, let \( \epsilon = 2/4113 \) and let \( s : \{5, 6, \ldots\} \rightarrow \mathbb{R} \) be the function defined by \( s(5) = 4/4113, s(6) = 72/4113, s(7) = 540/4113, s(8) = 2184/4113 \) and \( s(l) = l - 8 \) for \( l \geq 9 \). Based on this function, we assign weights to the faces. Let \( G \) be a graph embedded in \( \Sigma \) with rings \( \mathcal{R} \). For an internal face \( f \) of \( G \), we define \( w(f) = s(|f|) \) if \( f \) is open 2-cell and \( |f| \geq 5 \), and \( w(f) = |f| \) otherwise. We define \( w(G, \mathcal{R}) \) as the sum of \( w(f) \) over all internal faces \( f \) of \( G \).

Let \( \Pi \) be a surface with boundary and \( c \) a simple curve intersecting the boundary of \( \Pi \) exactly in its ends. The topological space obtained from \( \Pi \) by cutting along \( c \) (i.e., duplicating every point of \( c \) and turning both copies into boundary points) is a union of at most two surfaces. If \( \Pi_1, \ldots, \Pi_k \) are obtained from \( \Pi \) by repeating this construction, we say that they are fragments of \( \Pi \).

Consider a graph \( H \) embedded in a surface \( \Pi \) with rings \( \mathcal{Q} \), and let \( f \) be an internal face of \( H \). For each facial walk \( t \) of \( f \), we perform the following: if \( t \) consists only of a vertex ring incident with the cuff \( C \), then we remove \( C \) from \( f \). Otherwise, we add a simple closed curve tracing \( t \) (if an edge appears twice in \( t \), then it will correspond to two disjoint parts of the curve). We define \( \Pi_f \) to be the resulting surface. Note that the cuffs of \( \Pi_f \) correspond to the facial walks of \( f \).

Let \( G \) be a graph embedded in \( \Sigma \) with rings \( \mathcal{R} \). Let \( J \) be a subgraph of \( G \) and let \( S \) be a subset of faces of \( J \) such that

\[
J \text{ is equal to the union of the boundaries of the faces in } S, \text{ each isolated vertex of } J \text{ is a vertex ring and whenever } C \text{ is a cuff intersecting a face } f \in S, \text{ then } C \text{ is incident with a vertex ring belonging to } J.
\]

We define \( G[S] \) to be the subgraph of \( G \) consisting of \( J \) and all the vertices and edges drawn inside the faces of \( S \). Let \( C_1, C_2, \ldots, C_k \) be the boundary walks of the faces in \( S \) (in case that a vertex ring \( R \in \mathcal{R} \) forms a component of a boundary of a face in \( S \), we consider \( R \) itself to be such a walk). We would like to view \( G[S] \) as a graph with rings \( C_1, \ldots, C_k \). However, the \( C_i \)'s do not necessarily have to be disjoint, and they do not have to be cycles or isolated vertices. To overcome this difficulty, we proceed as follows: Let \( Z \) be the set of cuffs incident with the vertex rings that form a component of \( J \) by themselves, and let \( \bar{Z} = \bigcup_{R \in Z} \bar{R} \). Suppose that \( S = \{f_1, \ldots, f_m\} \). For \( 1 \leq i \leq m \), let \( \Sigma_i' \) be a surface with boundary \( B_i \) such that \( \Sigma_i' \setminus B_i \) is homeomorphic to \( f_i \). Let \( \theta_i : \Sigma_i' \setminus B_i \rightarrow f_i \) be a homeomorphism that extends to a continuous mapping \( \theta_i : \Sigma_i' \rightarrow \overline{f_i} \), where \( \overline{f_i} \) denotes the closure of \( f_i \). Let \( \Sigma_i = \Sigma_i' \setminus \theta_i^{-1}(\bar{Z} \cap f_i) \), and let \( G_i \) be the inverse image of \( G \cap \overline{f_i} \) under \( \theta_i \). Then \( G_i \) is a graph normally embedded in \( \Sigma_i \). We say that the set of embedded graphs \( \{G_i : 1 \leq i \leq m\} \) is a \( G \)-expansion of \( S \). Note that there is a one-to-one correspondence between the boundary walks of the faces of \( S \) and the rings of the graphs in the \( G \)-expansion.
of $S$; however, each vertex of $J$ may be split to several copies. For $1 \leq i \leq m$, we let $R_i$ be the set of rings of $G_i$ consisting of the facial rings formed by the cycles contained in the boundary of $\Sigma_i'$ and of the vertex rings formed by the vertices contained in $\theta_i(Z \cap f_i)$, where each vertex ring is weak if and only if the corresponding vertex ring of $R$ is weak. We say that the rings in $R_i$ are the natural rings of $G_i$.

Let now $G'$ be another $\mathcal{R}$-critical graph embedded in $\Sigma$ with rings $\mathcal{R}$. Suppose that there exists a collection $\{(J_f, S_f) : f \in F(G')\}$ of subgraphs $J_f$ of $G$ and sets $S_f$ of faces of $J_f$ satisfying (1) and a set $X \subset F(G)$ such that

- for every $f \in F(G')$, the boundary of $S_f$ is not equal to the union of $\mathcal{R}$,
- for every $f \in F(G')$, the surfaces embedding the components of the $G$-expansion of $S_f$ are fragments of $\Sigma_f$,
- for every face $h \in F(G) \setminus X$, there exists unique $f \in F(G')$ such that $h$ is a subset of a member of $S_f$, and
- if $X \neq \emptyset$, then $X$ consists of a single closed 2-cell face of length 6.

We say that $X$ together with this collection forms a cover of $G$ by faces of $G'$. We define the elasticity $e(f)$ of a face $f \in F(G')$ to be $\left(\sum_{h \in S_f} |h|\right) - |f|$.

We now want to bound the weight of $G$ by the weight $G'$. To this end, we define a contribution $c(f')$ of a face $f'$ of $G'$ that bounds the difference between the weight of $f'$ and the weight of the corresponding subgraph of $G$. We only define the contribution in the case that every face of $G'$ is either closed 2-cell of length at least 5 or omnipresent. The contribution $c(f')$ of an omnipresent face $f'$ of $G'$ is defined as follows. Let $G'_1, G'_2, \ldots, G'_k$ be the components of $G'$ such that $G'_i$ contains the ring $R_i \in \mathcal{R}$. If there exist distinct indices $i$ and $j$ such that $G'_i \neq R_i$ and $G'_j \neq R_j$, then $c(f') = 1$. Otherwise, suppose that $G'_i = R_i$ for $i \geq 2$. If $G'_i$ satisfies (E0), (E1), (E2) or (E3), then $c(f') = -\infty$. If $G'_i$ satisfies (E4) or (E5), then $c(f') = 5 - e_l(f') - 5s(5)$, otherwise $c(f') = 5 - e_l(f') + 5s(5)$.

For a closed 2-cell face, the definition of the contribution can be found in [7]; here, we only use its properties given by the following theorem, which was proved at the end of [7].

**Theorem 3.3.** Let $G$ be a well-behaved graph embedded in a surface $\Sigma$ with rings $\mathcal{R}$ satisfying (10)–(19) and let $M$ be a subgraph of $G$ that captures $(\leq 4)$-cycles. Assume that either $g > 0$ or $|\mathcal{R}| > 1$ and that $w(G, \mathcal{R}) > 8g + 8|\mathcal{R}| + (2/3 + 26e)((\mathcal{R}) + 20E(M))/3 - 16$. If $G$ is $\mathcal{R}$-critical, then there exists an $\mathcal{R}$-critical graph $G'$ embedded in $\Sigma$ with rings $\mathcal{R}$ such that $|E(G')| < |E(G)|$ and the following conditions hold.

(a) If $G$ has girth at least five, then there exists a set $Y \subseteq V(G')$ of size at most two such that $G' - Y$ has girth at least five.

(b) If $C'$ is a $(\leq 4)$-cycle in $G'$, then $C'$ is non-contractible and $G$ contains a non-contractible cycle $C$ of length at most $|C'| + 3$ such that $C \not\subset M$. Furthermore, all ring vertices of $C'$ belong to $C$, and if $C'$ is a triangle disjoint from the rings and its vertices have distinct pairwise non-adjacent neighbors in a ring $R$ of length 6, then $G$ contains edges $e$ and $e'$ with $e, e' \in V(C) \setminus V(R)$ and $r, r' \in V(R) \setminus V(C)$ such that $r$ and $r'$ are non-adjacent.
Theorem 3.4. Let $G'$ be a graph of girth at least 5 embedded in the disk with one ring $R_1$ of length $l \geq 5$. If $G$ is $R$-critical, then

(c) $G'$ has an internal face that either is not closed 2-cell or has length at least 6.

(d) There exists $X \subset F(G)$ and a collection $\{J_f, S_f : f \in F(G')\}$ forming a cover of $G$ by faces of $G'$, such that $\sum_{f \in F(G')} e(f) \leq 10$, and if $f$ is an omnipresent face, then $\sum_{f \in F(G')} e(f) \leq 5$. Furthermore, if every internal face of $G'$ is closed 2-cell or omnipresent and $G'$ satisfies (16), then $\sum_{f \in F(G')} c(f) \geq |X|s(6)$.

(c) If $f \in F(G')$ is closed 2-cell and $G_1, \ldots, G_k$ are the components of the $G$-expansion of $S_f$, where for $1 \leq i \leq k$, $G_i$ is embedded in the disk with one ring $R_i$, then $\sum_{i=1}^k w(G_i, \{R_i\}) \leq s(|f|) - c(f)$.

The main result of [7] bounds the weight of graphs embedded in the disk with one ring.

**Theorem 3.4.** Let $G$ be a graph of girth at least 5 embedded in the disk with one ring $R$ of length $l \geq 5$. If $G$ is $R$-critical, then

- $w(G, \{R\}) \leq s(l - 3) + s(5)$, and furthermore,
- if $R$ does not satisfy (E1), then $w(G, \{R\}) \leq s(l - 4) + 2s(5)$,
- if $(G, R)$ is not very exceptional, then $w(G, \{R\}) \leq s(l - 5) + 5s(5)$, and
- if $(G, R)$ is not exceptional, then $w(G, \{R\}) \leq s(l - 5) - 5s(5)$.

We will also need the following property of critical graphs.

**Lemma 3.5.** Let $G$ be a graph in a surface $\Sigma$ with rings $\mathcal{R}$, and assume that $G$ is $\mathcal{R}$-critical. Let $C$ be a non-facial cycle in $G$ bounding an open disk $\Delta \subseteq \Sigma$ disjoint from the rings, and let $G'$ be the graph consisting of the vertices and edges of $G$ drawn in the closure of $\Delta$. Then $G'$ may be regarded as graph embedded in the disk with one ring $C$, and as such it is $C$-critical.

This together with Theorem 3.4 implies that property (19) holds for all embedded critical graphs without contractible ($\leq 4$)-cycles. Lemma 3.5 is a special case of the following result.

**Lemma 3.6.** Let $G$ be a graph in a surface $\Sigma$ with rings $\mathcal{R}$, and assume that $G$ is $\mathcal{R}$-critical. Let $J$ be a subgraph of $G$ and $S$ be a subset of faces of $J$ satisfying (1). Let $G'$ be an element of the $G$-expansion of $S$ and $\mathcal{R}'$ its natural rings. If $G'$ is not equal to the union of the rings in $\mathcal{R}'$, then $G'$ is $\mathcal{R}'$-critical.

**Proof.** Consider any edge $e' \in E(G')$ that does not belong to any of the rings in $\mathcal{R}'$. By the definition of $G$-expansion, there is a unique edge $e \in E(G)$ corresponding to $e'$. Since $G$ is $\mathcal{R}$-critical, there exists a precoloring $\psi$ of $\mathcal{R}$ that does not extend to a 3-coloring of $G$, but extends to a 3-coloring $\phi$ of $G - e$. We define a precoloring $\psi'$ of $\mathcal{R}'$. Each ring vertex $v' \in V(G')$ corresponds to a unique vertex $v \in V(G)$. If $v'$ is a weak vertex ring, then $v$ is a weak vertex ring and we set $\psi'(v') = \psi(v)$. Otherwise, we set $\psi'(v') = \phi(v)$. Observe that $\phi$ corresponds to a 3-coloring of $G' - e'$ that extends $\psi'$. If $\psi$ extends to a 3-coloring $\phi'$ of $G'$, then define $\phi_1$ in the following way. If $v \in V(G)$ corresponds to no vertex of $G'$, then $\phi_1(v) = \phi(v)$. If $v \in V(G)$ corresponds to at least
Let $\Delta$ be the boundary of an open disk or a disjoint union of two open disks disjoint from $\Sigma$ of $X$ embedded in $\Sigma$. Let us split the vertices of $G$ of vertices. Let $\Sigma$ be a graph in a surface $\Sigma$ with rings $R$, and assume that $G$ is $\mathcal{R}$-critical. Let $c$ be a simple closed curve in $\Sigma$ intersecting $G$ in a set $X$ of vertices. Let $\Sigma_0$ be one of the surfaces obtained from $\Sigma$ by cutting along $c$. Let us split the vertices of $G$ along $c$, let $G'$ be the part of the resulting graph embedded in $\Sigma_0$, let $X'$ be the set of vertices of $G'$ corresponding to the vertices of $X$ and let $R' \subseteq R$ be the rings of $G$ that are contained in $\Sigma_0$. Let $\Delta$ be an open disk or a disjoint union of two open disks disjoint from $\Sigma_0$ such that the boundary of $\Delta$ is equal to the cuff(s) of $\Sigma_0$ corresponding to $c$. Let $\Sigma' = \Sigma_0 \cup \Delta$. Let $Y$ consist of all vertices of $X'$ that are not incident with a cuff in $\Sigma'$. For each $y \in Y$, choose an open disk $\Delta_y \subset \Delta$ such that the closures of the disks are pairwise disjoint and the boundary of $\Delta_y$ intersects $G'$ exactly in $y$. Let $\Sigma'' = \Sigma' \setminus \bigcup_{y \in Y} \Delta_y$ and $\mathcal{R}' = \mathcal{R}' \cup Y$, where the elements of $Y$ are considered to be non-weak vertex rings of the embedding of $G'$ in $\Sigma''$. If $G'$ is not equal to the union of the rings in $\mathcal{R}'$, then $G'$ is $\mathcal{R}'$-critical.

In particular, if $G'$ is a component of a $\mathcal{R}$-critical graph, $\mathcal{R}'$ are the rings contained in $G'$ and $G'$ is not equal to the union of $\mathcal{R}'$, then $G'$ is $\mathcal{R}'$-critical.

4 \((\leq 4)\)-cycles on a cylinder

The most technically difficult part of the proof of Theorem 1.3 is dealing with long cylindrical subgraphs of the considered graph. We work out the details of this situation in the following two sections. We start with the case of a graph embedded in the cylinder with rings of length at most four. We will need the following result on graphs embedded in the disk with a ring of length at most twelve, which follows from the results of Thomassen [15].

**Theorem 4.1.** Let $G$ be a graph of girth 5 embedded in the disk with a ring $R$ such that $|R| \leq 12$. If $G$ is $R$-critical and $R$ is an induced cycle, then

- (a) $|R| \geq 9$ and $G - V(R)$ is a tree with at most $|R| - 8$ vertices, or
- (b) $|R| \geq 10$ and $G - V(R)$ is a connected graph with at most $|R| - 5$ vertices containing exactly one cycle, and the length of this cycle is 5, or
- (c) $|R| = 12$ and every second vertex of $R$ has degree two and is contained in a facial 5-cycle.

A graph $H$ embedded in the cylinder with (vertex-disjoint) facial rings $C_1$ and $C_2$ of length at most 4 is basic if every contractible cycle in $H$ has length at least five, $H$ is $\{C_1, C_2\}$-critical, and one of the following holds:

10
Consider a basic 2-connected triangle-free graph. We can cut the graph along a shortest path between $C_1$ and $C_2$, resulting in a graph embedded in a disk bounded by a cycle $C$ of length 10. Note that the resulting graph is $C$-critical by Lemma 3.6. A straightforward case analysis using Theorem 4.1 shows that every 2-connected triangle-free basic graph is a subgraph of one of the graphs drawn in Figure 2.

Observe furthermore that these graphs have the following properties.

Let $C_1$ and $C_2$ be the rings of a triangle-free 2-connected basic graph $H$. There exists a 3-coloring $\psi$ of $C_1$, vertices $v_1, v_2 \in V(C_2)$ and colors $c_1 \neq c_2$ such that if $\phi$ is a 3-coloring of $C_1 \cup C_2$ matching $\psi$ on $C_1$ and satisfying $\phi(v_i) \neq c_i$ for $i \in \{1, 2\}$, then $\phi$ extends to a 3-coloring of $H$.

(2)

The colorings for (2) are indicated in Figure 3.

\[\text{Figure 2: Maximal basic graphs.}\]
and let \( v \) be the vertex indicated in the figure. If \( H = B_4 \), then let \( c \) be the unique color distinct from \( c_1 \) and \( c_2 \) and let \( v \) be the vertex indicated in the figure.

Consider any 3-coloring \( \psi \) of \( C_2 \). For each vertex \( w \in V(C_1) \), let \( L'_\psi(w) \subseteq \{1, 2, 3\} \) be the list consisting of all colors not used by \( \psi \) on the neighbors of \( w \). Let \( L_\psi(w) = L'_\psi(w) \) if \( w \neq v_i \) for \( i \in \{1, 2\} \), and \( L_\psi(w) = L'_\psi(w) \setminus \{v_i\} \) if \( w = v_i \).

Suppose first that \( H \) is \( B_1 \), \( B_2 \) or \( B_3 \). Note that the sum of the sizes of the lists \( L_\psi \) is at least 8 and each of the lists has size at least one. Therefore, \( C_1 \) can be colored from the lists given by \( L_\psi \), unless \( C_1 \) contains two adjacent vertices with the same list of size one. That is only possible if \( H \) is \( B_1 \) or \( B_3 \) and \( v_1 \) and \( v_2 \) are as depicted in Figure 2. However, then we can choose the vertex \( v \) as indicated in the figure and set \( c = c_2 \). This ensures that \( c_2 \) belongs to \( L_\psi(v_1) \setminus L_\psi(v_2) \), and thus \( L_\psi(v_1) \neq L_\psi(v_2) \). Therefore, \( \psi \) extends to a 3-coloring \( \phi \) of \( H \) satisfying \( \phi(v_1) \neq c_1 \) and \( \phi(v_2) \neq c_2 \).

Suppose now that \( H \) is \( B_4 \) or \( B_5 \). Let us first consider the case that \( \psi(x_1) \neq \psi(x_2) \). If \( H = B_4 \), then we can by symmetry assume that \( \{v_1, v_2\} \neq \{w_1, y_1\} \), as otherwise we can swap the labels \( x_1 \) with \( x_2 \) and \( y_1 \) with \( y_2 \). Let \( L(w) = L_\psi(v) \) for \( w \in V(C_1) \setminus \{y_1\} \) and \( L(y_1) = L_\psi(y_1) \setminus \{\psi(x_2)\} \) and observe that any coloring of \( C_1 \) from lists given by \( L \) extends to a 3-coloring \( \phi \) of \( H \) matching \( \psi \) on \( C_2 \) and satisfying \( \phi(v_1) \neq c_1 \) and \( \phi(v_2) \neq c_2 \). Again, the sum of the sizes of the lists \( L \) is at least 8 and each of the lists has size at least one, thus such a coloring exists unless \( C_1 \) contains two adjacent vertices with the same list of size one. This is not possible, since if \( H = B_4 \), then \( \{v_1, v_2\} \neq \{w_1, y_1\} \).

Finally, let us consider the case that \( \psi(x_1) = \psi(x_2) \). If there exists a coloring \( \psi' \) of \( C_1 \) from lists \( L_\psi \) such that \( \psi'(y_1) \neq \psi'(y_2) \), then the union of \( \psi \) and \( \psi' \)
extends to a 3-coloring $\phi$ of $H$, which clearly satisfies $\phi(v_1) \neq c_1$ and $\phi(v_2) \neq c_2$. Let us find such a coloring $\psi'$. If $C_1$ contains a vertex $w \notin \{y_1, y_2\}$ such that $|L_\psi(w)| = 3$, then it suffices to color the vertices of $Y = V(C_1) \setminus \{w\}$ by pairwise distinct colors from their lists and then color $w$ differently from its neighbors. Such a coloring of $Y$ always exists, since $\sum_{y \in Y} |L_\psi(y)| \geq 6$, all the lists have size at least one and at most three and if all of them have size two, then $v_1$ and $v_2$ belong to $Y$ and $L_\psi(v_1) = \{1, 2, 3\} \setminus \{c_1\}$ is different from $L_\psi(v_2) = \{1, 2, 3\} \setminus \{c_2\}$. Therefore, we can assume that all vertices in $V(C_1) \setminus \{y_1, y_2\}$ have lists of size at most two.

Suppose that either $H = B_5$, or $H = B_4$ and $|L_\psi(w_1)| = 2$. In this case $|L_\psi(w_1)| = |L_\psi(w_2)| = 2$ and by symmetry, we can assume that $|L_\psi(y_1)| = 3$ and $|L_\psi(y_2)| \geq 2$. Let us choose a color $\psi'(w_1) = \psi'(w_2) \in L_\psi(w_1) \cap L_\psi(w_2)$ and then color $y_1$ and $y_2$ by distinct colors from $L(y_1) \setminus \{\psi'(w_1)\}$ and $L(y_2) \setminus \{\psi'(w_1)\}$, respectively.

Therefore, we can assume that $H = B_4$ and $|L_\psi(w_1)| = 1$. Note that $|L_\psi(w_2)| = 2$ and $|L_\psi(y_1)| = |L_\psi(y_2)| = 3$. By symmetry between $v_1$ and $v_2$, we can assume that $v_1 = w_1$ and $v_2 = w_2$. The coloring $\psi'$ exists unless $L_\psi(w_2) = \{1, 2, 3\} \setminus L_\psi(w_1)$. However, this is prevented by the choice of $c$.

For a 4-cycle $C = x_1x_2x_3x_4$, the type of its 3-coloring $\lambda$ is the set of the vertices $x_i$ of $C$ such that $\lambda(x_i) \neq \lambda(x_{i+2})$ (where $x_5 = x_1$ and $x_6 = x_2$). Note that the type of $\lambda$ is $\emptyset$, $\{x_1, x_3\}$ or $\{x_2, x_4\}$. In (2), any coloring of the same type as $\psi$ has the same property, possibly with different colors $c_1$ and $c_2$.

Let $G$ and $H$ be graphs with common rings $\{C_1, C_2\}$. We say that $H$ subsumes $G$ if every precoloring of $C_1 \cup C_2$ that extends to a 3-coloring of $H$ also extends to a 3-coloring of $G$.

**Lemma 4.2.** Let $G$ be a graph embedded in the cylinder with facial rings $\{R_1, R_2\}$ of length at most 4. If every cycle of length at most 4 in $G$ is non-contractible, then there exists a basic graph $H$ with rings $\{R_1, R_2\}$ that subsumes $G$. Furthermore, either $H = G$ or $|V(H)| + |E(H)| < |V(G)| + |E(G)|$.

**Proof.** Suppose for a contradiction that $G$ is a counterexample such that $|V(G)| + |E(G)|$ is minimal. It follows that $G$ is $\{R_1, R_2\}$-critical, 2-connected and triangle-free, and in particular $|R_1| = |R_2| = 4$. Let $R_1 = a_1a_2a_3a_4$ and $R_2 = b_1b_2b_3b_4$, where the labels are assigned in the clockwise order. Since $G$ is triangle-free and all 4-cycles are non-contractible, it follows that every internal vertex has at most one neighbor in each of the rings.

Suppose that $G$ contains a 5-face $C = v_1v_2v_3v_4v_5$ such that all its vertices are internal and have degree three. For $1 \leq i \leq 5$, let $x_i$ be the neighbor of $v_i$ different from $v_{i-1}$ and $v_{i+1}$ (where $v_0 = v_5$ and $v_6 = v_1$). Observe that if $x_1 = x_3$, then $x_2 \neq x_4$, thus by symmetry assume that $x_1 \neq x_3$. Let $G' = (G - V(C)) + x_1x_3$. Suppose that $K'$ is a cycle of length at most 4 in $G'$ that contains the edge $x_1x_3$. Then $G$ contains a cycle $K$ of length at most 7 obtained from $K'$ by replacing $x_1x_3$ by $x_1v_1v_2v_3x_3$. Since $v_1$ and $v_2$ have neighbors on the opposite sides of this path, $K$ does not bound a face. By Theorem 4.1, we conclude that $K$ and $K'$ are non-contractible. Therefore, all (≤4)-cycles in $G'$ are non-contractible. Furthermore, every precoloring of $R_1$ and $R_2$ that extends to a 3-coloring of $G'$ also extends to a 3-coloring of $G$ (the 3-coloring of $G'$ assigns different colors to $x_1$ and $x_3$, thus it can be extended
to $C$). Thus, $G'$ subsumes $G$, and consequently it contradicts the minimality of $G$. We conclude that

**every 5-face in $G$ is incident with a ring vertex or a vertex of degree at least 4.**

(4)

It follows that the distance between $R_1$ and $R_2$ is at least two: otherwise, if say $a_4$ is adjacent to $b_1$, then apply Theorem 4.1 to the graph obtained from $G$ by cutting open along the walk $a_1a_2\ldots a_1b_2\ldots b_1$. Outcome (b) is excluded by (4), thus $G - V(R_1 \cup R_2)$ would have at most two vertices and $G$ would be basic.

Suppose that $G$ contains a face $C = v_1v_2\ldots v_k$ of length $k \geq 7$. We may assume that $v_1$ is an internal vertex. Let $G'$ be the graph obtained from $G$ by identifying $v_1$ with $v_3$ to a vertex $v$. Consider a cycle $K' \subseteq G'$ of length at most 4 that does not appear in $G$. Such a cycle corresponds to a cycle $K$ in $G$ of length at most 6, obtained by replacing $v$ by $v_1v_2v_3$. Since $v_1$ is an internal vertex, $v_2$ cannot be a ring vertex of degree two. It follows that $K$ does not bound a face and it is non-contractible by Theorem 4.1. Therefore, all ($\leq 4$)-cycles in $G'$ are non-contractible. Furthermore, every 3-coloring of $G'$ extends to a 3-coloring of $G$, and we obtain a contradiction with the minimality of $G$. Therefore, each face of $G$ has length at most 6.

Suppose that $G$ contains a face $C = v_1v_2\ldots v_6$ of length 6. We can assume that $v_1$ is an internal vertex. If $v_3$ or $v_5$ is an internal vertex, then let $G'$ be the graph obtained from $G$ by identifying $v_1$, $v_3$ and $v_5$ to a single vertex. As in the previous paragraph, we obtain a contradiction. It follows that $v_3$ and $v_5$ are ring vertices, and by a symmetrical argument, two of $v_2$, $v_4$ and $v_6$ are ring vertices. If $v_2$ is internal, then since the distance between $R_1$ and $R_2$ is at least two, we can assume that $V(R_1) = \{v_2, v_4, v_5, v_6\}$, and thus $v_2$ and $v_6$ are adjacent. In this situation, we consider the graph obtained from $G$ by identifying $v_2$ with $v_6$ and $v_2$ with $v_4$ (which is isomorphic to $G - \{v_3, v_5\}$, and thus contains no contractible ($\leq 4$)-cycles), and again obtain a contradiction with the minimality of $G$. Thus $v_2$ is not internal, and by symmetry, $v_6$ is not internal either. Therefore, $v_4$ is internal and $v_2$ and $v_6$ are ring vertices. Since the distance between $R_1$ and $R_2$ is at least two, we may assume that $v_2 = a_2$, $v_3 = a_3$, $v_5 = b_4$ and $v_6 = b_1$. We apply Theorem 4.1 to the 10-cycle $B = a_1a_2v_1b_1b_2b_3b_4v_3a_3a_4$. The case (b) is excluded by (4), thus either $B$ is not induced or (a) holds. If $B$ is not induced, then its chord joins $v_1$ with $v_4$. However, then the precolorings $\psi$ of the rings that do not extend to 3-colorings of $G$ satisfy $\psi(a_2) = \psi(b_4)$, and we can set $H$ to be the graph consisting of $R_1$, $R_2$ and the edge between $a_2$ and $b_4$. Therefore, $B$ is an induced cycle and $G - V(B)$ is a tree $F$ with at most two vertices. If $F$ has only one vertex $w$, then $w$ cannot be adjacent to both $v_1$ and $v_4$, hence one of these vertices has degree two, which is a contradiction. If $V(F) = \{x,y\}$, then since $v_1$ and $v_4$ have degree at least three, we can assume that $x$ is adjacent to $v_1$ and $a_4$ and $y$ is adjacent to $b_2$ and $v_4$. However, by identifying $v_1$ with $b_4$ and $v_4$ with $a_2$, we obtain a graph isomorphic to the graph $B_5$ of Figure 2, which subsumes $G$.

Therefore, **all internal faces of $G$ have length 5.**

(5)

Together with Lemma 3.5 and Theorem 4.1, this implies that
Suppose that $G$ contains a 4-cycle $C = v_1v_2v_3v_4$ different from $R_1$ and $R_2$. By the assumptions, $C$ is non-contractible; for $i \in \{1, 2\}$, let $G_i$ be the subgraph of $G$ drawn between $R_i$ and $C$ and let $d_i$ be the distance between $R_i$ and $C$.

Let us first consider the case that $d_i \geq 1$ and $G_i$ is not basic for some $i \in \{1, 2\}$. By the minimality of $G$, there exists a basic graph $G_i'$ that subsumes $G_i$ (considered to be embedded in a cylinder with rings $R_i$ and $C$) such that $|V(G_i')| + |E(G_i')| < |V(G_i)| + |E(G_i)|$. Let $G' = G_i' \cup G_{3-i}$ and observe that $G'$ subsumes $G$ and $|V(G')| + |E(G')| < |V(G) + |E(G)|$. Note that every contractible cycle in $G'$ has length at least five, since neither $G_i'$ nor $G_{3-i}$ contains a contractible cycle ($\leq 4$)-cycle and $G_{3-i}$ is triangle-free. Therefore, by the minimality of $G$, there exists a basic graph $H$ which subsumes $G'$ and $|V(H)| + |E(H)| \leq |V(G')| + |E(G')|$. However, then $H$ also subsumes $G$, which is a contradiction.

We conclude that if $d_i \geq 1$, then $G_i$ is a basic graph, and since it is 2-connected and triangle-free, it follows that $d_i = 1$. Let us choose the labels of $R_1$ and $R_2$ and the cycle $C$ so that $d_1$ is as small as possible. In particular, $d_1 \leq d_2$. Let us discuss the possible cases:

- $d_1 = d_2 = 0$: Since the distance between $R_1$ and $R_2$ is at least two, we conclude that $|V(R_1) \cap V(C)| = |V(R_2) \cap V(C)| = 1$. We can assume that $v_1 = a_1$ and $v_3 = b_3$. By Theorem 4.1, the open disks bounded by the closed walks $a_1v_2v_3v_4a_4a_3a_2$ and $b_3b_4b_1b_2b_3v_4a_1v_2$ contain no vertices, and since $v_2$ and $v_4$ have degree at least three, we may assume that $v_2$ is adjacent to $a_4$ and $v_4$ to $b_2$. However, then $G$ contains a triangle $a_1v_2a_4$, which is a contradiction.

- $d_1 = 0$, $d_2 = 1$: We may assume that $a_3 = v_1$. Since $G$ is triangle-free, (6) implies that $|V(C) \cap V(R_1)| = 1$ and $a_3v_3 \in E(G)$. Since $d_2 = 1$, $G_2$ is a basic graph, and by (5), we conclude that $G_2$ is isomorphic to $B_4$ or $B_5$ from Figure 2. Let $w_1w_2 = G_2 = V(C \cup R_2)$. Up to symmetry, there are two cases to consider:

  - $b_3$ is adjacent to $v_3$. Since $v_2$ and $v_4$ have degree at least three, we can assume that $w_1$ is adjacent to $v_4$ and $b_2$ and $w_2$ is adjacent to $v_2$ and $b_4$. In this case, we let $H$ be the graph consisting of $R_1$, $R_2$ and a vertex $z$, with $a_1$ adjacent to $b_2$ and $z$ to $b_2$, $b_3$ and $a_3$.

  - $b_3$ is adjacent to $v_4$. Since $v_2$ has degree at least three, we can assume that $w_1$ is adjacent to $b_1$ and $v_4$ and $w_2$ is adjacent to $b_3$ and $v_2$. We let $H$ be the graph consisting of $R_1$, $R_2$, adjacent vertices $z_1$ and $z_2$, and edges $a_1b_3$, $a_1z_1$, $b_1z_1$, $b_3z_2$ and $a_3z_2$.

- $d_1 = 1$, $d_2 = 1$: By the choice of $C$, $G$ does not contain a 4-cycle distinct from $R_1$ and $R_2$ that intersects one of them. Additionally, all internal faces of $G$ have length 5 and $G_1$ and $G_2$ are basic graphs, hence we can assume that $a_1$ is adjacent to $v_1$ and $G_1 - (R_1 \cup C) = w_1w_2$ with $w_1$ adjacent to $a_4$ and $v_4$ and $w_2$ adjacent to $a_2$ and $v_2$, and $G_2$ is isomorphic

$G$ contains no contractible cycles of length 6 or 7, and the disk bounded by any contractible 8-cycle $K$ of $G$ consists of two 5-faces separated by a chord of $K$. (6)
to \( G \). Since \( v_3 \) has degree at least three, \( v_1 \) cannot have a neighbor in \( R_2 \), thus there are up to symmetry two possible cases:

- \( b_1 \) is adjacent to \( v_2 \), \( G_2 - V(R_2 \cup C) = w_3 w_4 \), and \( w_3 v_1, w_3 b_2, w_4 v_3, w_4 b_4 \in E(G) \). Then, let \( H \) be the graph consisting of \( R_1 \), \( R_2 \) and the edge \( a_1 b_2 \).

- \( b_1 \) is adjacent to \( v_3 \), \( G_2 - V(R_2 \cup C) = w_3 w_4 \), and \( w_3 v_2, w_3 b_2, w_4 v_4, w_4 b_4 \in E(G) \). But then every precoloring of \( R_1 \) and \( R_2 \) extends to a 3-coloring of \( G \), contrary to the assumption that \( G \) is \( \{ R_1, R_2 \} \)-critical.

Therefore,

\( R_1 \) and \( R_2 \) are the only 4-cycles in \( G \).

Suppose that \( G \) has a face \( C = v_1 v_2 v_3 v_4 v_5 \) such that \( v_2, \ldots, v_5 \) are internal vertices of degree three. For \( 2 \leq i \leq 5 \), let \( x_i \) be the neighbor of \( v_i \) that is not incident with \( C \). By (7), the vertices \( x_i \) are distinct. If at least one of \( x_3 \) and \( x_4 \) is internal, then let \( G' \) be the graph obtained from \( G - \{ v_2, \ldots, v_5 \} + x_2 x_5 \) by identifying \( x_3 \) with \( x_4 \) to a new vertex \( x \). Observe that every 3-coloring of \( G' \) extends to a 3-coloring of \( G \). Furthermore, suppose that \( K' \) is a cycle of length at most 4 in \( G' \) that does not appear in \( G \), and let \( K \) be the corresponding cycle in \( G \) obtained by replacing \( x_2 x_5 \) by \( x_2 v_2 v_3 x_5 \) or \( x_3 v_3 v_4 x_4 \) or both. If \( |K| \leq 7 \), then since \( K \) cannot bound a face, Theorem 4.1 implies that \( K \) and \( K' \) are non-contractible. If \( |K| \geq 8 \), then \( K \) contains both \( x_2 v_2 v_3 x_5 \) and \( x_3 v_3 v_4 x_4 \), and since \( |K'| \leq 4 \) and \( G \) is embedded in the cylinder, it follows that \( x_4 \) is adjacent to \( x_3 \) or \( x_3 \) is adjacent to \( x_2 \). This is excluded by (7).

Therefore, \( G' \) is a smaller counterexample than \( G \), which is a contradiction. Let us now consider the case that both \( x_3 \) and \( x_4 \) are ring vertices. Here, we exclude the possibility that \( x_3 \) and \( x_4 \) belong to different rings: If that were the case, then we can assume that \( x_3 = a_1 \) and \( x_4 = b_1 \). Since all internal faces of \( G \) have length 5, it follows that \( x_4 \) and \( x_5 \) have a common neighbor \( v \). We apply Theorem 4.1 to the disk bounded by the closed walk \( a_1 a_2 a_3 a_4 a_1 b_1 b_2 b_3 b_4 b_1 v \) of length 12. By (4), the case (b) is excluded. Since \( v \) has degree at least three, \( a_1 v b_1 \) cannot be incident with two 5-faces and the case (c) is excluded as well. Therefore, \( G - V(R_1 \cup R_2) - \{v\} \) is a tree with four vertices \( v_2, v_3, v_4 \) and \( v_5 \). By (7), \( v \) is not equal to \( x_2, x_3 \) or \( v_2 \), hence two of these vertices belong to the same ring. Since \( G \) is triangle-free, (7) implies that no internal vertex has two neighbors in the same ring, thus we can assume that \( v_2 \in V(R_1) \) and \( x_2, x_3 \in V(R_2) \). However, the path \( x_2 v_2 v_3 v_4 x_5 \) together with a subpath of \( R_2 \) forms a cycle that separates \( v_2 \) from \( R_1 \), which contradicts the assumption that \( G \) is embedded in the cylinder. Therefore,

if \( C = v_1 v_2 v_3 v_4 v_5 \) is a face such that \( v_2, \ldots, v_5 \) are internal vertices of degree three, then for some \( i \in \{1, 2\} \), both \( v_3 \) and \( v_4 \) have a neighbor in \( R_i \).

(8)

Let us now assign charge to vertices and faces of \( G \) as follows: each face \( f \) gets the initial charge \( |f| - 4 \) and each vertex \( v \) gets the initial charge \( \deg(v) - 4 \). The sum of the initial charges is \(-8\). Let us redistribute the charge: each 5-face sends \( 1/3 \) to each incident vertex \( v \) such that \( v \) is internal and has degree three. Furthermore, for each ring vertex \( w \) of degree two, if there exists a face \( f = v_1 v_2 v_3 v_4 v_5 \) such all vertices incident with \( f \) except for \( v_1 \) are internal of
degree three and if $v_{34}v_4$ is incident with the same face as $w$, then $w$ sends 1/3 to $f$. Note that after this procedure, all vertices and faces have non-negative charge, with the following exceptions: the ring vertices of degree two have charge at most $-7/3$ and the ring vertices of degree three have charge $-1$. For $i \in \{1, 2\}$, let $c_i$ be the sum of the charges of the vertices of $R_i$, together with the charges of the faces that share an edge with $R_i$ (such a face cannot share an edge with $R_{3-i}$, since the distance between $R_1$ and $R_2$ is at least two and all internal faces have length 5). Note that $c_1 + c_2 \leq -8$, and we may assume that $c_1 \leq -4$.

For $i \in \{1, 2, 3, 4\}$, let $f_i$ denote the face sharing the edge $a_i a_{i+1}$ with $R_1$. If a vertex $a_i$ has degree three, then let $x_i$ denote its internal neighbor. Since $G$ is 2-connected, at most two vertices of $R_1$ have degree two. Let us discuss several cases.

- $R_1$ contains two vertices of degree two: Since all faces have length 5 and $G$ is triangle-free, these two vertices are non-adjacent, say $a_2$ and $a_4$. Similarly, since $G$ does not contain a 4-cycle different from $R_1$ and $R_2$, both $a_1$ and $a_3$ have degree at least four, and since the sum of the charges of the vertices of $R_1$ is at most $-4$, we conclude that $\deg(a_1) = \deg(a_3) = 4$. Let $f_2 = a_1 a_2 a_3 x_2 x'_1$ and $f_4 = a_1 a_4 a_3 x'_3 x'_1$. Note that both $f_2$ and $f_4$ send charge to at most two vertices, hence their final charge is 1/3, and since $c_1 \leq -4$, it follows that the charge of $a_2$ and $a_4$ is $-7/3$. Therefore, the vertices $x'_1$, $x''_1$, $x'_3$, $x''_3$ and their neighbors distinct from $a_1$ and $a_3$ are internal vertices of degree three. However, these vertices form an 8-cycle, contradicting the criticality of $G$.

- $R_1$ contains one vertex of degree two, say $a_2$, and $a_1$, $a_3$ and $a_4$ have degree three: by (5), $x_1$ is adjacent to $x_3$, $x_1$ and $x_4$ have a common neighbor $x_{41}$ and $x_3$ and $x_4$ have a common neighbor $x_{43}$. Suppose that $x_1$ and $x_3$ have degree three. The path $x_{41}x_1x_3x_{43}$ is a part of a boundary of a 5-face $f$; let $y$ be the fifth vertex of $f$. Then $x_{41}x_4x_{43}y$ is a 4-cycle, contradicting (7). Therefore, we may assume that $x_1$ has degree greater than three. This implies that $a_2$ does not send any charge and its final charge is $-2$. Furthermore, $f_2$ has charge at least 2/3 and $f_4$ has charge at least 1/3, and thus $c_1 = -4$. Furthermore, $x_3$, $x_4$, $x_{41}$ and $x_{43}$ are internal and have degree three.

- $R_1$ contains one vertex of degree two, say $a_2$, and at least one vertex of $R_1$ has degree at least four: note that the sum of the charges of $a_2$ and $f_2$ is at least $-2$. It follows that exactly one vertex of $R_1$ has degree four, two vertices have degree three, and $c_1 = -4$.

- $R_1$ contains no vertices of degree two. Since $c_1 \leq -4$, it follows that all vertices of $R_1$ have degree three and all internal vertices of the faces sharing an edge with $R_1$ have degree three. But then $G$ contains an 8-cycle of internal vertices of degree 3, contradicting the criticality of $G$.

We conclude that $c_1 = -4$, and by symmetry, $c_2 = -4$. It follows that all charges that are not counted in $c_1$ and $c_2$ are equal to zero. Let us now go over the possible cases for the neighborhood of $R_1$ again, keeping the notation established in the previous paragraph:

- $R_1$ contains one vertex of degree two, say $a_2$, and $a_1$, $a_3$ and $a_4$ have degree three: Since all internal vertices have zero charge, $x_1$ has degree exactly
four. Let \( y_1, y_{41} \) and \( y_{43} \) be the neighbors of \( x_1, x_{41} \) and \( x_{43} \), respectively, not incident with \( f_2, f_3 \) and \( f_4 \). By (5), \( y_{43} \) is adjacent to \( y_1 \) and to \( y_{41} \), and the vertices \( y_1 \) and \( y_{41} \) have a common neighbor \( z \) distinct from \( y_{43} \). By (7), we have \( R_2 = y_1y_{43}y_{41}z \). However, then we can set \( H \) to be the graph consisting of \( R_1, R_2 \) and a vertex \( w \), with edges \( a_4y_{41}, wy_1, wa_1 \) and \( wa_4 \).

- \( R_1 \) contains one vertex of degree two, say \( a_2 \), one vertex of degree four and two of degree three. Let \( a_i \) be the vertex of degree four and \( x'_i \) and \( x''_i \) its internal neighbors. Since \( c_1 = -4 \), all internal vertices incident with the faces \( f_2, f_3 \) and \( f_4 \) have degree three, and by (5) they form a path \( P \) with ends \( x'_i \) and \( x''_i \). Furthermore, \( x'_i \) and \( x''_i \) have adjacent neighbors \( y'_i \) and \( y''_i \). We let \( G' \) consist of \( G - V(P) \) and a new vertex \( w \) adjacent to \( y'_i, y''_i \) and \( a_i \), and observe that every 3-coloring of \( G' \) extends to a 3-coloring of \( G \). This contradicts the minimality of \( G \).

\[ \square \]

A finite graph \( G \) is an \((e_1, e_2)\)-chain if either \( G \) is the complete graph on four vertices and \( e_1 \) and \( e_2 \) form a matching in \( G \); or, there exists an \((e_1, u_1u_2)\)-chain \( H \) and \( G \) consists of \( H - u_1u_2 \), vertices \( y_1, y_2 \) and \( u'_2 \) and edges \( y_1y_2, u_2u'_2, u_3y_1, u_1y_2, u'_2y_1 \) and \( u'_2y_2 \), where \( e_2 = y_1y_2 \). Let us note that each \((e_1, e_2)\)-chain is a planar graph with chromatic number 4 containing exactly four triangles (two incident with each of \( e_1 \) and \( e_2 \)), and all other faces of \( G \) have length 5. The graph \( G \) can be embedded in the Klein bottle by putting crosscaps on the edges \( e_1 \) and \( e_2 \); we call such an embedding canonical. Note that no cycle of length less than 5 is contractible in a canonical embedding of \( G \). Thomas and Walls [12] proved the following:

**Theorem 4.3.** If \( G \) is a 4-critical graph embedded in the Klein bottle so that no cycle of length at most 4 is contractible, then \( G \) is a canonical embedding of an \((e_1, e_2)\)-chain, for some edges \( e_1, e_2 \in E(G) \).

For the torus, Thomassen [13] showed that the situation is even simpler.

**Theorem 4.4.** If \( G \) is embedded in the torus so that no cycle of length at most 4 is contractible, then \( G \) is 3-colorable.

Also, the results of Akmanov [2] imply the following:

**Theorem 4.5.** There exists no \( R \)-critical graph \( G \) embedded in the cylinder with one ring \( R \) (and the other cuff disjoint from \( G \)) such that \( R \) is a cycle of length at most four and every cycle of length at most four in \( G \) is non-contractible.

Let us now give a description of \( \{R_1, R_2\} \)-critical graphs on a cylinder, where each of \( R_1 \) and \( R_2 \) is either a vertex ring or a triangle:

**Lemma 4.6.** Let \( G \) be an \( \{R_1, R_2\} \)-critical graph embedded in the cylinder, where each of \( R_1 \) and \( R_2 \) is either a vertex ring or a facial ring of length three. If every cycle of length at most 4 in \( G \) is non-contractible, then one of the following holds:

- \( G \) consists of \( R_1, R_2 \) and an edge between them, or
Figure 4: Arbitrarily large critical graph with rings of length four.

- $R_1$ and $R_2$ are triangles and $G$ consists of $R_1$, $R_2$ and two edges between them, or
- $R_1$ and $R_2$ are triangles and $G$ consists of $R_1$, $R_2$ and two adjacent vertices of degree three, each having a neighbor in $R_1$ and in $R_2$.

Proof. By Theorem 4.5, we have that $G$ is connected. We may assume that at least one of $R_1$ and $R_2$ is a triangle, since if both $R_1$ and $R_2$ are vertex rings, we can add a triangle containing $R_2$ to $G$. If the distance between $R_1$ and $R_2$ is at most two, then $G$ satisfies one of the conclusions of the lemma by Lemma 3.5 applied to the closed walk tracing the rings and the shortest path between them and by Theorem 4.1 (in the cases (b) and (c), the corresponding graphs are not $(R_1, R_2)$-critical). Therefore, assume that the distance between $R_1$ and $R_2$ is at least three.

Since $G$ is $(R_1, R_2)$-critical, there exists a precoloring $\psi$ of $R_1 \cup R_2$ that does not extend to a 3-coloring of $G$. We identify the vertices of $R_1$ and $R_2$ to which $\psi$ assigns the same color and we obtain a graph $G'$ embedded in the torus or in the Klein bottle (in the latter case, we can assume that both $R_1$ and $R_2$ are triangles). Note that $G'$ has no loops, since $R_1$ and $R_2$ are not adjacent. Observe also that $G'$ contains no contractible ($\leq 4$)-cycle. Since $G'$ is not 3-colorable, Theorems 4.3 and 4.4 imply that $G'$ is embedded in the Klein bottle and contains a canonical embedding of an $(e_1, e_2)$-chain as a subgraph. Therefore, $G'$ contains a separating non-contractible 4-cycle $C$. The subgraph of $G$ corresponding to $C$ contains at least two paths joining $R_1$ and $R_2$. However, this implies that the distance between $R_1$ and $R_2$ is at most two, which is a contradiction.

Finally, we give a similar result for $(R_1, R_2)$-critical graphs, where each of $R_1$ and $R_2$ has length at most four. A broken chain is a graph obtained from an $(e_1, e_2)$-chain by removing the edges $e_1$ and $e_2$, see Figure 4 for an illustration (the top of the picture is identified with the bottom, giving an embedding in the cylinder). Note that in any 3-coloring of the graph depicted in Figure 4, if $A$ and $B$ have different colors, then the colors of $C$ and $D$ differ as well. Repeating this observation, we conclude that if the colors of $A$ and $B$ differ, then the colors of the corresponding vertices of the rightmost cycle differ as well. Consequently, this gives an example of an $(R_1, R_2)$-critical graph embedded in the cylinder, where $R_1$ and $R_2$ are arbitrarily distant 4-cycles.

Dvořák and Lidický [8] gave a complete list of such $(R_1, R_2)$-critical graphs embedded in a cylinder without contractible ($\leq 4$)-cycles (other than broken chains, there are only finitely many). However, their proof is computer assisted. In this paper, we give a much weaker bound on the size of the graphs, which
however suffices for our purposes. We start with the case that there are many \((\leq 4)\)-cycles separating \(R_1\) from \(R_2\).

**Lemma 4.7.** Let \(G\) be an \(\{R_1, R_2\}\)-critical graph embedded in the cylinder \(\Sigma\), where each of \(R_1\) and \(R_2\) is either a vertex ring or a facial ring of length at most four. Suppose that every cycle of length at most 4 in \(G\) is non-contractible. If \(G\) contains at least 34 cycles of length at most 4, then \(G\) is a broken chain.

**Proof.** The graph \(G\) is connected by Theorem 4.5. Let \(C_1\) and \(C_2\) be distinct cycles of length at most 4 in \(G\). We claim that \(C_1\) bounds a closed disk in \(\Sigma\) that contains \(C_2\). Indeed, otherwise each of the open disks in \(\Sigma\) bounded by \(C_1\) contains a vertex of \(C_2\), and we conclude that the set \(X = V(C_1) \cap V(C_2)\) has size two. But then there exist three disjoint paths of length at most two between the vertices of \(X\), and one of the \((\leq 4)\)-cycles formed by these paths is contractible in \(\Sigma\), contradicting the assumptions.

We write \(C_1 < C_2\) if the closed disk bounded by \(C_1\) in \(\Sigma + R_2\) contains \(C_2\). Note that \(<\) is a linear ordering of the cycles of length at most four in \(G\). Let \(K_1, K_2, \ldots, K_m\) be the list of all cycles of length at most four in \(G\) sorted according to this ordering. Suppose that \(K_i\) and \(K_j\) are triangles for some \(i < j\). By Theorem 4.1, if \(V(K_i) \cap V(K_j) \neq \emptyset\), then \(j = i + 1\). If \(K_i\) and \(K_j\) are vertex-disjoint, then Lemma 4.6 implies that \(j \leq i + 3\).

For any \(i < j\), if \(V(K_i) \cap V(K_j) \neq \emptyset\), then by Theorem 4.1 the area between \(K_i\) and \(K_j\) consists either of one face or of two 5-faces, thus either \(j = i + 1\), or \(j = i + 2\) and \(K_i+1\) is a triangle. In particular, \(K_i\) and \(K_{i+3}\) are vertex-disjoint.

For \(i < j\), let \(G_{ij}\) be the subgraph of \(G\) drawn between \(K_i\) and \(K_j\). Note that if \(K_i\) and \(K_j\) are vertex-disjoint, then \(G_{ij}\) is \(\{K_i, K_j\}\)-critical. In that case Lemma 4.2 implies that \(G_{ij}\) is subsumed by a \(\{K_i, K_j\}\)-critical basic graph \(H_{ij}\).

If \(K_i\) and \(K_j\) are not vertex-disjoint, then we define \(H_{ij} = G_{ij}\).

Consider indices \(i < j < k\) and a graph \(B \in \{G_{ij}, H_{ij}\}\), and suppose that \(B \cup H_{jk}\) contains a contractible cycle \(C\) of length at most 4. By the definition of a basic graph, \(C \not\subseteq B\) and \(C \not\subseteq H_{jk}\); hence, \(C\) has length 4 and \(C = v_1v_2v_3v_4\), where \(v_2, v_3 \in V(K_j), v_1 \in V(B) \setminus V(K_j)\) and \(v_3 \in V(H_{jk}) \setminus V(K_j)\). Furthermore, \(v_2\) and \(v_3\) must be consecutive vertices of \(K_j\), thus both \(B\) and \(H_{jk}\) contain a triangle incident with an edge of \(K_j\). Therefore, we have the following. \(\text{For any } i < j < k \text{ and a graph } B \in \{G_{ij}, H_{ij}\}, \text{ if } B \cup H_{jk} \text{ contains a contractible cycle of length at most } 4, \text{ then both } B \text{ and } H_{jk} \text{ contain a triangle with an edge in } K_j.\)

\((9)\)

Let \(U\) be the set of indices \(i\) such that either there exists a triangle \(K_t\) with \(t \geq \max(1, i - 1)\), or there exists \(j\) such that \(i < j \leq m\) and \(H_{ij}\) contains a cutvertex or a triangle. Let \(L\) be the set of indices \(j\) such that either there exists a triangle \(K_t\) with \(t \leq \min(m, j + 1)\), or there exists \(i\) such that \(1 \leq i < j\) and \(H_{ij}\) contains a cutvertex or a triangle. Suppose that \(a \in L\) and \(b \in U\) satisfy \(b \geq a + 6\). If there exists \(t \leq \min(m, a + 1)\) such that \(K_t\) is a triangle, then let \(G_1 = G_{1a}\). Otherwise, there exists \(k < a\) such that \(H_{ka}\) contains a cutvertex or a triangle, and we set \(G_1 = G_{1k} \cup H_{ka}\). Similarly, if there exists \(t \geq \max(1, b - 1)\) such that \(K_t\) is a triangle, then let \(G_2 = G_{bm}\); otherwise let \(G_2 = H_{bd} \cup G_{tm}\) for \(l > b\) such that \(H_{bl}\) contains a cutvertex or a triangle. Let \(G' = G_1 \cup G_{ab} \cup G_2\). Note that \(G'\) contains no contractible \((\leq 4)\)-cycles.
by (9), the definition of $U$ and $L$ and the choice of $G_1$ and $G_2$. Furthermore, observe that $G'$ contains triangles or cutvertices $T_1$ and $T_2$ separated by at least three $(\leq 4)$-cycles belonging to $G_{ab}$. By Lemma 4.6, every precoloring of $T_1 \cup T_2$ extends to a 3-coloring of the subgraph of $G'$ between $T_1$ and $T_2$. Theorem 4.5 implies that every precoloring of $R_i$ extends to a 3-coloring of the subgraph of $G'$ between $R_i$ and $T_j$, for $i \in \{1, 2\}$. Therefore, every precoloring of $R_1 \cup R_2$ also extends to a 3-coloring of $G'$. However, $G'$ subsumes $G$, hence every precoloring of $R_1 \cup R_2$ also extends to a 3-coloring of $G$. This contradicts the criticality of $G$. We conclude that if $a \in L$ and $b \in U$, then $b \leq a + 5$. Let $X$ be the set of indices $i$ such that $i \in U$ and $i + 2 \in L$. Observe that if $X$ is nonempty, then $\max X - \min X \leq 7$. Let us also remark that if $K_i$ is a triangle, then $j \in X$ for $i - 3 \leq j \leq i + 1$.

Since $m \geq 34$, there exist indices $1 \leq i_1 < i_2 < \ldots < i_8 \leq m$ such that $i_{j+1} = i_j + 2$ for $1 \leq j \leq 4$ and for $5 \leq j \leq 8$, $i_5 \geq i_4 + 9$ and $i_1, i_2, i_3, i_5, i_6, i_7 \not\in X$. Note that the definition of $X$ ensures that $G_{i_1i_4}$ and $G_{i_5i_8}$ are triangle-free (and consequently, $K_{i_1}$ and $K_{i_5}$ are vertex-disjoint for $1 \leq j \leq 7$) and that $H_{i_1i_4}$ is 2-connected and triangle-free for $1 \leq j \leq 3$ and $5 \leq j \leq 7$.

Combining (2) and (3) shows that there exists a precoloring $\psi$ of $K_{i_1}$, a vertex $v \in V(K_{i_1})$ and a color $c$ such that every precoloring $\phi$ of $K_{i_1} \cup K_{i_4}$ that matches $\psi$ on $K_{i_1}$ and satisfies $\phi(v) \neq c$ extends to a 3-coloring of $H_{i_1i_4} \cup H_{i_1i_4}$. Furthermore, an inspection of the basic 2-connected triangle-free graphs shows that that every 3-coloring of $K_{i_1}$ extends to a 3-coloring of $H_{i_1i_2}$ that assigns $v$ a color different from $c$. It follows that every precoloring $\phi$ of $K_{i_1} \cup K_{i_4}$ that matches $\psi$ on $K_{i_1}$ extends to a 4-coloring of $H_{i_1i_4}$, and thus also to a 3-coloring of $G_{i_1i_4}$. In fact, it is sufficient to assume that $\phi$ has the same type $S_1$ on $K_{i_1}$ as $\psi$; thus, every precoloring of $K_{i_1} \cup K_{i_4}$ whose type on $K_{i_1}$ is $S_1$ extends to a 3-coloring of $G_{i_1i_4}$. Symmetrically, there exists a type $S_2$ such that every precoloring of $K_{i_5} \cup K_{i_8}$ whose type on $K_{i_5}$ is $S_2$ extends to a 3-coloring of $G_{i_5i_8}$.

Let $G' = G_{i_1i_4}$ with rings $L_1 = K_{i_4} = a_1a_2a_3a_4$ and $L_2 = K_{i_8} = b_1b_2b_3b_4$. Since $i_5 \geq i_4 + 9$, the distance between $L_1$ and $L_2$ is at least three. Let $G''$ be the graph obtained from the embedding of $G'$ in the cylinder in the following way: Cap the holes of the cylinder by disks. If $S_1 = \{a_i, a_{i+2}\}$ for some $i \in \{1, 2\}$, then add the edge $a_ia_{i+2}$ to the face bounded by $L_1$ and add a crosscap to the middle of this edge. If $S_1 = \emptyset$, then identify $a_1$ with $a_3$ to a vertex $a_{13}$ and $a_2$ with $a_4$ to a vertex $a_{24}$. Observe that at most two vertices of $L_1$ are adjacent with a $(\leq 4)$-cycle distinct from $L_1$ in $G'$, and if there are two such vertices, then they are adjacent. By symmetry, we can assume that $L_1$ is in the only $(\leq 4)$-cycle incident with at least one of $a_2$ and $a_3$. We add a crosscap on the edge $a_{13}a_{24}$ and draw the edges from $a_{13}$ to the neighbors of $a_3$ and the edges from $a_{24}$ to the neighbors of $a_2$ through the crosscap. Transform $L_2$ in the same way according to $S_2$. Note that $G''$ is embedded in the Klein bottle and it has no loops.

Consider a cycle $C$ of length at most 4 in $G''$. Since the distance between $L_1$ and $L_2$ is at least three, we may assume that $C$ does not contain any of the vertices $b_1, \ldots, b_4$. Let us first consider the case that $S_1 = \{a_i, a_{i+2}\}$ for some $i \in \{1, 2\}$. If $C$ does not contain the edge $a_ia_{i+2}$, then $C$ is non-contractible in $G$, and thus it separates the crosscaps in $G''$. If $C$ contains the edge $a_ia_{i+1}$, then $C$ is one-sided. Suppose now that $S_1 = \emptyset$, as in the construction of $G''$, we assume that $L_1$ is the only $(\leq 4)$-cycle incident with at least one of $a_2$ and $a_3$ in $G'$. If $C$ contains the edge $a_{13}a_{24}$, then $C$ corresponds to a $(\leq 4)$-cycle in $G'$ containing one of the edges of $L_1$, which necessarily must
be $a_1a_4$; hence, no other edge of $C$ passes through the crosscap and $C$ is one-sided. If $C$ contains neither $a_{13}$ nor $a_{24}$, then $C$ is non-contractible in $G$ and separates the crosscaps in $G''$. If $C$ contained both $a_{13}$ and $a_{24}$, but not the edge $a_{13}a_{24}$, then since $a_2$ and $a_3$ are not incident with $(\leq 4)$-cycles in $G'$, we conclude that $a_1a_4$ is incident with two triangles in $G'$. However, then either one of the triangles or the 4-cycle contained in their union is contractible in $G$, which is a contradiction. It remains to consider the case that $C$ contains exactly one of $a_{13}$ and $a_{24}$. By symmetry, assume that $C$ contains $a_{13}$. Let $e_1$ and $e_2$ be the edges incident with $a_{13}$ in $C$, and let $e_1$ and $e_2$ be the corresponding edges in $G$. Since no $(\leq 4)$-cycle different from $L_1$ is incident with $a_3$, we may assume that $e_1$ is incident with $a_1$. If $e_2$ is incident with $a_4$, then $C$ is one-sided. If $e_2$ is incident with $a_1$, then $C$ separates the crosscaps. We conclude that every $(\leq 4)$-cycle in $G''$ is non-contractible.

If $G''$ is 3-colorable, then the corresponding 3-coloring of $G'$ has type $S_1$ on $L_1$ and type $S_2$ on $L_2$. It follows that every precoloring of $K_{14}\cup K_{18}$ extends to a 3-coloring of the subgraph $G'_{1418}$, contradicting the criticality of $G$.

Therefore, $G''$ is not 3-colorable and it contains a 4-critical subgraph $F$. By Theorem 4.3, $F$ is a $(x_1x_2, y_1y_2)$-chain, for some vertices $x_1, x_2, y_1, y_2 \in V(G'')$, and its embedding derived from the embedding of $G''$ is canonical. Suppose that $S_1 = \emptyset$ and that $L_1$ is the only $(\leq 4)$-cycle in $G'$ incident with $a_2$ and $a_3$. By symmetry, we can assume that $x_1 = a_{13}$ and $x_1x_2$ corresponds to an edge $a_3v$ in $G'$. Since $x_1x_2$ is incident with two triangles $x_1x_2v_1$ and $x_1x_2v_2$ in $F$, but $a_3$ is not incident with a triangle, we have $a_1v_1, a_1v_2 \in E(G')$. Let us remark that $x_2 \neq a_{24}$, as otherwise we would similarly have $a_4v_1, a_4v_2 \in E(G')$ and at least one of the cycles $a_1a_4v_1$, $a_1a_4v_2$ and $a_1v_1a_4v_2$ would be contractible in $G$. Hence, both $v_1$ and $v_2$ are adjacent to the vertex $v$ in $G'$. Since the 4-cycle $a_1v_1v_2$ is non-contractible in $G'$, using Theorem 4.1 we can assume that $a_1a_2a_3v_1$ and $a_1a_2a_3v_2$ are faces of $G'$ and $a_2$ and $a_4$ have degree two. On the other hand, if $S_1 = \{a_i, a_{i+2}\}$ for some $i \in \{1, 2\}$, then one of $x_1x_2, y_1y_2$ is equal to $a_ia_{i+2}$, and since this edge is incident with two triangles in $F$, it follows that $L_1$ is a subgraph of $F$. A symmetrical claim holds at $L_2$. Theorem 4.1 implies that every face of $F$ not incident with $x_1x_2$ and $y_1y_2$ is also a face of $G'$. Let us recall that $F$ is a $(x_1x_2, y_1y_2)$-chain, and consequently observe that in all the cases, $G'$ is a broken chain.

Choose the labeling of $L_1$ and $L_2$ so that $a_1$ and $b_1$ are vertices of degree four in $G'$. Observe that a precoloring $\psi$ of $L_1 \cup L_2$ extends to a 3-coloring of $G'$ if and only if $\psi(a_1) \neq \psi(a_3)$ or $\psi(b_1) \neq \psi(b_3)$. Let $X_1$ be the subgraph of $G$ drawn between $R_1$ and $K_{14}$, and let $X_2$ be the subgraph of $G$ drawn between $R_1$ and $K_{18}$. Consider a precoloring $\phi$ of $R_1 \cup K_{18}$ that does not extend to a 3-coloring of $X_2$. Let $X'_1$ be the graph obtained from $X_1$ in the following way: first, we add the edge $a_1a_4$ and put a crosscap on it. If $R_1$ is a vertex ring or a triangle, then we paste a crosscap over the cuff incident with $R_1$. If $R_1$ is a 4-cycle, then we either add an edge between two of its vertices or identify its opposite vertices according to the type of $\phi$ on $R_1$ and put a crosscap in the appropriate place, using the same rules as in the construction of $G''$. Note that $X'_1$ is embedded in the Klein bottle so that all contractible cycles have length at least five. If $X'_1$ is 3-colorable, then its 3-coloring corresponds to a 3-coloring of $X_1$ that matches $\phi$ on $R_1$ and assigns $a_1$ and $a_3$ different colors. Hence, this coloring extends to a 3-coloring of $X_2$ that matches $\phi$ on $R_1 \cup K_{18}$, which is a contradiction.
Therefore, \( X'_1 \) is not 3-colorable, and by Theorem 4.3, \( X'_1 \) contains a canonical embedding of an \((e_1, e_2)\)-chain \( F_1 \), for some vertices \( e_1, e_2 \in E(X'_1) \). Since \( F_1 \) contains four one-sided triangles, it follows that \(|R_1| = 4\). However, as in the analysis of \( G' \), we conclude that then \( X_1 \) is a broken chain. By symmetry, the subgraph of \( G \) drawn between \( K_{i_5} \) and \( R_2 \) is a broken chain as well. This implies that \( G \) is a broken chain. \( \square \)

The case of a cylinder with two rings of length at most four is now easy to handle using Theorem 3.3, thanks to the bound on the size of a subgraph that captures \((\leq 4)\)-cycles given by Lemma 4.7. We will need the following observation.

**Lemma 4.8.** Let \( G \) be an \( R \)-critical graph embedded in a surface \( \Sigma \) with rings \( \mathcal{R} \) so that every \((\leq 4)\)-cycle is non-contractible, let \( G' \) be another \( R \)-critical graph embedded in \( \Sigma \) with rings \( \mathcal{R} \) and let \( X \subset F(G) \) and \{\( (J_f, S_f) : f \in F(G') \)\} be a cover of \( G \) by faces of \( G' \). Let \( f \) be an open \( 2 \)-cell face of \( G' \) and let \( G_1, \ldots, G_k \) be the components of the \( G \)-expansion of \( S_f \), where for \( 1 \leq i \leq k \), \( G_i \) is embedded in the disk with one ring \( R_i \). In this situation, \( \sum_{i=1}^{k} w(G_i, \{R_i\}) \leq s(|f|) + \text{el}(f) \).

**Proof.** By Theorem 3.4 and Lemma 3.5, we have
\[
\sum_{i=1}^{k} w(G_i, \{R_i\}) \leq \sum_{i=1}^{k} s(|R_i|).
\]
Note that we have \( s(x) + s(y) \leq s(x + y) \leq s(x) + y \) for every \( x, y \geq 5 \); hence,
\[
\sum_{i=1}^{k} s(|R_i|) \leq s \left( \sum_{i=1}^{k} |R_i| \right) = s(|f| + \text{el}(f)) \leq s(|f|) + \text{el}(f).
\]
\( \square \)

Let \( \text{cyl} \) be a function satisfying the following for all non-negative integers \( x \) and \( y \):

- \( \text{cyl}(0, 0) = 0 \)
- \( \text{cyl}(x, y) = \text{cyl}(y, x) \)
- if \( x > 0 \), then \( \text{cyl}(x, y) \geq \text{cyl}(0, y) + x + 13 \)
- if \( x, y > 1 \), then \( \text{cyl}(x, y) \geq \text{cyl}(1, x) + \text{cyl}(1, y) + 19 \)
- for any non-negative integer \( y' < y \), we have
  \[
  \text{cyl}(x, y) \geq \text{cyl}(x, y') + s(y - y' + 8) \geq \text{cyl}(x, y') + 1
  \]
- \( \text{cyl}(x, y) \geq s(x + y + 11) \)
- if \( x \geq 4 \), then \( \text{cyl}(x, y) \geq 886 \)
- \( 2\text{cyl}(6, 7) \leq \text{cyl}(7, 7) \)
- if \( x \leq 4 \) and \( 5 \leq y \leq 6 \), then
  \[
  \text{cyl}(x, y) \geq (2/3 + 52\epsilon)(x + y) + 20(40 + 5\text{cyl}(4, 4)/s(5) + 692)/3
  \]
Proof. We proceed by induction, and assume that the claim holds for all graphs
satisfying (I0), (I1) and (I2). By Theorem 4.5, if (I3) is false, then (I5) is false, so (I6) hold.

We are now ready to prove the main result of this section.

Theorem 4.9. Let \( G \) be a graph embedded in the cylinder with rings \( R_1 \) and \( R_2 \) of length at most four, where \( R_2 \) is a facial ring. Suppose that every \((\leq 4)\)-cycle in \( G \) is non-contractible. If \( G \) is \( \{R_1, R_2\}\)-critical and not a broken chain, then \( w(G, \{R_1, R_2\}) \leq \text{cyl}(|R_1|, |R_2|) \).

Proof. We proceed by induction, and assume that the claim holds for all graphs
with fewer than \( |E(G)| \) edges. By Lemma 4.6, we can assume that \( |R_2| = 4 \). By Theorem 4.5, \( G \) is connected. By Lemma 3.1, \( G \) satisfies (I0), (I1) and (I2). Furthermore, we already observed that every critical graph satisfies (I0), and (I6) hold trivially.

If (I3) is false, then \( G \) contains a cutvertex \( v \). Observe that \( v \) separates \( R_1 \) from \( R_2 \). Cut the cylinder along a non-contractible simple closed curve intersecting \( G \) only in \( v \), and let \( G_1 \) and \( G_2 \) be the pieces of \( G \) drawn in the resulting cylinders, where \( R_1 \subseteq G_1 \) and \( R_2 \subseteq G_2 \). If \( G_1 = R_1 \), then note that \( R_1 \) is a facial ring, as \( v \) is a cutvertex. The graph \( G_2 \) is \( \{v, R_2\}\)-critical, hence by induction \( w(G_2, \{v, R_2\}) \leq \text{cyl}(1, |R_2|) \). But then \( w(G, \{R_1, R_2\}) \leq w(G_2, \{v, R_2\}) + |R_1| \leq \text{cyl}(1, |R_2|) + |R_1| \leq \text{cyl}(|R_1|, |R_2|) \) as required.

On the other hand, if \( G_1 \neq R_1 \), then \( G_1 \) is \( \{R_1, v\}\)-critical. By Lemma 4.6, we have \( |R_1| \geq 1 \). Furthermore, if \( |R_1| \leq 3 \), then \( G_1 \) consists of \( R_1 \) and an edge between \( v \) and a vertex \( w \) of \( R_1 \). If that is the case, then \( G_2 \) is \( \{v, R_2\}\)-critical, where \( v \) is taken as a weak vertex ring: consider any edge \( e \) of \( G_2 \) not belonging to \( R_2 \). Since \( G \) is \( \{R_1, R_2\}\)-critical, there exists a precoloring \( \phi \) of \( R_1 \) and \( R_2 \) that extends to a coloring of \( G - e \), but not to \( G \). Let \( \psi \) be the precoloring of \( v \) and \( R_2 \) such that \( \psi(v) = \phi(w) \) and \( \psi \) matches \( \phi \) on \( R_2 \). Note that \( \psi \) extends to a coloring of \( G_2 - e \), but not to \( G_2 \). Since the choice of \( e \) was arbitrary, this shows that \( G_2 \) is \( \{v, R_2\}\)-critical with \( v \) weak. By induction, we have \( w(G_2, \{v, R_2\}) \leq \text{cyl}(0, |R_2|) \). Hence, if \( f \) is the cuff face of \( v \) in \( G_2 \), then \( w(G, \{R_1, R_2\}) \leq \text{cyl}(0, |R_2|) + s(|f|) \leq \text{cyl}(0, |R_2|) + 2cyl(1, 4) + 8 \leq \text{cyl}(4, 4) \). Therefore, we can assume that (I3) holds.

If (I5) is false, then since (I3) holds, we can assume that the two vertices \( r_1 \) and \( r_2 \) belong to a ring of length four, say to \( R_2 \). If the internal face incident with \( r_1r_2 \) has length five, then \( r_1r_2 \) is a part of a triangle \( T \) separating \( R_1 \) from \( R_2 \). By applying induction to the subgraph of \( G \) drawn between \( R_1 \) and \( T \), we conclude that \( w(G, \{R_1, R_2\}) \leq \text{cyl}(|R_1|, 3) + s(5) \leq \text{cyl}(|R_1|, |R_2|) \). Otherwise, we apply induction to the graph obtained by contracting the edge \( r_1r_2 \), and obtain \( w(G, \{R_1, R_2\}) \leq \text{cyl}(|R_1|, 3) + 1 \leq \text{cyl}(|R_1|, |R_2|) \). Hence, assume that (I5) holds.

Suppose now that the distance between \( R_1 \) and \( R_2 \) is at most four. If \( R_1 \) is a vertex ring, then add a triangle \( R_1' \) forming a boundary of the cuff incident with
$R_1$ and note that this operation does not decrease the weight of $G$; otherwise, let $R'_1 = R_1$. Next, we use Lemma 3.5 to the closed walk consisting of $R'_1$, $R_2$ and the shortest path between $R_1$ and $R_2$ traversed twice. By Theorem 3.4, we have $w(G, \{R_1, R_2\}) \leq s(|R_1| + 3) + |R_2| + 8 \leq \text{cyl}(|R_1|, |R_2|)$. Therefore, assume that (17) holds.

Consider a path $P$ of length at most four with both ends being ring vertices. By the previous paragraph, both ends of $P$ belong to the same ring $R$. By (15), $P$ has length at least three. Since $G$ is embedded in the cylinder, there exists a subpath $Q$ of $R$ such that $P \cup Q$ is a contractible cycle. Note that $|P \cup Q| \leq |P| + 3 \leq 7$, and by Theorem 4.1, $P \cup Q$ bounds a face. Therefore, $G$ is well-behaved and satisfies (14).

Let $M$ be the subgraph of $G$ consisting of edges incident with $(\leq 4)$-cycles. Since $G$ is not a broken chain, Lemma 4.7 implies that $|E(M)| \leq 132$. Note that $M$ captures $(\leq 4)$-cycles of $G$. If the assumptions of Theorem 3.3 are not satisfied, then $w(G, \{R_1, R_2\}) \leq (2/3 + 26\epsilon)(\ell((\{R_1, R_2\}) + 20\epsilon(E(M))/3 < 886 \leq \text{cyl}(|R_1|, |R_2|))$. Therefore, assume the contrary.

Then, there exists an $\{R_1, R_2\}$-critical graph $G'$ embedded in the cylinder with rings $R_1$ and $R_2$ such that $|E(G')| < |E(G)|$, satisfying conditions (a)–(e) of Theorem 3.3. By (b), all $(\leq 4)$-cycles in $G'$ are non-contractible. By Theorem 4.5, $G'$ is connected, and thus all its faces are open 2-cell. Let $X \subset F(G)$ and $(J_f, S_f) : f \in F(G')$ be the cover of $G$ by faces of $G'$ as in (d). For $f \in F(G')$, let $G_1^f, \ldots, G_k^f$ be the components of the $G$-expansion of $S_f$. Since $\Sigma_f$ is a disk and all surfaces of the $G$-expansion of $S_f$ are fragments of $\Sigma_f$, it follows that for $1 \leq i \leq k_f$, $G_i^f$ is embedded in the disk with one ring $R_i^f$. By the definition of a cover of $G$ by faces of $G'$, we have

$$w(G, \{R_1, R_2\}) = \sum_{f \in F(G)} w(f) = \sum_{f \in X} w(f) + \sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \{R_i^f\}).$$

Suppose first that $G'$ is 2-connected, and thus its embedding is closed 2-cell. By Theorem 3.3(c), $G'$ has a face of length at least 6, hence $G'$ is not a broken chain. Therefore, by induction we have $w(G', \{R_1, R_2\}) \leq \text{cyl}(|R_1|, |R_2|)$. Since each internal face of $G'$ is closed 2-cell, Theorem 3.3(e) implies that

$$\sum_{i=1}^{k_f} w(G_i^f, \{R_i^f\}) \leq s(|f|) - c(f)$$

for every $f \in F(G')$, and consequently (using Theorem 3.3(d) in the last inequality), we have

$$\sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \{R_i^f\}) \leq \sum_{f \in F(G')} s(|f|) - c(f) = w(G', \{R_1, R_2\}) - \sum_{f \in F(G')} c(f) \leq w(G', \{R_1, R_2\}) - |X|s(6).$$

25
Putting the inequalities together, we obtain
\[
w(G, \{R_1, R_2\}) \leq w(G', \{R_1, R_2\}) + \left( \sum_{f \in X} w(f) \right) - |X| s(6)
\]
\[
= w(G', \{R_1, R_2\}) \leq \text{cyl}(|R_1|, |R_2|),
\]
since the face in \(X\) (if any) has length 6 by the definition of a cover of \(G\) by faces of \(G'\).

Finally, let us consider the case that \(G'\) is not 2-connected. Let \(v\) be a cutvertex in \(G'\) and let \(G_1\) and \(G_2\) be the subgraphs of \(G'\) intersecting in \(v\) such that \(G' = G_1 \cup G_2\), \(R_1 \subseteq G_1\) and \(R_2 \subseteq G_2\). Similarly to the analysis of the property (I3) for \(G\), we show that \(|R_1| \geq 1\), if \(|R_1| \leq 3\), then \(w(G', \{R_1, R_2\}) \leq \text{cyl}(0, |R_2|) + 2 + |R_1| \leq \text{cyl}(|R_1|, |R_2|) - 11\), and if \(|R_1| = 4\), then \(w(G', \{R_1, R_2\}) \leq 2\text{cyl}(1, 4) + 8 \leq \text{cyl}(4, 4) - 11\). By Lemma 4.8, we have
\[
\sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G'_i, \{R'_i\}) \leq w(G', \{R_1, R_2\}) + \sum_{f \in F(G')} e(f) \leq w(G', \{R_1, R_2\}) + 10.
\]
Combining the inequalities, we have
\[
w(G, \{R_1, R_2\}) \leq w(G', \{R_1, R_2\}) + 10 + \sum_{f \in X} w(f)
\]
\[
\leq \text{cyl}(|R_1|, |R_2|) - 1 + \sum_{f \in X} w(f)
\]
\[
< \text{cyl}(|R_1|, |R_2|).
\]
\[\square\]

5 Narrow cylinder

In this section, we consider graphs embedded in the cylinder with two rings of length at most 7. First, let us state an auxiliary result that will also be useful in the case of general surfaces. Consider a graph embedded in a surface \(\Sigma\). If \(K_1\) and \(K_2\) are two cycles surrounding a cuff \(C\) and \(\Delta_1\) and \(\Delta_2\) are the open disks bounded by \(K_1\) and \(K_2\), respectively, in \(\Sigma + C\), then we say that \(K_1\) and \(K_2\) are incomparable if \(\Delta_1 \not\subseteq \Delta_2\) and \(\Delta_2 \not\subseteq \Delta_1\).

**Lemma 5.1.** Let \(G\) be a graph in a surface \(\Sigma\) with rings \(R\), such that \(G\) is \(R\)-critical and every \((\leq 4)\)-cycle is non-contractible. Let \(K_0\) be a cycle in \(G\) of length at most seven surrounding a ring \(R\), let \(C\) be the cuff incident with \(R\) and let \(\Delta\) be the closed disk in \(\Sigma + C\) bounded by \(K_0\). In this situation, at most \(10|K_0|\) edges of \(G\) drawn outside of \(\Delta\) are incident with \((\leq 7)\)-cycles surrounding \(R\) that are incomparable with \(K_0\).

**Proof.** Let \(X\) be the set of edges drawn outside of \(\Delta\) that are incident with \((\leq 7)\)-cycles surrounding \(R\) and incomparable with \(K_0\). Initially, we give each edge of \(X\) charge 1 and all the edges of \(K_0\) charge 0. Next, we aim to move the charge from \(X\) to \(K_0\).
For an edge $x \in X$, choose a $(\leq 7)$-cycle $K$ surrounding $R$ incomparable with $K_0$ and containing $x$. Note that at least one edge of $E(K) \setminus E(K_0)$ is drawn in $\Delta$. Let $K = P_1 \cup P_2$, where $P_1$ and $P_2$ are paths intersecting only in their endvertices such that $P_1$ has ends in $K_0$ and otherwise is drawn in the interior of $\Delta$. Let $K_0 = P_1 \cup P_3$, where $P_1$ and $P_3$ are paths sharing endvertices with $P_1$ and $P_2$ and the cycle $P_1 \cup P_3$ surrounds $R$. For $1 \leq i \leq 4$, let $m_i$ be the length of $P_i$. Since all $(\leq 4)$-cycles are non-contractible, we have $m_2 + m_3 \geq 5$ and $m_1 + m_4 \geq 5$. Since $m_1 + m_2 + m_3 + m_4 = |K_0| + |K| \leq 14$, it follows that $m_2 + m_3 \leq 9$ and $m_1 + m_4 \leq 9$. Note that the closed walk $P_2 \cup P_3$ is contractible. Let $K'$ be the symmetric difference of $P_2$ and $P_3$. Note that $K'$ is a union of a contractible cycles, and since every contractible cycle has length at least five, it follows that $K'$ has only one component. We distribute the charge of $x$ among the edges of $K' \cap K_0$ evenly.

Let us consider the case that $K'$ does not bound a face of $G$. By Theorem 4.1, it follows that $|K'| \geq 8$ and at most one vertex is contained in the open disk bounded by $K'$. Since $m_2 + m_3 \leq 9$, we have that $P_2$ and $P_3$ are edge-disjoint and $K' = P_2 \cup P_3$. Since $m_1 \geq 1$, we have $m_2 \leq 6$ and thus $m_3 \geq 2$. Suppose first that only one edge $e$ is drawn in the open disk bounded by $K'$. Let $f_1$ and $f_2$ be the faces of $G$ in this disk, and assume that $f_1$ shares no edge with $P_3$. In this situation, we say that the charge sent from the edges of $f_1 \cap K$ to the edges of $P_2$ passes through $e$. Note that if $m_3 = 2$, then $m_1 = 1$, $m_2 = 6$, $|K'| = 8$ and $|f_1| = |f_2| = 5$, thus in this case 4 units of charge pass through $e$; otherwise, at most 5 units of charge pass through $e$.

Suppose now that a vertex is contained in the open disk bounded by $K'$, and let $f_1$, $f_2$ and $f_3$ be the faces of $G$ in this disk. Note that $|f_1| = |f_2| = |f_3| = 5$ and $m_3 \geq 3$. For $1 \leq i < j \leq 3$, let $e_{ij}$ be the common edge of $f_i$ and $f_j$. Let us first consider the case that two of these faces, say $f_2$ and $f_3$, share an edge with $P_3$. We let the amount of charge of the edges of $f_1 \cap K$ proportional to $|E(f_i) \cap E(P_3)|$ pass through $e_{1i}$, for $i \in \{2, 3\}$. The other possible case is that only one face, say $f_3$, shares edges with $P_3$; in this situation, the charge of $f_1 \cap K$ passes through $e_{13}$ and the charge of $f_2 \cap K$ passes through $e_{23}$.

Let us now analyze the final charge of the edges of $K_0$. Consider $e \in K_0$, let $f$ be the face of $G$ incident with $e$ that is not a subset of $\Delta$ and let $m = |E(f) \cap E(K_0)|$. If $|f| \geq 10$, then no charge is sent to $e$. If $7 \leq |f| \leq 9$, then no charge passes through edges of $f$, and thus $e$ has charge at most $8$. Similarly, if $|f| = 6$ and $m = 1$, then no charge passes through edges of $f$, and thus $e$ has charge at most $5$. If $|f| = 6$ and $m \geq 2$, then at most four units pass through each edge of $E(f) \setminus E(K_0)$ and each of these edges additionally sends one unit of charge. This charge is evenly divided between the edges of $E(f) \cap E(K_0)$, hence $e$ has charge at most 10. If $|f| = 5$ and $m = 1$, then at most one unit passes through each edge of $E(f) \setminus E(K_0)$, and $e$ has charge at most 8. If $|f| = 5$ and $m \geq 2$, then at most five units of charge pass through each edge of $E(f) \setminus E(K_0)$ and each of these edges additionally sends one unit of charge. This charge is evenly divided between the edges of $E(f) \cap E(K_0)$, and thus $e$ has charge at most 9.

Therefore, each edge of $K_0$ has charge at most 10, and $|X| \leq 10|K_0|$. □

We will also need several related claims regarding cycles near a ring.

**Lemma 5.2.** Let $G$ be a graph embedded in the cylinder with rings $R_1$ and $R_2$, such that $G$ is $(R_1, R_2)$-critical and every $(\leq 4)$-cycle is non-contractible.
If $|R_1| = 4$, then at most 93 edges of $G$ are incident with a non-contractible $(\leq 7)$-cycle that shares a vertex with $R_1$.

**Proof.** Let $C_1$ be the cuff incident with $R_1$. Let $K_0$ be a non-contractible $(\leq 7)$-cycle that shares a vertex with $R_1$ such that the closed disk $\Delta$ bounded by $K_0$ in $\Sigma + \hat{C}_1$ is as large as possible. Observe that every edge of $G$ incident with a non-contractible $(\leq 7)$-cycle that shares a vertex with $R_1$ is either drawn in $\Delta$ or is incident with a non-contractible $(\leq 7)$-cycle that is incomparable with $K_0$. Theorem 4.1 implies that at most 23 edges of $G$ are drawn in $\Delta$. Together with Lemma 5.1, the claim follows. \qed

If $R$ is a 6-cycle and $C$ is a $(\leq 6)$-cycle in a graph $G$, then we say that $C$ is bound to $R$ if either $|V(C) \cap V(R)| \geq 3$ or $G$ contains edges $cr$ and $c'r'$ with $c, c' \in V(C) \setminus V(R)$ and $r, r' \in V(R) \setminus V(C)$ such that $r$ and $r'$ are non-adjacent.

**Lemma 5.3.** Let $G$ be a graph embedded in the cylinder with rings $R_1$ and $R_2$, such that every $(\leq 4)$-cycle is non-contractible. Suppose that $|R_1| = 6$ and $R_1$ is an induced cycle, and let $X \subseteq E(G)$ be the set of edges incident with non-contractible $(\leq 6)$-cycles bound to $R_1$. If $G$ is $\{R_1, R_2\}$-critical, then $|X| \leq 346$.

**Proof.** Let $C$ be a non-contractible $(\leq 6)$-cycle in $G$ that is bound to $R_1$. If $|V(C) \cap V(R_1)| \geq 3$, then for every edge $e \in E(C) \setminus E(R_1)$, there exists a contractible cycle $K$ in $C \cup R_1$ of length at most 8 that contains $e$, sharing at least $|K| - 4$ edges with $R_1$.

If $G$ contains edges $cr$ and $c'r'$ with $c, c' \in V(C) \setminus V(R)$ and $r, r' \in V(R) \setminus V(C)$ such that $r$ and $r'$ are non-adjacent, then $G$ contains two contractible cycles $K_1$ and $K_2$ bounding disjoint open disks, such that $E(C) \subseteq E(K_1) \cup E(K_2)$ and $|K_1| + |K_2| = |R_1| + |C| + 4 \leq 16$. Furthermore, for $i \in \{1, 2\}$, the cycle $K_i$ shares at least $\max(2, |K_i| - 7)$ edges with $R_1$. Since $|K_{3-\epsilon}| \geq 5$, we have $|K_i| \leq 11$.

By Theorem 4.1, we conclude that each edge $e \in X \setminus E(R_1)$ satisfies at least one of the following claims:

1. $e$ is incident with a face $f_1$ of length at most 11 sharing at least two edges with $R_1$, or
2. $e$ is incident with a face $f_1$ of length at most 8 sharing an edge with $R_1$, or
3. there exist faces $f_1$ and $f_2$ sharing an edge such that $e$ is incident with $f_2$, $f_1$ shares $k \in \{2, 3, 4\}$ edges with $R_1$ and $|f_1| + |f_2| \leq 9 + k$, or
4. there exist faces $f_1$, $f_2$ and $f_3$ of length five such that $e$ is incident with $f_3$, $f_1$ shares at least 3 edges with $R_1$ and $f_2$ shares an edge both with $f_1$ and with $f_3$, or
5. there exist faces $f_1$, $f'_1$ and $f_2$ such that $e$ is incident with $f_2$, $f_2$ shares an edge with both $f_1$ and $f'_1$, each of $f_1$ and $f'_1$ shares $k \in \{1, 2\}$ edges with $R_1$, $|f_1| = |f'_1| = 5$ and $|f_2| = 3 + 2k$, or
6. there exist faces $f_1$ and $f_2$ with $|f_1|, |f_2| \leq 6$ such that $e$ is incident with $f_2$, $f_1$ shares at least $|f_1| - 3$ edges with $R_1$, and there exists a path $P$ consisting of at most four internal vertices of degree three which joins a vertex incident with $f_1$ with a vertex incident with $f_2$ and not incident with $e$. 

28
In each of the cases, we say that the edges of $E(f_1 \cup f'_1) \cap E(R_1)$ see the edge $e$. For each $e \in X \setminus E(R_1)$, let $n(e)$ denote the number of edges of $R_1$ that see it. Consider an edge $e' \in E(R_1)$ and let

$$N(e') = \sum_{e' \text{ sees } e} \frac{1}{n(e')}.$$  

Let $f_1$ be the internal face incident with $e'$. If $|f_1| > 11$, then $N(e') = 0$. If $9 \leq |f_1| \leq 11$, then $e'$ sees at most 9 edges by the first claim, which are also seen by another edge of $R_1$, and thus $N(e') \leq 9/2$. Suppose that $|f_1| \leq 8$. If $f_1$ shares $k \in \{2, 3, 4\}$ edges with $R_1$, then it sees at most $(|f_1| - k)(9 + k - |f_1|)$ edges satisfying the first three claims, and each of those is seen by at least $k$ other edges; hence, these edges contribute at most $(|f_1| - k)(9 + k - |f_1|)/k \leq 10$ to $N(e')$. Therefore, if $|f_1| \geq 7$, then $N(e') \leq 10$.

Suppose that $|f_1| \leq 6$ and consider the edges seen by $e'$ and satisfying the last claim. There are at most two choices for the first vertex of $P$, the second vertex is uniquely determined and there are two choices for each of the third and fourth vertex, giving at most $2 + 2 + 4 + 8 = 16$ possibilities for the last vertex of $P$ and at most 64 edges seen by $e'$. Since $|f_1|$ shares at least 2 edges with $R_1$, this contributes at most 32 to $N(e')$.

Finally, if $|f_1| = 5$, then the edges satisfying the fourth and the fifth claim contribute at most $32/3 + 4$ to $N(e')$. Consequently, we have $N(e') \leq 170/3$. The sum of $N(e')$ over all edges of $R_1$ gives an upper bound on $|X \setminus E(R_1)|$, hence $|X| \leq 346$. \hfill \qed

Similarly, we can prove the following.

**Lemma 5.4.** Let $G$ be a graph embedded in the cylinder with rings $R_1$ and $R_2$, such that every $(\leq 4)$-cycle is non-contractible. Suppose that $|R_1| = 7$ and $R_1$ is an induced cycle, and let $X \subseteq E(G)$ be the set of edges incident with non-contractible 7-cycles that share at least four vertices with $R_1$. If $G$ is $(R_1, R_2)$-critical, then $|X| \leq 35$.

The first paper of this series [5] together with the results of Gimbel and Thomassen [9] and Aksenov et al. [1] implies the following.

**Theorem 5.5.** Let $G$ be a graph embedded in the cylinder with one ring $R$ of length at most 7 (the other cuff does not correspond to a ring). Suppose that all $(\leq 4)$-cycles in $G$ are non-contractible and that $G$ has girth at least $|R| - 3$. If $G$ is $R$-critical and $R$ is an induced cycle, then $|R| = 6$ and $G$ contains a triangle $C$ such that all vertices of $C$ are internal and have mutually distinct and non-adjacent neighbors in $R$.

Finally, we can prove the main result of this section.

**Lemma 5.6.** Let $G$ be a graph embedded in the cylinder with rings $R_1$ and $R_2$, where $|R_1| \leq |R_2|$ and $5 \leq |R_2| \leq 7$. Suppose that every $(\leq 4)$-cycle in $G$ is non-contractible. Furthermore, assume that the following conditions hold:

- if $|R_1| = 4$, then all other 4-cycles in $G$ are vertex-disjoint with $R_1$,
- if $|R_1| \geq 5$, then $G$ contains no $(\leq 4)$-cycle,
• if \(|R_2| = 7\), then \(G\) contains no triangle distinct from \(R_1\), and \(R_2\) is an induced cycle, and

• if \(|R_2| = 6\), then \(R_2\) is an induced cycle and \(G\) contains no triangle \(T\) such that all vertices of \(T\) are internal and they have mutually distinct and non-adjacent neighbors in \(R_2\).

If \(G\) is \(|\{R_1, R_2\}|\)-critical, then \(w(G, \{R_1, R_2\}) \leq \text{cyl}(|R_1|, |R_2|)\).

Proof. By induction, we can assume that the claim holds for all graphs with fewer than \(|E(G)|\) edges. By Theorems 4.5 and 5.5, \(G\) is connected. Note that \(G\) satisfies (I0), (I1), (I2), (I6), (I8) and (I9). The cases that \(G\) is not 2-connected or that the distance between \(R_1\) and \(R_2\) is at most four are dealt with in the same way as in the proof of Theorem 4.9, hence assume that (I3) and (I7) hold.

If \(P\) is a path of length at most four with both ends being ring vertices, then both ends belong to the same ring \(R_i\) for some \(i \in \{1, 2\}\). Since \(G\) is embedded in the cylinder, there exists a subpath \(Q\) of \(R_i\) such that \(P \cup Q\) is a contractible cycle. Let us consider the case that \(|Q| > |P|\), and let \(Q'\) be the path with edge set \(E(R_i) \setminus E(Q)\). Note that \(Q' \cup P\) is a non-contractible cycle shorter than \(|R_i|\).

We apply induction to the subgraph of \(G\) between \(R_{3-i}\) and \(Q' \cup P\). Furthermore, we use Theorem 3.4 to bound the weight of the subgraph embedded in the disk bounded by \(Q \cup P\). We conclude that \(w(G, \{R_1, R_2\}) \leq \text{cyl}(|R_{3-i}|, |Q' \cup P|) + s(|Q \cup P|) - s(|Q' \cup P| + 2|P|) \leq \text{cyl}(|R_{3-i}|, |R_i|)\), since \(2|P| \leq 8\).

Therefore, we can assume that \(|Q| \leq |P|\) for each such path \(P\). This implies that (I4) holds. Furthermore, \(|P \cup Q| \leq 8\), and by Theorem 4.1, at most two faces of \(G\) are in the disk bounded by \(P \cup Q\).

Suppose that (I5) is false, and the ring \(R_i\) for some \(i \in \{1, 2\}\) contains adjacent vertices \(r_1\) and \(r_2\) of degree two. By (I3), we have \(|R_i| \geq 4\), and by the previous paragraph, the internal face incident with \(r_1, r_2\) has length at least 6. We apply induction to the graph obtained by contracting the edge \(r_1, r_2\), and obtain \(w(G, \{R_1, R_2\}) \leq \text{cyl}(|R_{3-i}|, |R_i| - 1) + 1 \leq \text{cyl}(|R_1|, |R_2|)\). Hence, assume that (I5) holds. Together with the observations from the previous paragraph, this implies that \(G\) is well-behaved.

If \(|R_1| = |R_2| = 7\) and \(G\) contains a non-contractible \((\leq 6)\)-cycle, then by induction we have \(w(G, \{R_1, R_2\}) \leq 2\text{cyl}(6, 7) \leq \text{cyl}(7, 7)\), hence we can assume that if \(|R_1| = |R_2| = 7\), then all non-contractible cycles have length at least seven.

Let \(k = 6\) if \(|R_2| = 7\) and \(k = 4\) otherwise. Let \(M\) be the subgraph of \(G\) containing

• edges incident with non-contractible \((\leq k)\)-cycles,

• if \(|R_1| = 4\), then also include edges incident with non-contractible \((\leq 7)\)-cycles sharing a vertex with \(R_1\),

• if \(|R_i| = 6\) for some \(i \in \{1, 2\}\), then include all edges of non-contractible \((\leq 6)\)-cycles bound to \(R_i\), and

• if \(|R_i| = 7\) for some \(i \in \{1, 2\}\), then include all edges of non-contractible 7-cycles that share at least four vertices with \(R_i\).
Let us bound the number of edges of $M$. Note that if $|R_2| < 7$, then $k = 4$ and by the assumptions, if $G$ contains a $(\leq k)$-cycle, then $|R_1| \leq 4$. Suppose that there exists a non-contractible $(\leq k)$-cycle $C$, and choose $C$ so that the closed subset $\Sigma'$ of $\Sigma$ between $R_1$ and $C$ is as large as possible. Note that $|R_1| \leq k < |R_2|$. If $|R_1| = k = 4$, then no 4-cycle distinct from $R_1$ contains a vertex of $V(R_1)$, hence the subgraph $G'$ of $G$ embedded in $\Sigma'$ is not a broken chain. All non-contractible $(\leq k)$-cycles in $G$ are either drawn in $\Sigma'$ or are incomparable with $C$. By Theorem 3.4 (when $R_1$ and $C$ intersect), Theorem 4.9 (when $R_1$ and $C$ are vertex-disjoint and $|C| \leq 4$) or by induction (when $R_1$ and $C$ are vertex-disjoint and $5 \leq |C| \leq 6$), we conclude that the total weight of the internal faces of $G'$ is at most $\max(\cyl(k, k), s(2k)) = \cyl(k, k)$. In particular, at most $5\cyl(k, k)/s(5)$ edges of $G$ are drawn in $\Sigma'$, and by Lemma 5.1, at most $10k + 5\cyl(k, k)/s(5)$ edges of $G$ are incident with non-contractible $(\leq k)$-cycles. By Lemmas 5.2, 5.3 and 5.4, we have $|E(M)| \leq 10k + 5\cyl(k, k)/s(5) + 692$.

Note that $M$ captures $(\leq 4)$-cycles of $G$, and that $(2/3 + 26k)(|R_1| + |R_2|) + 20(10k + 5\cyl(k, k)/s(5) + 692)/3 \leq \cyl(|R_1|, |R_2|)$; therefore, we can assume that we can apply Theorem 3.3. Let $G'$ be the $(R_1, R_2)$-critical graph embedded in the cylinder with rings $R_1$ and $R_2$ such that $|E(G')| < |E(G)|$, satisfying the conditions of Theorem 3.3. In particular, (b) implies that all $(\leq 4)$-cycles in $G'$ are non-contractible; and furthermore, using the choice of $M$ we have

- if $|R_2| = 7$, then $G'$ contains no triangle distinct from $R_1$,
- if $|R_1| = 4$, then no 4-cycle in $G'$ distinct from $R_1$ contains a vertex of $V(R_1)$,
- if $6 \leq |R_i| \leq 7$ for some $i \in \{1, 2\}$, then $R_i$ is an induced cycle in $G'$, and
- if $|R_i| = 6$ for some $i \in \{1, 2\}$, then $G'$ contains no triangle $T$ such that all vertices of $R_i$ are internal and have non-adjacent neighbors in $R_i$.

These constraints enable us to apply Theorem 5.5 to show that $G'$ is connected. It follows that all its faces are open 2-cell. Furthermore, the assumptions on non-contractible cycles from the statement of Lemma 5.6 are satisfied for $G'$, except that $G'$ can contain non-contractible $(\leq 4)$-cycles even if $|R_1| \geq 5$.

Let $X \subset F(G)$ and $\{(J_f, S_f) : f \in F(G')\}$ be the cover of $G$ by faces of $G'$ as in Theorem 3.3(d). For $f \in F(G')$, let $G_1^f, \ldots, G_{k_j}^f$ be the components of the $G$-expansion of $S_f$, where for $1 \leq i \leq k_j$, $G_i^f$ is embedded in the disk with one ring $R_i^f$. We have

$$w(G, (R_1, R_2)) = \sum_{f \in F(G)} w(f) = \sum_{f \in X} w(f) + \sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \{R_i^f\}).$$

The case that $G'$ is not 2-connected is dealt with in the same way as in the proof of Theorem 4.9; hence, assume that $G$ is 2-connected. If $G'$ does not satisfy the assumptions of Lemma 5.6, then $|R_1| \geq 5$ and $G'$ contains a $(\leq 4)$-cycle. Let $C_1$ and $C_2$ be the $(\leq 4)$-cycles in $G'$ such that the closed subset $\Sigma' \subseteq \Sigma$ between $C_1$ and $C_2$ is as large as possible, and observe that all $(\leq 4)$-cycles in $G'$ belong to the subgraph $G_c$ of $G'$ embedded in $\Sigma'$. By Theorem 3.3(a), if $G_c$ is a broken chain, then it has at most four internal faces. Therefore, Theorem 4.9 implies that the total weight of the internal faces of $G_c$ is at most $\cyl(4, 4)$. Applying
induction to the subgraphs of $G'$ between $R_1$ and $C_1$ and between $R_2$ and $C_2$, we have $w(G', \{R_1, R_2\}) \leq \text{cyl}(4, |R_1|) + \text{cyl}(4, |R_2|) + \text{cyl}(4, 4) \leq \text{cyl}(|R_1|, |R_2|)$.

If $G'$ satisfies the assumptions of Lemma 5.6, then the same inequality $w(G', \{R_1, R_2\}) \leq \text{cyl}(|R_1|, |R_2|)$ follows by induction. Since each face of $G'$ is closed 2-cell, we conclude that $w(G, \{R_1, R_2\}) \leq w(G', \{R_1, R_2\})$ as in the proof of Theorem 4.9.

\end{proof}

6 Graphs on surfaces

Let $\text{gen}(g, t, t_0, t_1)$ be a function defined for non-negative integers $g, t, t_0$ and $t_1$ such that $t \geq t_0 + t_1$ as

$$\text{gen}(g, t, t_0, t_1) = 120g + 48t - 4t_1 - 5t_0 - 120.$$ 

Let $\text{surf}(g, t, t_0, t_1)$ be a function defined for non-negative integers $g, t, t_0$ and $t_1$ such that $t \geq t_0 + t_1$ as

- $\text{surf}(g, t, t_0, t_1) = \text{gen}(g, t, t_0, t_1) + 116 - 42t = 8 - 4t_1 - 5t_0$ if $g = 0$ and $t = t_0 + t_1 = 2$,

- $\text{surf}(g, t, t_0, t_1) = \text{gen}(g, t, t_0, t_1) + 114 - 42t = 6t - 4t_1 - 5t_0 - 6$ if $g = 0$, $t \leq 2$ and $t_0 + t_1 < 2$, and

- $\text{surf}(g, t, t_0, t_1) = \text{gen}(g, t, t_0, t_1)$ otherwise.

We will need the following properties of the function $\text{surf}$:

**Lemma 6.1.** If $g, g', t, t_0, t_1, t'_0, t'_1$ are non-negative integers, then the following holds:

(a) Assume that if $g = 0$ and $t \leq 2$, then $t_0 + t_1 < t$. If $t \geq 2$, $t'_0 \leq t_0$, $t'_1 \leq t_1$ and $t'_0 + t'_1 \geq t_0 + t_1 - 2$, then $\text{surf}(g, t - 1, t'_0, t'_1) < \text{surf}(g, t, t_0, t_1)$.

(b) If $g' < g$ and either $g' > 0$ or $t \geq 2$, then $\text{surf}(g, t, t_0, t_1) \leq \text{surf}(g, t, t_0, t_1) - 120(g - g') + 32$.

(c) Let $g''$, $t'$, $t'_0$, $t'_1$ and $t''_1$ be nonnegative integers satisfying $g = g' + g''$, $t = t' + t''$, $t_0 = t'_0 + t''_0$, $t_1 = t'_1 + t''_1$, either $g'' > 0$ or $t'' \geq 1$, and either $g' > 0$ or $t' \geq 2$. Then, $\text{surf}(g', t', t'_0, t'_1) + \text{surf}(g'', t'', t''_0, t''_1) \leq \text{surf}(g, t, t_0, t_1) - \delta$, where $\delta = 16$ if $g'' = 0$ and $t'' = 1$, and $\delta = 56$ otherwise.

(d) If $g \geq 2$, then $\text{surf}(g - 2, t, t_0, t_1) \leq \text{surf}(g, t, t_0, t_1) - 124$

**Proof.** Let us consider the claims separately.

(a) If $g = 0$ and $t = 2$, then $\text{surf}(g, t, t_0, t_1) \geq 1$, while $\text{surf}(g, t - 1, t'_0, t'_1) = 0$. If $g = 0$ and $t = 3$, then $\text{surf}(g, t, t_0, t_1) \geq 9$ and $\text{surf}(g, t - 1, t'_0, t'_1) \leq 6$.

Finally, if $g > 0$ or $t > 3$, then $\text{surf}(g, t, t_0, t_1) = \text{gen}(g, t, t_0, t_1)$ and $\text{surf}(g, t - 1, t'_0, t'_1) = \text{gen}(g, t - 1, t'_0, t'_1) = \text{gen}(g, t - 1, t'_0, t'_1) = \text{gen}(g, t - 1, t'_0, t'_1) = 48 - 5(t_0 - t''_0) - 4(t_0 - t''_0) \geq 48 - 5(t_0 + t_1 - t'_0 - t'_1) \geq 38$.

(b) If $g' > 0$ or $t > 2$, then $\text{surf}(g', t, t_0, t_1) = \text{gen}(g', t, t_0, t_1)$ and we have $\text{surf}(g', t, t_0, t_1) = \text{surf}(g, t, t_0, t_1) - 120(g - g')$. If $g' = 0$ and $t = 2$, then $\text{surf}(g', t, t_0, t_1) - \text{surf}(g, t, t_0, t_1) + 120(g - g') \leq 116 - 42t = 32$. 

32
(c) Suppose first that \( g'' = 0 \) and \( t'' = 1 \), i.e., we have \( g = g' \) and \( t = t' + 1 \).
If \( g > 0 \), then \( \text{surf}(g, t, t_0, t_1) - \text{surf}(g', t', t'_0, t'_1) = \text{gen}(g, t, t_0, t_1) + (4t'_1 + 5t'_0) - \text{gen}(g', t', t'_0, t'_1) = 48 \). If \( g = 0 \), then \( t' \geq 2 \) and we have \( \text{surf}(g', t', t'_0, t'_1) = \text{gen}(g', t', t'_0, t'_1) + 32 \). Hence, \( \text{surf}(g, t, t_0, t_1) - \text{surf}(g', t', t'_0, t'_1) = \text{gen}(g, t, t_0, t_1) + (4t'_1 + 5t'_0) - \text{gen}(g', t', t'_0, t'_1) + 32 \). It follows that \( \text{surf}(g, t, t_0, t_1) - \text{surf}(g', t', t'_0, t'_1) = \text{gen}(g, t, t_0, t_1) - \text{gen}(g', t', t'_0, t'_1) - \text{gen}(g', t'', t'_0, t'_1) - 64 = 120 - 64 = 56 \).

(d) We have \( \text{surf}(g, t, t_0, t_1) - \text{surf}(g - 2, t, t_0, t_1) \leq \text{gen}(g, t, t_0, t_1) - (\text{gen}(g - 2, t, t_0, t_1) + 116) = 124 \).

\( \square \)

Consider a graph \( H \) embedded in a surface \( \Pi \) with rings \( Q \) and let \( f \) be an internal face of \( H \). Let us recall that \( \Pi_f \) is the surface corresponding to the face in the natural way, as defined in Section 3. Let \( a_0 \) and \( a_1 \) be the number of weak and non-weak rings, respectively, that form one of the facial walks of \( f \) by themselves. Let \( a \) be the number of facial walks of \( f \). We define \( \text{surf}(f) = \text{surf}(g(\Pi_f), a, a_0, a_1) \).

Let \( G_1 \) be a graph embedded in \( \Sigma_1 \) with rings \( R_1 \) and \( G_2 \) a graph embedded in \( \Sigma_2 \) with rings \( R_2 \). Let \( m(G_i) \) denote the number of edges of \( G_i \) that are not contained in the boundary of \( \Sigma_i \). Let us write \((G_1, \Sigma_1, R_1) \prec (G_2, \Sigma_2, R_2)\) to denote that the quadruple \((g(\Sigma_1), |R_1|, m(G_1), |E(G_1)|)\) is lexicographically smaller than \((g(\Sigma_2), |R_2|, m(G_2), |E(G_2)|)\).

A graph \( G \) embedded in a surface \( \Sigma \) with rings \( R \) has internal girth at least five if every \((\leq 4)\)-cycle in \( G \) is equal to one of the rings. Let \( t_0(R) \) and \( t_1(R) \) be the number of weak and non-weak vertex rings in \( R \), respectively. In order to prove Theorem 1.3, we show the following more general claim.

**Theorem 6.2.** There exists a constant \( \eta \) with the following property. Let \( G \) be a graph embedded in a surface \( \Sigma \) with rings \( R \). If \( G \) is \( R \)-critical and has internal girth at least five, then \( w(G, R, \Sigma) \leq (\ell(R) + \eta) \text{surf}(g(\Sigma), |R|, t_0(R), t_1(R)) \).

**Proof.** Let \( \eta = 1867 + 67 \text{cyl}(7, 7)/s(5) \). We proceed by induction and assume that the claim holds for all graphs \( G' \) embedded in surfaces \( \Sigma' \) with rings \( R' \) such that \((G', \Sigma', R') \prec (G, \Sigma, R)\). Let \( g = g(\Sigma), t_0 = t_0(R) \) and \( t_1 = t_1(R) \).

By Theorem 3.4, the claim holds if \( g = 0 \) and \( |R| = 1 \), hence assume that \( g > 0 \) or \( |R| > 1 \). Similarly, if \( g = 0 \) and \( |R| = 2 \), then we can assume that \( t_0 + t_1 \leq 1 \) by Lemma 4.6. By Lemma 3.1, Lemma 3.2 and Theorem 4.1, \( G \) satisfies (I0), (I1), (I2), (I6) and (I9).

Suppose now that there exists a path \( P \) of length at most four with ends in distinct rings \( R_1, R_2 \in R \). By choosing the shortest such path, we can assume that \( P \) intersects no other rings. If \( R_1 \) or \( R_2 \) is a vertex ring, first replace it by a facial ring of length three by adding new vertices and edges in the incident cuff. Let \( J = P \cup \bigcup_{R \in R} R \) and let \( S = \{j\} \), where \( j \) is the face of \( J \) incident with edges of \( P \). Let \( \{G'\} \) be the \( G \)-expansion of \( S \), let \( \Sigma' \) be the surface in that \( G' \) is embedded and let \( \mathcal{R}' \) be the natural rings of \( G' \).
Note that \( g(\Sigma') = g, |R'| = |R| - 1, \ell(R') \leq \ell(R) + 14 \) and \( t_0(R') + t_1(R') \geq t_0 + t_1 - 2. \) Since \( (G', \Sigma', R') \prec (G, \Sigma, R), \) by induction and by Lemma 6.1(a) we have \( w(G, R) = w(G', R') \leq gsurf(g, |R| - 1, t_0(R'), t_1(R')) + \ell(R) + 14 < gsurf(g, |R|, t_0, t_1) + \ell(R). \) Therefore, we can assume that no such path exists, and in particular, (I7) holds.

Next, we aim to prove property (I3). For later use, we will consider a more general setting.

Let \( H \) be a graph embedded in \( \Pi \) with rings \( \mathcal{Q} \) such that at least one internal face of \( H \) is not open 2-cell and no face of \( H \) is omnipresent. If \( H \) is \( \mathcal{Q} \)-critical, has internal girth at least five and \((H, \Pi, \mathcal{Q}) \preceq (G, \Sigma, R),\) then

\[
w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + \eta \left( surf(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 7 - \sum_{h \in F(H)} surf(h) \right).
\]

(10)

**Proof.** We prove the claim by induction. Consider for a moment a graph \( H' \) of internal girth at least 5 embedded in a surface \( \Pi' \) with rings \( \mathcal{Q}' \) with \((H', \Pi', \mathcal{Q}') \prec (H, \Pi, \mathcal{Q}),\) such that either \( H' = \mathcal{Q}' \) or \( H' \) is \( \mathcal{Q}' \)-critical. We claim that

\[
w(H', \mathcal{Q}') \leq \ell(\mathcal{Q}') + \eta \left( surf(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}')) - \sum_{h \in F(H')} surf(h) \right).
\]

(11)

The claim obviously holds if \( H' = \mathcal{Q}' \), since \( w(H', \mathcal{Q}') \leq \ell(\mathcal{Q}') \) in that case; hence, it suffices to consider the case that \( H' \) is \( \mathcal{Q}' \)-critical. If at least one internal face of \( H' \) is not open 2-cell and no face of \( H' \) is omnipresent, then this follows by an inductive application of (10) (we could even strengthen the inequality by 7\( \mathcal{C} \)). If all internal faces of \( H' \) are open 2-cell, then note that \( \text{surf}(h) = 0 \) for every \( h \in F(H') \), and since \((H', \Pi', \mathcal{Q}') \prec (G, \Sigma, R),\) we can apply Theorem 6.2 inductively to obtain (11). Finally, suppose that \( H' \) has an omnipresent face \( f \), let \( \mathcal{Q}' = \{Q_1, \ldots, Q_t\} \) and for \( 1 \leq i \leq t \), let \( f_i \) be the boundary walk of \( f \) such that \( f_i \) and \( Q_i \) are contained in a closed disk \( \Delta_i \subset \Pi' + Q_i \). Since all components of \( H' \) are planar and contain only one ring, Lemma 3.2 implies that all internal faces of \( H' \) distinct from \( f \) are closed 2-cell. Furthermore, each vertex ring forms component of the boundary of \( f \) by itself, hence \( \text{surf}(f) = \text{surf}(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}')) \). If \( Q_i \) is a facial ring, then by applying Theorem 3.4 to the subgraph \( H'_i \) of \( H' \) embedded in \( \Delta_i \), we conclude that the weight of \( H'_i \) is at most \( s(|Q_i|) \) and that \( |f_i| \leq |Q_i| \). Note that \( s(|Q_i|) - s(|f_i|) \leq |Q_i| - |f_i| \). Therefore, we again obtain (11):

\[
w(H', \mathcal{Q}') \leq |f| + \sum_{i=1}^{t} s(|Q_i|) - s(|f_i|) \leq |f| + \sum_{i=1}^{t} |Q_i| - |f_i|
\]

\[
= \ell(\mathcal{Q}')
\]

\[
= \ell(\mathcal{Q}') + \eta \left( surf(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}')) - \sum_{h \in F(H')} surf(h) \right).
\]

Let us now return to the graph \( H \). Since \( H \) is \( \mathcal{Q} \)-critical, Theorem 1.1 implies that no component of \( H \) is a planar graph without rings. Let \( f \) be a face of \( H \)
which is not open 2-cell. Since $H$ has such a face and $f$ is not omnipresent, we have $g(\Pi) > 0$ or $|Q| > 2$. Let $c$ be a simple closed curve in $f$ infinitesimally close to a facial walk $W$ of $f$. Cut $\Pi$ along $c$ and cap the resulting holes by disks ($c$ is always a 2-sided curve). Let $\Pi_1$ be the surface obtained this way that contains $W$, and if $c$ is separating, then let $\Pi_2$ be the other surface. Since $f$ is not omnipresent, we can choose $W$ so that either $g(\Pi_1) > 0$ or $\Pi_1$ contains at least two rings of $Q$. Let us discuss several cases:

- **The curve $c$ is separating and $H$ is contained in $\Pi_1$.** In this case $f$ has only one facial walk, and since $f$ is not open 2-cell, $\Pi_2$ is not the sphere. It follows that $g(\Pi_1) = g(\Pi) - g(\Pi_2) < g(\Pi)$, and thus $(H, \Pi_1, Q) \not\prec (H, \Pi, Q)$. Note that the weights of the faces of the embedding of $H$ in $\Pi$ and in $\Pi_1$ are the same, with the exception of $f$ whose weight in $\Pi$ is $|f|$ and weight in $\Pi_1$ is $w((f)) \geq |f| - 2$. By (11), we have
  \[
  w(H, Q) \leq \ell(Q) + 8 + \eta \left( \text{surf}(g(\Pi_1), |Q|, t_0(Q), t_1(Q)) + \sum_{h \in F(H)} \text{surf}(h) \right).
  \]
  Note that $\text{surf}(f) = 120 g(\Pi_2) - 72$. By Lemma 6.1(b), we conclude that
  \[
  w(H, Q) \leq \ell(Q) + 8 + \eta \left( \text{surf}(g(\Pi), |Q|, t_0(Q), t_1(Q)) - 40 + \sum_{h \in F(H)} \text{surf}(h) \right).
  \]

- **The curve $c$ is separating and $\Pi_2$ contains a nonempty part $H_2$ of $H$.** Let $H_1$ be the part of $H$ contained in $\Pi_1$. Let $Q_i$ be the subset of $Q$ belonging to $\Pi_i$ and $f_i$ the face of $H_i$ corresponding to $f$, for $i \in \{1, 2\}$. Note that $f_1$ is an open disk, hence $\text{surf}(f_1) = 0$. Using (11), we get
  \[
  w(H, Q) \leq w(f) - w(f_1) - w(f_2) + \ell(Q_1) + \ell(Q_2) + \eta \sum_{i=1}^2 \text{surf}(g(\Pi_i), |Q_i|, t_0(Q_i), t_1(Q_i)) + \eta \left( \text{surf}(f) - \text{surf}(f_2) - \sum_{h \in F(H)} \text{surf}(h) \right).
  \]
  Note that $w(f) - w(f_1) - w(f_2) \leq 16$ and $\ell(Q_1) + \ell(Q_2) = \ell(Q)$. Also, $\text{surf}(f) - \text{surf}(f_2) \leq 48$, and when $g(\Pi_f) = 0$ and $f$ has only two facial walks, then $\text{surf}(f) - \text{surf}(f_2) \leq 6$.
  By Lemma 6.1(c), we have
  \[
  \sum_{i=1}^2 \text{surf}(g(\Pi_i), |Q_i|, t_0(Q_i), t_1(Q_i)) \leq \text{surf}(g(\Pi), |Q|, t_0(Q), t_1(Q)) - \delta,
  \]
  where $\delta = 16$ if $g(\Pi_2) = 0$ and $|Q_2| = 1$ and $\delta = 56$ otherwise. Note that if $g(\Pi_2) = 0$ and $|Q_2| = 1$, then $g(\Pi_f) = 0$ and $f$ has only two facial walks. We conclude that $\text{surf}(f) - \text{surf}(f_2) - \delta \leq -8$. Therefore, $w(H, Q) \leq \ell(Q) + 16 + \eta \left( \text{surf}(g(\Pi), |Q|, t_0(Q), t_1(Q)) - 8 - \sum_{h \in F(H)} \text{surf}(h) \right).

- **The curve $c$ is not separating.** Let $f_1$ be the face of $H$ (in the embedding in $\Pi_1$) bounded by $W$ and $f_2$ the other face corresponding to $f$. Again,
Furthermore, $w$ face of $\alpha$ $w$ suffices to prove the first inequality of the claim.

Since $c$ is two-sided, $g(\Pi_1) = g(\Pi) - 2$, and $\operatorname{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) = \operatorname{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 124$

by Lemma 6.1(d). Since $\ell$ satisfies $w(f) - \operatorname{surf}(f_2) \leq 148$ and $w(f) - w(f_1) - w(f_2) \leq 16$, we have $w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + 16 + \eta \left( \operatorname{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 76 + \sum_{h \in F(H)} \operatorname{surf}(h) \right)$.

The results of all the subcases imply (10).

Let $H$ be a graph embedded in $\Sigma$ with rings $\mathcal{R}$ and let $f$ be an omnipresent face of $H$. If $H$ is $\mathcal{R}$-critical, has internal girth at least five and no component of $H$ satisfies (E1), (E2) or (E3), then

$$w(H, \mathcal{R}) \leq \ell(\mathcal{R}) - \kappa = \ell(\mathcal{R}) - \kappa + \eta \left( \operatorname{surf}(g, |\mathcal{R}|, t_0, t_1) - \sum_{h \in F(H)} \operatorname{surf}(h) \right),$$

where $\kappa = 5 - 5s(5)$ if $H$ has exactly one component not equal to a ring and this component is exceptional, $\kappa = 5 + 5s(5)$ if $H$ has exactly one component not equal to a ring and this component is not exceptional, and $\kappa = 6$ otherwise. (12)

Proof. Since $H$ is $\mathcal{R}$-critical and $f$ is an omnipresent face, each component of $H$ is planar and contains exactly one ring. In particular, all internal faces of $H$ distinct from $f$ are closed 2-cell. For $R \in \mathcal{R}$, let $H_R$ be the component of $H$ containing $R$. Exactly one boundary walk $W$ of $f$ belongs to $H_R$. Cutting along $W$ and capping the hole by a disk, we obtain an embedding of $H_R$ in a disk with one ring $R$. Let $f_R$ be the face of this embedding bounded by $W$. Note that either $H_R = R$ or $H_R$ is $\{R\}$-critical. If $R$ is a vertex ring, then we have $H_R = R$; hence, every vertex ring in $\mathcal{R}$ forms a facial walk of $f$, and $\operatorname{surf}(f) = \operatorname{surf}(g, |\mathcal{R}|, t_0, t_1)$. Consequently, $\operatorname{surf}(g, |\mathcal{R}|, t_0, t_1) = \sum_{h \in F(H)} \operatorname{surf}(h)$, and it suffices to prove the first inequality of the claim.

Suppose that $H_R \neq R$ for a ring $R \in \mathcal{R}$. Since $H_R$ does not satisfy (E1), (E2) or (E3), Theorem 3.4 implies that $w(H_R, \{R\}) \leq s(|R| - 5) + \alpha$, where $\alpha = 5s(5)$ if $H_R$ satisfies (E4) or (E5) and $\alpha = -5s(5)$ otherwise. Since $f_R$ is a face of $H_R$ and $s(y) - s(x) > 5s(5)$ for every $y > x \geq 5$, we have $|f_R| \leq |R| - 5$. Furthermore, $w(H_R, \{R\}) - w(f_R) \leq s(|R| - 5) + \alpha - s(|f_R|) \leq |R| - |f_R| - 5 + \alpha$. 36
Summing over all the rings, we obtain

\[ w(H, \mathcal{R}) = w(f) + \sum_{R \in \mathcal{R}} (w(H_R, \{R\}) - w(f_R)) \]

\[ \leq |f| + \sum_{R \in \mathcal{R}} (|\mathcal{R}| - |f_R|) - \kappa \]

\[ = \ell(\mathcal{R}) - \kappa. \]

Let \( H \) be an \( \mathcal{R} \)-critical graph embedded in \( \Sigma \) with rings \( \mathcal{R} \) so that all internal faces of \( H \) are open 2-cell. If \( H \) is \( \mathcal{R} \)-critical, has internal girth at least five, \( |E(H)| \leq |E(G)| \) and an internal face \( f \) of \( H \) is not closed 2-cell, then

\[ w(H, \mathcal{R}) \leq \ell(\mathcal{R}) + \eta \left( \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 1/2 \right). \]

(13)

Proof. Since \( f \) is not closed 2-cell, there exists a vertex \( v \) appearing at least twice in the facial walk of \( f \). There exists a simple closed curve \( c \) going through the interior of \( f \) and joining two of the appearances of \( v \). Cut the surface along \( c \) and patch the resulting hole(s) by disk(s). Let \( v_1 \) and \( v_2 \) be the two vertices to that \( v \) is split. For each of \( v_1 \) and \( v_2 \), if it is not incident with a cuff, drill a new hole next to it in the incident patch.

If \( c \) is separating, then let \( H_1 \) and \( H_2 \) be the resulting graphs embedded in the two surfaces \( \Sigma_1 \) and \( \Sigma_2 \) obtained by this construction; if \( c \) is not separating, then let \( H_1 \) be the resulting graph embedded in a surface \( \Sigma_1 \). We choose the labels so that \( v_1 \in V(H_1) \). If \( c \) is two-sided, then let \( f_1 \) and \( f_2 \) be the faces to that \( f \) is split by \( c \), where \( f_1 \) is a face of \( H_1 \). If \( c \) is one-sided, then let \( f_1 \) be the face in \( \Sigma_1 \) corresponding to \( f \). Note that \( |f_1| + |f_2| = |f| \) in the former case, and thus \( w(f) - w(f_1) - w(f_2) \leq 16 \). In the latter case, we have \( w(f) = w(f_1) \).

If \( c \) is separating, then for \( i \in \{1, 2\} \), let \( \mathcal{R}_i \) consist the rings of \( \mathcal{R} \setminus \{v\} \) contained in \( \Sigma_i \), and if none of these rings contains \( v \), then also of the vertex ring \( v_i \). Here, \( v_i \) is weak if \( v \) is an internal vertex, \( \Sigma_{3-i} \) is a cylinder and the ring of \( G_{3-i} \) distinct from \( v_{3-i} \) is a vertex ring. If \( c \) is not separating, then let \( \mathcal{R}_1 \) consist of the rings of \( \mathcal{R} \setminus \{v\} \) as well as those of \( v_1 \) and \( v_2 \) that are not incident with any of the rings in this set. In this case, we treat \( v_1 \) and \( v_2 \) as non-weak vertex rings.

Suppose first that \( c \) is not separating. Note that \( H_1 \) has at most two more rings (of length 1) than \( H \) and \( g(\Sigma_1) \in \{g - 1, g - 2\} \) (depending on whether \( c \) is one-sided or not), and that \( H_1 \) has at least two rings. If \( H_1 \) has only one more ring than \( H \), then

\[ \text{surf}(g(\Sigma_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) \leq \text{surf}(g - 1, |\mathcal{R}| + 1, t_0, t_1 + 1) \]

\[ \leq \text{gen}(g - 1, |\mathcal{R}| + 1, t_0, t_1 + 1) + 32 \]

\[ = \text{gen}(g, |\mathcal{R}|, t_0, t_1) - 44 \]

\[ = \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 44. \]

Let us now consider the case that \( H_1 \) has two more rings than \( H \) (i.e., that \( v \) is an internal vertex). If \( g(\Sigma_1) = 0 \) and \( |\mathcal{R}_1| = 2 \), then note that both
rings of $H_1$ are vertex rings. Lemma 4.6 implies that $H_1$ has only one edge; but the corresponding edge in $H$ would form a loop, which is a contradiction. Consequently, we have $g(\Sigma_1) > 1$ or $|R_1| \geq 3$, and

$$\text{surf}(g(\Sigma_1), |R_1|, t_0(R_1), t_1(R_1)) \leq \text{surf}(g - 1, |R| + 2, t_0, t_1 + 2) = \text{surf}(g, |R|, t_0, t_1) - 32.$$  

By induction, we can apply Theorem 6.2 inductively for $H$, concluding that $w(H, R) \leq \ell(R) + 18 + \eta \left( \text{surf}(g, |R|, t_0, t_1) - 32 \right)$, and the claim follows.

Next, we consider the case that $c$ is separating. Let us remark that $G_i$ is $R_i$-critical for $i \in \{1, 2\}$. This is obvious unless $v_i$ is a weak vertex ring. However, in that case Lemma 4.6 implies that $G_{3-i}$ is a single edge, and the $R_i$-criticality of $G_i$ is argued in the same way as in the proof of Theorem 4.9. Thus, we can apply Theorem 6.2 inductively for $G_1$ and $G_2$, and we have

$$w(H, R) = w(H_1, R_1) + w(H_2, R_2) + w(f) - w(f_1) - w(f_2) \leq \ell(R) + 18 + \eta \sum_{i=1}^{2} \text{surf}(g(\Sigma_i), |R_i|, t_0(R_i), t_1(R_i))$$

Therefore, it suffices to prove that

$$\sum_{i=1}^{2} \text{surf}(g(\Sigma_i), |R_i|, t_0(R_i), t_1(R_i)) \leq \text{surf}(g, |R|, t_0, t_1) - 1. \quad (14)$$

Note that if $g(\Sigma_i) = 0$, then $|R_i| \geq 2$ for $i \in \{1, 2\}$. If $|R_1| + |R_2| = |R| + 1$, we have

$$\sum_{i=1}^{2} \text{surf}(g(\Sigma_i), |R_i|, t_0(R_i), t_1(R_i)) \leq \sum_{i=1}^{2} (\text{gen}(g(\Sigma_i), |R_i|, t_0(R_i), t_1(R_i)) + 32)$$

$$= \text{gen}(g, |R|, t_0, t_1) - 12$$

$$= \text{surf}(g, |R|, t_0, t_1) - 12.$$  

Therefore, we can assume that $|R_1| + |R_2| = |R| + 2$, i.e., $v$ is an internal vertex. Suppose that for both $i \in \{1, 2\}$, we have $g(\Sigma_i) > 0$ or $|R_i| > 2$. Then,

$$\sum_{i=1}^{2} \text{surf}(g(\Sigma_i), |R_i|, t_0(R_i), t_1(R_i)) = \sum_{i=1}^{2} \text{gen}(g(\Sigma_i), |R_i|, t_0(R_i), t_1(R_i))$$

$$= \text{surf}(g, |R|, t_0, t_1) - 32.$$  

and the claim follows.

Hence, we can assume that say $g(\Sigma_1) = 0$ and $|R_1| = 2$. Then, $R_1 = \{v_1, R_1\}$ for some ring $R_1$, $g(\Sigma_2) = g$ and $|R_2| = |R|$. Since $G_1$ is $R_1$-critical, Lemma 4.6 implies that $R_1$ is not a weak vertex ring. If $R_1$ is a vertex ring, then $v_2$ is a weak vertex ring of $R_2$ which replaces the non-weak vertex ring $R_1$. Therefore, $\text{surf}(g(R_2), |R_2|, t_0(R_2), t_1(R_2)) = \text{surf}(g, |R|, t_0, t_1) - 1$. Furthermore, $\text{surf}(g(R_1), |R_1|, t_0(R_1), t_1(R_1)) = \text{surf}(0, 2, 0, 2) = 0$, and the claim follows.

Finally, consider the case that $R_1$ is a facial ring. By symmetry, we can assume that if $g(\Sigma_2) = 0$ and $|R_2| = 2$, then $R_2$ contains a facial ring. Since $R_2$
is obtained from \( R \) by replacing a facial ring \( R_1 \) by a non-weak vertex ring \( v_2 \), we have \( \text{surf}(g(R_1), |R_1|, t_0(R_1), t_1(R_1)) = \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 4 \). Furthermore, \( \text{surf}(g(R_1), |R_1|, t_0(R_1), t_1(R_1)) = \text{surf}(0, 2, 0, 1) = 2 \). Consequently,

\[
\sum_{i=1}^{2} \text{surf}(g(\Sigma_i), |R_i|, t_0(R_i), t_1(R_i)) \leq \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 2.
\]

Therefore, inequality (14) holds.

By (10), (12) and (13), we can assume that \( G \) satisfies (13).

Suppose that \( G \) contains a path \( P \) of length at most six joining two distinct vertices \( u \) and \( v \) of a ring \( R \in \mathcal{R} \), such that \( V(P) \cap V(R) = \{u, v\} \) and \( R \cup P \) contains no contractible cycle. Since the distance between any two rings in \( G \) is at least five, all vertices of \( V(P) \setminus \{u, v\} \) are internal. Let \( J \) be the subgraph of \( G \) consisting of \( P \) and of the union of the rings, and let \( S \) be the set of internal faces of \( J \). Clearly, \( S \) and \( J \) satisfy (1). Let \( \{G_1, \ldots, G_k\} \) be the \( G \)-expansion of \( S \), and for \( 1 \leq i \leq k \), let \( \Sigma_i \) be the surface in that \( G_i \) is embedded and let \( R_i \) be the natural rings of \( G_i \). Note that \( \sum_{i=1}^{k} t_0(R_i) = t_0 \) and \( \sum_{i=1}^{k} t_1(R_i) = t_1 \). Let \( r = \left( \sum_{i=1}^{k} |\Sigma_i| \right) - |\mathcal{R}| \) and observe that either \( r = 0 \) and \( k = 1 \), or \( r = 1 \) and \( 1 \leq k \leq 2 \) (depending on whether the curve in \( \hat{\Sigma} \) corresponding to a cycle in \( R \cup P \) distinct from \( R \) is one-sided, two-sided and non-separating or two-sided and separating). Furthermore, \( \sum_{i=1}^{k} g(\Sigma_i) = g + 2k - r - 3 \).

We claim that \( (G_i, \Sigma_i, R_i) \prec (G, \Sigma, \mathcal{R}) \) for \( 1 \leq i \leq k \). This is clearly the case, unless \( g(\Sigma_i) = g \). Then, we have \( k = 2 \), \( r = 1 \) and \( g(\Sigma_{3-i}) = 0 \). Since \( R \cup P \) contains no contractible cycle, \( \Sigma_{3-i} \) is not a disk, hence \( |\Sigma_{3-i}| \geq 2 \) and \( |\mathcal{R}_{3-i}| < |\mathcal{R}| \), again implying \( (G_i, \Sigma_i, R_i) \prec (G, \Sigma, \mathcal{R}) \).

By induction, we have \( w(G_i, R_i) \leq \ell(R_i) + g(\Sigma_i), |R_i|, t_0(R_i), t_1(R_i)) \), for \( 1 \leq i \leq k \). Since every internal face of \( G \) is an internal face of \( G_i \) for some \( i \in \{1, \ldots, k\} \) and \( \sum_{i=1}^{k} \ell(R_i) \leq \ell(\mathcal{R}) + 12 \), we conclude that \( w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + 12 + g \sum_{i=1}^{k} \text{surf}(g(\Sigma_i), |R_i|, t_0(R_i), t_1(R_i)) \). Note that for \( 1 \leq i \leq k \), we have that \( \Sigma_i \) is not a disk and \( R_i \) contains at least one facial ring, and thus \( \text{surf}(g(\Sigma_i), |R_i|, t_0(R_i), t_1(R_i)) \leq \text{gen}(g(\Sigma_i), |R_i|, t_0(R_i), t_1(R_i)) + 30 \). Therefore,

\[
\sum_{i=1}^{k} \text{surf}(g(\Sigma_i), |R_i|, t_0(R_i), t_1(R_i))
= \sum_{i=1}^{k} (\text{gen}(g(\Sigma_i), |R_i|, t_0(R_i), t_1(R_i)) + 30)
\leq \text{surf}(g, |\mathcal{R}|, t_0, t_1) + 120(2k - r - 3) + 48r - 120(k - 1) + 60
= \text{surf}(g, |\mathcal{R}|, t_0, t_1) + 120k - 72r - 180
\leq \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 12.
\]

The inequality of Theorem 6.2 follows; therefore, we can assume that

if \( P \) is a path of length at most six joining two distinct vertices of a ring \( R \), then \( R \cup P \) contains a contractible cycle.

Let us note that since \( g > 0 \) or \( |\mathcal{R}| \geq 2 \), this contractible cycle is unique. (15)
Consider now a path $P$ of length at most four, such that its ends $u$ and $v$ are ring vertices and all other vertices of $P$ are internal. By (I7), both ends of $P$ belong to the same ring $R$; let $P$, $P_1$ and $P_2$ be the paths in $R \cup P$ joining $u$ and $v$. By the previous paragraph, we can assume that $P \cup P_2$ is a contractible cycle. Suppose that the disk bounded by $P \cup P_2$ neither is a face nor consists of two 5-faces. By Theorem 4.1, we have $|P \cup P_2| \geq 9$. Let $J$, $S$, $G_i$, $\Sigma_i$ and $\mathcal{R}_i$ (for $i \in \{1, 2\}$) be defined as in the previous paragraph, where $\Sigma_2$ is a disk and $\mathcal{R}_2$ consists of a single ring corresponding to $P \cup P_2$. Since $g(\Sigma_2) = g$, $|\mathcal{R}_2| = |\mathcal{R}|$ and $|E(G_i)| < |E(G)|$, by induction we have $w(G_i, \{\mathcal{R}_2\}) \leq \ell(\mathcal{R}_2) + |\mathcal{R}_2|$. Note that $\ell(\mathcal{R}_2) = \ell(\mathcal{R}) + |P| - |P_2|$. Furthermore, Theorem 3.4 implies $w(G_2, \{\mathcal{R}_2\}) \leq s(|P| + |P_2|) = |P| + |P_2| - 8$. Therefore, $w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + \eta g(\mathcal{R}, t_0, t_1) + 2|P| - 8$. Since $|P| \leq 4$, the claim of Theorem 6.2 follows. Therefore, we can assume that the disk bounded by $P \cup P_1$ is either a face or consists of two 5-faces. The same calculation also excludes the possibility that $|P| \leq 2$, since $s(|P| + |P_2|) \leq |P| + |P_2| - 4$ for any $P$ and $P_2$ such that $|P| + |P_2| \geq 5$. In particular, we can assume that (I4) holds for $G$.

Suppose that $G$ contains two adjacent vertices $r_1$ and $r_2$ of degree two. By (I7), both $r_1$ and $r_2$ are incident with the same facial ring $R$. By (I3), we have $|R| \geq 4$. By (I4), the internal face $f$ incident with $r_1r_2$ has length at least six. Let $G'$ be the graph obtained from $G$ by contracting $r_1r_2$, let $R'$ be the set of rings of $G'$ obtained from $R$ by contracting edge $r_1r_2$ in $R$, and let $f'$ be the face of $G'$ corresponding to $f$. Observe that $G'$ is $R'$-critical. Suppose that $G'$ contains a $(\leq 4)$-cycle $C'$ distinct from the rings. Then $G$ contains a $(\leq 5)$-cycle $C$ distinct from the rings containing $r_1r_2$. Since $G$ has internal girth at least five, we have $|C| = 5$, and we obtain a contradiction with (I4).

Therefore, $G'$ has internal girth at least five. By induction, we have $w(G', R') = \ell(R') + \eta g(R', t_0, t_1)$, and since $\ell(R) = \ell(R') + 1$, we have $w(f) \leq w(f') + 1$. $G$ satisfies the inequality of Theorem 6.2. Therefore, assume that $G$ satisfies (I5). Together with the previous paragraph, this implies that $G$ is well-behaved.

Suppose that $G$ contains a non-contractible cycle $C$ of length at most seven that does not surround any of the rings. By (I7), $C$ intersects at most one ring, and by (I5), $C$ shares at most one vertex with this ring. Let $s = 1$ if $C$ intersects a ring, and $s = 0$ otherwise. Let $J$ be the subgraph of $G$ consisting of $C$ and of the union of the rings, and let $S$ be the set of internal faces of $J$. Clearly, $S$ and $J$ satisfy (1). Let $\{G_1, \ldots, G_k\}$ be the $G$-expansion of $S$, and for $1 \leq i \leq k$, let $\Sigma_i$ be the surface in that $G_i$ is embedded and let $R_i$ be the natural rings of $G_i$.

Let $r = \left(\sum_{i=1}^k |\mathcal{R}_i|\right) - |R|$. Note that either $r + s = 1$ and $k = 1$, or $r + s = 2$ and $1 \leq k \leq 4$. Observe that $\sum_{i=1}^k g(\Sigma_i) = g - s - r + 2k - 2$. Furthermore, $\sum_{i=1}^k t_0(\mathcal{R}_i) + \sum_{i=1}^k t_1(\mathcal{R}_i) \geq t_0 + t_1 - s$, and $\sum_{i=1}^k \ell(\mathcal{R}_i) \leq \ell(\mathcal{R}) + 14$. If $g(\Sigma_1) = g$, then $k = 2$ and $g(\Sigma_2) = 0$; furthermore, $\Sigma_2$ has at least two cuffs, and if $s = 0$, then it has at least three cuffs, since $C$ does not surround a ring. Thus, if $g(\Sigma_1) = g$, then $r = 2 - s$ and consequently $|\mathcal{R}_1| = |R| + r - |\mathcal{R}_2| = |R| + 2 - s - |\mathcal{R}_2| < |R|$. The same argument can be applied to $\Sigma_2$ if $k = 2$, hence $(G_1, \Sigma_i, \mathcal{R}_i) \prec (G, \Sigma, \mathcal{R})$ for $1 \leq i \leq k$.

By induction, we conclude that

$$w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + 14 + \eta \sum_{i=1}^k \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)).$$
For $1 \leq i \leq k$, let $\delta_i = 72$ if $g(\Sigma_i) = 0$ and $|R_i| = 1$, let $\delta_i = 30$ if $g(\Sigma_i) = 0$ and $|R_i| = 2$, and let $\delta_i = 0$ otherwise, and note that since $R_i$ contains a facial ring, we have $\text{surf}(g(\Sigma_i), |R_i|, t_0(\Sigma_i), t_1(\Sigma_i)) = \text{gen}(g(\Sigma_i), |R_i|, t_0(\Sigma_i), t_1(\Sigma_i)) + \delta_i$.

If $k = 2$, then recall that since $C$ does not surround a ring, we have either $g(\Sigma_i) > 0$ or $|R_i| \geq 3 - s$ for $i \in \{1, 2\}$; hence, if $s = 0$, then either $g(\Sigma_1) > 0$, or $|R_1| \geq 2$. Consequently, we have $\delta_1 \leq 30 + 42s$.

Combining the inequalities, we obtain $\sum_{i=1}^{k} \delta_i \leq 60 + 43s - 30k$, and

$$\sum_{i=1}^{k} \text{surf}(g(\Sigma_i), |R_i|, t_0(\Sigma_i), t_1(\Sigma_i)) = \sum_{i=1}^{k} \text{gen}(g(\Sigma_i), |R_i|, t_0(\Sigma_i), t_1(\Sigma_i)) + \delta_i$$

$$\leq \text{surf}(g, |\Sigma|, t_0, t_1) + 120(2k - r - s - 2) + 48r - 120(k - 1) + 5s + \sum_{i=1}^{k} \delta_i$$

$$= \text{surf}(g, |\Sigma|, t_0, t_1) + 90k - 72(r + s) - 60$$

$$\leq \text{surf}(g, |\Sigma|, t_0, t_1) - 24.$$ 

This implies the inequality of Theorem 6.2. Therefore, assume that every non-contractible cycle of length at most 7 surrounds a ring. In particular, $G$ satisfies (18).

For each ring $R \in \mathcal{R}$, let $M_R$ be the set of all edges incident with cycles of $G$ of length at most 7 that surround $R$, and let $C_R$ be such a cycle chosen so that the part $\Sigma_R$ of $\Sigma$ between $R$ and $C_R$ is as large as possible. By Lemma 5.1, at most 70 edges of $M_R$ are drawn outside of $\Sigma_R$. Let $K_R$ be a $(\leq 7)$-cycle in $G \cap \Sigma_R$ chosen so that the part $\Sigma_R^p$ of $\Sigma$ between $R$ and $K_R$ is as small as possible. Analogically to Lemma 5.1, we see that at most 70 edges of $M_R \cap \Sigma_R$ are drawn outside of $\Sigma_R^p$. We claim that at most $5\text{cyl}(7,7)/s(5)$ edges of $G$ are drawn in $\Sigma_R^p$. When $K_R$ and $C_R$ are vertex-disjoint, this follows from Lemma 5.6. When $K_R$ intersects $C_R$, this is implied by Lemma 3.5 and Theorem 3.4, since $\text{cyl}(7,7) > s(14)$. We conclude that $|M_R| \leq 140 + 5\text{cyl}(7,7)/s(5)$.

Let $M$ consist of all facial rings of length at most four and of all non-contractible cycles in $G$ of length at most 7. Observe that $M = \bigcup_{R \in \mathcal{R}} M_R$, and thus $|E(M)| \leq (140 + 5\text{cyl}(7,7)/s(5))|\mathcal{R}|$. Note that $M$ captures all $(\leq 4)$-cycles in $G$. If $w(G, \mathcal{R}) \leq 8g + 8|\mathcal{R}| + (2/3 + 26c)\ell(\mathcal{R}) + 20|E(M)|/3 - 16$, then $w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + \eta\text{surf}(g, |\mathcal{R}|, t_0, t_1)$ by the choice of $\eta$, and Theorem 6.2 is true. Therefore, assume that this is not the case, and thus the assumptions of Theorem 3.3 are satisfied.

Let $G'$ be the $\mathcal{R}$-critical graph embedded in $\Sigma$ such that $|E(G')| < |E(G)|$, satisfying the conditions of Theorem 3.3. In particular, (b) together with the choice of $M$ implies that $G'$ has internal girth at least five. Let $X \subset F(G)$ and $\{(S_f, R_f) : f \in F(G')\}$ be the cover of $G$ by faces of $G'$ as in Theorem 3.3(d). For $f \in F(G')$, let $G_f^{l_1}, \ldots, G_f^{l_{k_f}}$ be the $G$-expansion of $S_f$ and for $1 \leq i \leq k_f$, let $\Sigma_i^f$ be the surface in that $G_i^f$ is embedded and let $R_i^f$ denote the natural
rings of $G'_i$. We have

$$w(G, \mathcal{R}) = \sum_{f \in F(G')} w(f) = \sum_{f \in X} w(f) + \sum_{i=1}^{k_f} w(G'_i, \mathcal{R}'_i).$$  \tag{16}$$

Consider a face $f \in F(G')$. We have $g(\Sigma_f) \leq g$. If $g(\Sigma_f) = g$, then every component of $G'$ is $\mathcal{R}$-planar, and since $G'$ is $\mathcal{R}$-critical, each component of $G'$ contains at least one ring of $\mathcal{R}$; consequently, $f$ has at most $|\mathcal{R}|$ facial walks and $\Sigma_f$ has at most $|\mathcal{R}|$ cuffs. Since the surfaces embedding the components of the $G$-expansion of $S_f$ are fragments of $\Sigma_f$, we have $(G'_i, \Sigma'_i, \mathcal{R}'_i) \prec (G, \Sigma, \mathcal{R})$ for $1 \leq i \leq k_f$: otherwise, we would have $m(G'_i) = m(G)$, hence by the definition of $G$-expansion, the boundary of $S_f$ would have to be equal to the union of rings in $\mathcal{R}$, contrary to the definition of a cover of $G$ by faces of $G'$.

Therefore, we can apply Theorem 6.2 inductively for $G'_i$ and we get $w(G'_i, \mathcal{R}'_i) \leq \ell(\mathcal{R}'_i) + \eta_{\text{surf}}(g(\Sigma'_i), |\mathcal{R}'_i|, t_0(\mathcal{R}'_i), t_1(\mathcal{R}'_i))$. Observe that since $\{\Sigma'_1, \ldots, \Sigma'_i\}$ are fragments of $\Sigma_f$, we have

$$\sum_{i=1}^{k_f} \text{surf}(g(\Sigma'_i), |\mathcal{R}'_i|, t_0(\mathcal{R}'_i), t_1(\mathcal{R}'_i)) \leq \text{surf}(f),$$

and we obtain

$$\sum_{i=1}^{k_f} w(G'_i, \mathcal{R}'_i) \leq |f| + \text{el}(f) + \eta_{\text{surf}}(f).$$ \tag{17}$$

In case that $f$ is open 2-cell, all fragments of $f$ are disks and we can use Theorem 3.4 instead of Theorem 6.2, getting the stronger inequality $w(G'_i, \mathcal{R}'_i) \leq s(\ell(\mathcal{R}'_i))$ for $1 \leq i \leq k_f$. Summing these inequalities, we can strengthen (17) to

$$\sum_{i=1}^{k_f} w(G'_i, \mathcal{R}'_i) \leq w(f) + \text{el}(f) + \eta_{\text{surf}}(f).$$ \tag{18}$$

The inequalities (16), (18) and Theorem 3.3(d) imply that

$$w(G, \mathcal{R}) \leq |X|s(6) + \sum_{f \in F(G')} (w(f) + \text{el}(f) + \eta_{\text{surf}}(f))$$

$$\leq w(G', \mathcal{R}) + s(6) + 10 + \eta \sum_{f \in F(G')} \text{surf}(f).$$

If $G'$ has a face that is neither open 2-cell nor omnipresent, then (10) implies that

$$w(G', \mathcal{R}) \leq \ell(\mathcal{R}) + \eta \left( \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 7 - \sum_{f \in F(G')} \text{surf}(f) \right),$$

and consequently $G$ satisfies the outcome of Theorem 6.2. Therefore, we can assume that all internal faces of $G'$ are either open 2-cell or omnipresent. Similarly, using (13) we can assume that if no face of $G'$ is omnipresent, then all of them are closed 2-cell.
Suppose first that $G$ has no omnipresent face. Using (16) and Theorem 3.3(d) and (e) and applying Theorem 6.2 inductively for $G'$, we have

$$w(G, R) \leq |X|s(6) + \sum_{f \in F(G')} w(f) - c(f)$$

$$= w(G', R) + |X|s(6) - \sum_{f \in F(G')} c(f)$$

$$\leq w(G', R) \leq \ell(R) + \eta_{surf}(g, |R|, t_0, t_1).$$

It remains to consider the case that $G'$ has an omnipresent face $h$. Then, every component of $G$ is a plane graph with one ring, and by Lemma 3.2, we conclude that every internal face of $G$ different from $h$ is closed 2-cell and $G'$ satisfies (I6). By Theorem 3.3(d), we have $c(h) \neq -\infty$, hence no component of $G'$ satisfies (E1), (E2) or (E3). By (16) and Theorem 3.3(d) and (e) and by (18), we have

$$w(G, R) \leq |X|s(6) + \sum_{f \in F(G'), f \neq h} (w(f) - c(f)) + \sum_{i=1}^{k_h} w(G_{i}^h, R_{i}^h)$$

$$= w(G', R) + |X|s(6) + (c(h) - w(h)) - \sum_{f \in F(G')} c(f) + \sum_{i=1}^{k_h} w(G_{i}^h, R_{i}^h)$$

$$\leq w(G', R) + c(h) - w(h) + \sum_{i=1}^{k_h} w(G_{i}^h, R_{i}^h)$$

$$\leq w(G', R) + c(h) + \text{el}(h) + \eta_{surf}(g, |R|, t_0, t_1)$$

We use (12) to bound the weight of $G'$. We obtain ($\kappa$ is defined as in (12))

$$w(G, R) \leq \ell(R) + \eta_{surf}(g, |R|, t_0, t_1) + c(h) + \text{el}(h) - \kappa.$$ 

By Theorem 3.3(d), we have $\text{el}(h) \leq 5$. By the definition of $\kappa$ and of the contribution of $h$, it follows that $c(h) + \text{el}(h) \leq \kappa$. Therefore,

$$w(G, R) \leq \ell(R) + \eta_{surf}(g, |R|, t_0, t_1)$$

as required.

Let us remark that Theorem 6.2 implies the special case of Theorem 1.5 for graphs with no 4-cycles, by considering the triangles to be rings (we need to first split their vertices so that they become vertex-disjoint, then drill holes in them). Furthermore, Theorem 1.3 follows as a special case when the set of rings is empty.

References


43


