Three-coloring triangle-free graphs on surfaces III. Graphs of girth five

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Abstract

We show that the size of a 4-critical graph of girth at least five is bounded by a linear function of its genus. This strengthens the previous bound on the size of such graphs given by Thomassen. It also serves as the basic case for the description of the structure of 4-critical triangle-free graphs embedded in a fixed surface, presented in a future paper of this series.

1 Introduction

This paper is a part of a series aimed at studying the 3-colorability of graphs on a fixed surface that are either triangle-free, or have their triangles restricted in some way. Historically the first result in this direction is the following classical theorem of Grötzsch [10].

Theorem 1.1. *Every triangle-free planar graph is* 3*-colorable.*

Thomassen [13, 14, 16] found three reasonably simple proofs of this claim. Recently, two of us, in joint work with Kawarabayashi [4] were able to design a linear-time algorithm to 3-color triangle-free planar graphs, and as a by-product found perhaps a yet simpler proof of Theorem 1.1. The statement of Theorem 1.1 cannot be directly extended to any surface other than the sphere. In fact, for every non-planar surface Σ there are infinitely many 4-critical graphs that can be embedded in Σ. For instance, the graphs obtained from an odd cycle of length five or more by applying Mycielski's construction [3, Section 8.5] have that property. Thus an algorithm for testing 3-colorability of triangle-free graphs on a fixed surface will have to involve more than just testing the presence of finitely many obstructions.

The situation is different for graphs of girth at least five by another deep theorem of Thomassen [15], the following.

Theorem 1.2. *For every surface* Σ *there are only finitely many* 4*-critical graphs of girth at least five that can be embedded in* Σ*.*

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Thus the 3-colorability problem on a fixed surface has a linear-time algorithm for graphs of girth at least five, but the presence of cycles of length four complicates matters. Let us remark that there are no 4-critical graphs of girth at least five on the projective plane and the torus [13] and on the Klein bottle [12].

In his proof of Theorem 1.2, Thomassen does not give a specific bound on the size of a 4-critical graph of girth at least five embedded in Σ. It appears that if one was to extract a bound from the argument, that bound would be at least doubly-exponential in the genus of Σ. In this paper, we give a different proof of the result, which gives a linear bound.

Theorem 1.3. *There exists a constant C with the following property. If G is a 4-critical graph of Euler genus g and girth at least 5, then $|V(G)| \leq Cg$.*

Let us now outline the relationship of this result to the structure of triangle-free 4-critical graphs. The only non-planar surface for which the 3-colorability problem for triangle-free graphs is fully characterized is the projective plane. Building on earlier work of Youngs [17], Gimbel and Thomassen [9] obtained the following elegant characterization. A graph embedded in a surface is a *quadrangulation* if every face is bounded by a cycle of length four.

Theorem 1.4. *A triangle-free graph embedded in the projective plane is 3-colorable if and only if it has no subgraph isomorphic to a non-bipartite quadrangulation of the projective plane.*

For other surfaces there does not seem to be a similarly nice characterization, but in a later paper of this series we will present a polynomial-time algorithm to decide whether a triangle-free graph in a fixed surface is 3-colorable. The algorithm naturally breaks into two steps. The first is when the graph is a quadrangulation, except perhaps for a bounded number of larger faces of bounded size, which will be allowed to be precolored. In this case there is a simple topological obstruction to the existence of a coloring extension based on the so-called "winding number" of the precoloring. Conversely, if the obstruction is not present and the graph is highly "locally planar", then we can show that the precoloring can be extended to a 3-coloring of the entire graph. This can be exploited to design a polynomial-time algorithm. With additional effort the algorithm can be made to run in linear time.

The second step covers the remaining case, when the graph has either many faces of size at least five, or one large face, and the same holds for every subgraph. In that case, we reduce the problem to Theorem 1.3 and show that the graph is 3-colorable. More precisely, in a future paper of this series, we use Theorem 1.3 to derive the following cornerstone result.

Theorem 1.5. *There exists an absolute constant K with the following property. Let G be a graph embedded in a surface Σ of Euler genus γ with no separating cycles of length at most four, and let t be the number triangles in G. If G is 4-critical, then $\sum |f| \leq K(t+\gamma)$, where the summation is over all faces f of G of length at least five.*

The fact that the bound in Theorems 1.3 and 1.5 is linear is needed in our solution [6] of a problem of Havel [11], as follows.

Theorem 1.6. *There exists an absolute constant d such that if G is a planar graph and every two distinct triangles in G are at distance at least d, then G is 3-colorable.*

Our technique to prove Theorem 1.3 is a refinement of the standard method of reducible configurations. We show that every sufficiently generic graph G (i.e., a graph that is large enough and cannot be decomposed to smaller pieces along cuts simplifying the problem) embedded in a surface contains one of a fixed list of configurations. Each such configuration enables us to obtain a smaller 4-critical graph G' with the property that every 3-coloring of G' corresponds to a 3-coloring of G. Furthermore, we perform the reduction in such a way that a properly defined weight of G' is greater or equal to the weight of G. A standard inductive argument then shows that the weight of every 4-critical graph is bounded, which also restricts its size. This brief exposition however hides a large number of technical details that were mostly dealt with in the previous paper in the series [7]. There, we introduced this basic technique and used it to prove the following special case of Theorem 1.5.

Theorem 1.7. *Let G be a graph of girth at least 5 embedded in the plane and let C be a cycle in G. Suppose that there exists a precoloring ϕ of C by three colors that does not extend to a proper 3-coloring of G. Then there exists a subgraph $H \subseteq G$ such that $C \subseteq H$, $|V(H)| \leq 1715|C|$ and H has no proper 3-coloring extending ϕ.*

Further results of [7] needed in this paper are summarized in Section 3.

2 Definitions

In this section, we give a few basic definitions All graphs in this paper are simple, with no loops or parallel edges.

A *surface* is a compact connected 2-manifold with (possibly null) boundary. Each component of the boundary is homeomorphic to the circle, and we call it a *cuff*. For non-negative integers a, b and c, let $\Sigma(a, b, c)$ denote the surface obtained from the sphere by adding a handles, b crosscaps and removing interiors of c pairwise disjoint closed discs. A standard result in topology shows that every surface is homeomorphic to $\Sigma(a, b, c)$ for some choice of a, b and c. Note that $\Sigma(0, 0, 0)$ is a sphere, $\Sigma(0, 0, 1)$ is a closed disk, $\Sigma(0, 0, 2)$ is a cylinder, $\Sigma(1, 0, 0)$ is a torus, $\Sigma(0, 1, 0)$ is a projective plane and $\Sigma(0, 2, 0)$ is a Klein bottle. The *Euler genus* $g(\Sigma)$ of the surface $\Sigma = \Sigma(a, b, c)$ is defined as $2a + b$. For a cuff C of Σ, let $\hat{C}$ denote an open disk with boundary C such that $\hat{C}$ is disjoint from Σ, and let $\Sigma + \hat{C}$ be the surface obtained by gluing Σ and $\hat{C}$ together, that is, by closing C with a patch. Let $\hat{\Sigma} = \Sigma + \hat{C}_1 + \ldots + \hat{C}_c$, where $C_1, \ldots, C_c$ are the cuffs of Σ, be the surface without boundary obtained by patching all the cuffs.

Consider a graph G embedded in the surface Σ; when useful, we identify G with the topological space consisting of the points corresponding to the vertices of G and the simple curves corresponding to the edges of G. A *face* f of G is a maximal connected subset of $\hat{\Sigma} - G$ (let us remark that this definition is somewhat non-standard, and in particular the face is not necessarily a subset of the surface). The boundary of a face is equal to a union of closed walks of G, which we call the *boundary walks* of f (in case that G contains an isolated vertex, this vertex forms a boundary walk by itself).

An embedding of G in Σ is *normal* if each cuff C that intersects G either does so in exactly one vertex v or is equal to a cycle B in G. In the former case, we call v a *vertex ring* and the face of G that contains C the *cuff face* of v. In the latter case, note that B is the boundary walk of the face $\hat{C}$ of G; we say that B is a *facial ring.* A face of G is a *ring face* if it is equal $\hat{C}$ for some ring C, and *internal* otherwise (in particular, the cuff face of a vertex ring is internal). We write $F(G)$ for the set of internal faces of G. A vertex v of G is a *ring vertex* if v is incident with a ring (i.e., it is drawn in the boundary of Σ), and *internal* otherwise. A cycle K in G is *separating the surface* if $\hat{\Sigma}-K$ has two components, and K is *non-separating* otherwise. A cycle K is *contractible* if there exists a closed disk $\Delta \subseteq \Sigma$ with boundary equal to K. A cycle K *surrounds the cuff* C if K is not contractible in Σ, but it is contractible in $\Sigma + \hat{C}$. We say that K *surrounds a ring* R if K surrounds the cuff incident with R.

Let G be a graph embedded in a surface Σ, let the embedding be normal, and let $\mathcal{R}$ be the set of vertex rings and facial rings of this embedding. In those circumstances we say that G is a *graph in* Σ *with rings* $\mathcal{R}$. Furthermore, some vertex rings are designated as *weak vertex rings.* Unless explicitly specified otherwise, we assume that every cuff of Σ is incident with a ring in $\mathcal{R}$.

The length $|R|$ of a facial ring is the number of edges of R. For a vertex ring R, we define $|R| = 0$ if R is weak and $|R| = 1$ otherwise. For an internal face f, by $|f|$ we mean the sum of the lengths of the boundary walks of f (in particular, if an edge appears twice in the boundary walks, it contributes 2 to $|f|$); if a boundary walk consists just of a vertex ring R, it contributes $|R|$ to $|f|$. For a set of rings $\mathcal{R}$, let us define $\ell(\mathcal{R}) = \sum_{R\in\mathcal{R}} |R|$.

Let G be a graph with rings $\mathcal{R}$. A *precoloring* of $\mathcal{R}$ is a proper 3-coloring of the graph $H = \bigcup \mathcal{R}$. Note that H is a (not necessarily induced) subgraph of G. A precoloring ϕ of $\mathcal{R}$ *extends to a* 3*-coloring of* G if there exists a proper 3-coloring ψ of G such that $\phi(v) \neq \psi(v)$ for every weak vertex ring v and $\phi(v) = \psi(v)$ for every other vertex v incident with one of the rings. The graph G is $\mathcal{R}$*-critical* if $G \neq H$ and for every proper subgraph G' of G that contains $\mathcal{R}$, there exists a precoloring of $\mathcal{R}$ that extends to a 3-coloring of G', but not to a 3-coloring of G. If $\mathcal{R}$ consists of a single ring R, then we abbreviate $\{R\}$-critical to R*-critical.* For a precoloring ϕ of the rings, the graph G is ϕ*-critical* if $G \neq H$ and ϕ does not extend to a 3-coloring of G, but it extends to a 3-coloring of every proper subgraph of G that contains $\mathcal{R}$.

Let us remark that if G is ϕ-critical for some ϕ, then it is $\mathcal{R}$-critical, but the converse is not true (for example, consider a graph consisting of a single facial ring with two chords). On the other hand, if ϕ is a precoloring of the rings of G that does not extend to a 3-coloring of G, then G contains a (not necessarily unique) ϕ-critical subgraph.

Let G be a graph embedded in a disk with one ring R of length $l \geq 5$. We say that G is *exceptional* if it satisfies one of the conditions below (see Figure 1):

(E0) $G = R$,

(E1) $l \geq 8$ and $E(G) - E(R) = 1$,

(E2) $l \geq 9$, $V(G) - V(R)$ has exactly one vertex of degree three, and the internal faces of G have lengths $5, 5, l - 4$,

(E3) $l \geq 11$, $V(G) - V(R)$ has exactly one vertex of degree three, and the internal faces of G have lengths $5, 6, l - 5$,

(E4) $l \geq 10$, $V(G) - V(R)$ consists of two adjacent degree three vertices, and the internal faces of G have lengths $5, 5, 5, l - 5$,

(E5) $l \geq 10$, $V(G) - V(R)$ consists of five degree three vertices forming a facial cycle of length five, and the internal faces of G have lengths $5, 5, 5, 5, 5, l - 5$,

We say that G is *very exceptional* if it satisfies (E1), (E2) or (E3).

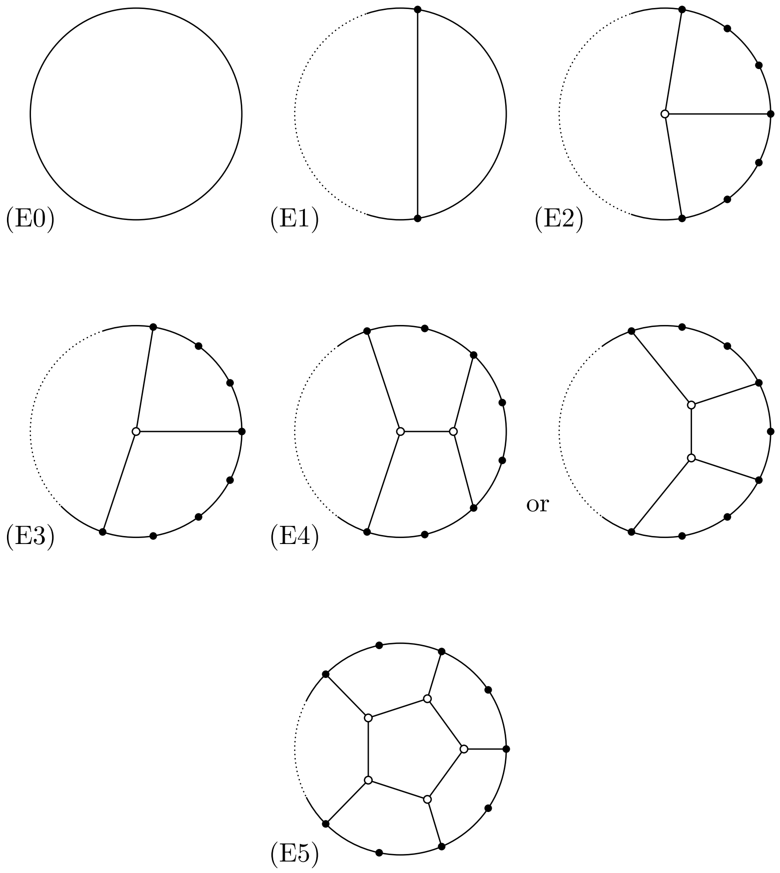

Figure 1: Exceptional graphs.

3 Definitions and results from [7]

Let G be a graph in a surface Σ with rings $\mathcal{R}$. A face is *open 2-cell* if it is homeomorphic to an open disk. A face is *closed 2-cell* if it is open 2-cell and bounded by a cycle. A face f is *omnipresent* if it is not open 2-cell and each of its boundary walks is either a vertex ring or a cycle bounding a closed disk $\Delta \subseteq \hat{\Sigma} \setminus f$ containing exactly one ring. We say that G has an *internal 2-cut* if there exist sets $A, B \subseteq V(G)$ such that $A \cup B = V(G)$, $|A \cap B| = 2$, $A - B \neq \emptyset \neq B - A$, A includes all vertices of $\mathcal{R}$, and no edge of G has one end in $A - B$ and the other in $B - A$.

We wish to consider the following conditions that the triple $G, \Sigma, \mathcal{R}$ may or may not satisfy:

(I0) every internal vertex of G has degree at least three,

(I1) G has no even cycle consisting of internal vertices of degree three,

(I2) G has no cycle C consisting of internal vertices of degree three, together with two distinct adjacent vertices $u, v \in V(G) - V(C)$ such that both u and v have a neighbor in C,

(I3) every internal face of G is closed 2-cell and has length at least 5,

(I4) if a path of length at most two has both ends in $\mathcal{R}$, then it is a subgraph of $\mathcal{R}$,

(I5) no two vertices of degree two in G are adjacent.

(I6) if Σ is the sphere and $|\mathcal{R}| = 1$, or if G has an omnipresent face, then G does not contain an internal 2-cut,

(I7) the distance between every two distinct members of $\mathcal{R}$ is at least four,

(I8) every cycle in G that does not separate the surface has length at least seven,

(I9) if a cycle C of length at most 9 in G bounds an open disk Δ in $\hat{\Sigma}$ such that Δ is disjoint from all rings, then Δ is a face, a union of a 5-face and a $(|C| - 5)$-face, or C is a 9-cycle and Δ consists of three 5-faces intersecting in a vertex of degree three.

Some of these properties are automatically satisfied by critical graphs; see [7] for the proofs of the following observations.

Lemma 3.1. *Let G be a graph in a surface Σ with rings $\mathcal{R}$. If G is $\mathcal{R}$-critical, then it satisfies* (I0), (I1) and (I2).

Lemma 3.2. *Let G be a graph in a surface Σ with rings $\mathcal{R}$. Suppose that each component of G is a planar graph containing exactly one of the rings. If G is $\mathcal{R}$-critical and contains no non-ring triangle, then each component of G is 2-connected and G satisfies (I6).*

Let G be a graph in a surface Σ with rings $\mathcal{R}$, and let P be a path of length at least one and at most four with ends $u, v \in V(\mathcal{R})$ and otherwise disjoint from $\mathcal{R}$. We say that P is *allowable* if

- u, v belong to the same ring of $\mathcal{R}$, say R,
- P has length at least three,
- there exists a subpath Q of R with ends u, v such that $P \cup Q$ is a cycle of length at most eight that bounds an open disk $\Delta \subset \Sigma$,
- if P has length three, then $P \cup Q$ has length five and Δ is a face of G, and

- if P has length four, then Δ includes at most one edge of G, and if it includes one, then that edge joins the middle vertex of P to the middle vertex of Q (which also has length four).

We say that G is *well-behaved* if every path P of length at least one and at most four with ends $u, v \in V(\mathcal{R})$ and otherwise disjoint from $\mathcal{R}$ is allowable.

Let M be a subgraph of G. A subgraph $M \subseteq G$ *captures* (≤ 4)*-cycles* if M contains all cycles of G of length at most 4 and furthermore, M is either null or has minimum degree at least two.

Throughout the rest of the paper, let $\epsilon = 2/4113$ and let let $s : \{5, 6, \ldots\} \to \mathbb{R}$ be the function defined by $s(5) = 4/4113$, $s(6) = 72/4113$, $s(7) = 540/4113$, $s(8) = 2184/4113$ and $s(l) = l - 8$ for $l \geq 9$. Based on this function, we assign weights to the faces. Let G be a graph embedded in Σ with rings $\mathcal{R}$. For an internal face f of G, we define $w(f) = s(|f|)$ if f is open 2-cell and $|f| \geq 5$, and $w(f) = |f|$ otherwise. We define $w(G, \mathcal{R})$ as the sum of $w(f)$ over all internal faces f of G.

Let Π be a surface with boundary and c a simple curve intersecting the boundary of Π exactly in its ends. The topological space obtained from Π by cutting along c (i.e., duplicating every point of c and turning both copies into boundary points) is a union of at most two surfaces. If $\Pi_1, \ldots, \Pi_k$ are obtained from Π by repeating this construction, we say that they are *fragments* of Π.

Consider a graph H embedded in a surface Π with rings $\mathcal{Q}$, and let f be an internal face of H. For each facial walk t of f, we perform the following: if t consists only of a vertex ring incident with the cuff C, then we remove $\hat{C}$ from f. Otherwise, we add a simple closed curve tracing t (if an edge appears twice in t, then it will correspond to two disjoint parts of the curve). We define Π_f to be the resulting surface. Note that the cuffs of Π_f correspond to the facial walks of f.

Let G be a graph embedded in Σ with rings $\mathcal{R}$. Let J be a subgraph of G and let S be a subset of faces of J such that

J is equal to the union of the boundaries of the faces in S, each isolated vertex of J is a vertex ring and whenever C is a cuff intersecting a face $f \in S$, then C is incident with a vertex ring belonging to J. (1)

We define $G[S]$ to be the subgraph of G consisting of J and all the vertices and edges drawn inside the faces of S. Let $C_1, C_2, \ldots, C_k$ be the boundary walks of the faces in S (in case that a vertex ring $R \in \mathcal{R}$ forms a component of a boundary of a face in S, we consider R itself to be such a walk). We would like to view $G[S]$ as a graph with rings $C_1, \ldots, C_k$. However, the C_i's do not necessarily have to be disjoint, and they do not have to be cycles or isolated vertices. To overcome this difficulty, we proceed as follows: Let Z be the set of cuffs incident with the vertex rings that form a component of J by themselves, and let $\hat{Z} = \bigcup_{R \in Z} \hat{R}$. Suppose that $S = \{f_1, \ldots, f_m\}$. For $1 \leq i \leq m$, let Σ'_i be a surface with boundary B_i such that $\Sigma'_i \setminus B_i$ is homeomorphic to f_i. Let $\theta_i : \Sigma'_i \setminus B_i \to f_i$ be a homeomorphism that extends to a continuous mapping $\theta_i : \Sigma'_i \to \overline{f_i}$, where $\overline{f_i}$ denotes the closure of f_i. Let $\Sigma_i = \Sigma'_i \setminus \theta_i^{-1}(\hat{Z} \cap f_i)$, and let G_i be the inverse image of $G \cap \overline{f_i}$ under θ_i. Then G_i is a graph normally embedded in Σ_i. We say that the set of embedded graphs $\{G_i : 1 \leq i \leq m\}$ is a *G-expansion of S*. Note that there is a one-to-one correspondence between the boundary walks of the faces of S and the rings of the graphs in the G-expansion of S; however, each vertex of J may be split to several copies. For $1 \leq i \leq m$, we let $\mathcal{R}_i$ be the set of rings of G_i consisting of the facial rings formed by the cycles contained in the boundary of Σ'_i, and of the vertex rings formed by the vertices contained in $\theta_i(Z \cap f_i)$, where each vertex ring is weak if and only if the corresponding vertex ring of $\mathcal{R}$ is weak. We say that the rings in $\mathcal{R}_i$ are the *natural rings* of G_i.

Let now G' be another $\mathcal{R}$-critical graph embedded in Σ with rings $\mathcal{R}$. Suppose that there exists a collection $\{(J_f, S_f) : f \in F(G')\}$ of subgraphs J_f of G and sets S_f of faces of J_f satisfying (1) and a set $X \subset F(G)$ such that

- for every $f \in F(G')$, the boundary of S_f is not equal to the union of $\mathcal{R}$,
- for every $f \in F(G')$, the surfaces embedding the components of the G-expansion of S_f are fragments of Σ_f,
- for every face $h \in F(G) \setminus X$, there exists unique $f \in F(G')$ such that h is a subset of a member of S_f, and
- if $X \neq \emptyset$, then X consists of a single closed 2-cell face of length 6.

We say that X together with this collection forms a *cover of G by faces of G'*. We define the *elasticity* $\mathrm{el}(f)$ of a face $f \in F(G')$ to be $\left(\sum_{h \in S_f} |h|\right) - |f|$.

We now want to bound the weight of G by the weight G'. To this end, we define a *contribution* $c(f')$ of a face f' of G' that bounds the difference between the weight of f' and the weight of the corresponding subgraph of G. We only define the contribution in the case that every face of G' is either closed 2-cell of length at least 5 or omnipresent. The contribution $c(f')$ of an omnipresent face f' of G' is defined as follows. Let $G'_1, G'_2, \ldots, G'_k$ be the components of G' such that G'_i contains the ring $R_i \in \mathcal{R}$. If there exist distinct indices i and j such that $G'_i \neq R_i$ and $G'_j \neq R_j$, then $c(f') = 1$. Otherwise, suppose that $G'_i = R_i$ for $i \geq 2$. If G'_1 satisfies (E0), (E1), (E2) or (E3), then $c(f') = -\infty$. If G'_1 satisfies (E4) or (E5), then $c(f') = 5 - \mathrm{el}(f') - 5s(5)$, otherwise $c(f') = 5 - \mathrm{el}(f') + 5s(5)$.

For a closed 2-cell face, the definition of the contribution can be found in [7]; here, we only use its properties given by the following theorem, which was proved at the end of [7].

Theorem 3.3. *Let G be a well-behaved graph embedded in a surface Σ with rings $\mathcal{R}$ satisfying* (I0)–(I9) *and let M be a subgraph of G that captures (≤ 4)-cycles. Assume that either $g > 0$ or $|\mathcal{R}| > 1$ and that $w(G, \mathcal{R}) > 8g + 8|\mathcal{R}| + (2/3 + 26\epsilon)\ell(\mathcal{R}) + 20|E(M)|/3 - 16$. If G is $\mathcal{R}$-critical, then there exists an $\mathcal{R}$-critical graph G' embedded in Σ with rings $\mathcal{R}$ such that $|E(G')| < |E(G)|$ and the following conditions hold.*

(a) If G has girth at least five, then there exists a set $Y \subseteq V(G')$ of size at most two such that $G' - Y$ has girth at least five.

(b) If C' is a (≤ 4)-cycle in G', then C' is non-contractible and G contains a non-contractible cycle C of length at most $|C'| + 3$ such that $C \not\subseteq M$. Furthermore, all ring vertices of C' belong to C, and if C' is a triangle disjoint from the rings and its vertices have distinct pairwise non-adjacent neighbors in a ring R of length 6, then G contains edges cr and $c'r'$ with $c, c' \in V(C) \setminus V(R)$ and $r, r' \in V(R) \setminus V(C)$ such that r and r' are non-adjacent.

(c) *G' has an internal face that either is not closed 2-cell or has length at least 6.*

(d) *There exists $X \subset F(G)$ and a collection $\{(J_f, S_f) : f \in F(G')\}$ forming a cover of G by faces of G', such that $\sum_{f \in F(G')} \mathrm{el}(f) \leq 10$, and if f is an omnipresent face, then $\mathrm{el}(f) \leq 5$. Furthermore, if every internal face of G' is closed 2-cell or omnipresent and G' satisfies (I6), then $\sum_{f \in F(G')} c(f) \geq |X|s(6)$.*

(e) *If $f \in F(G')$ is closed 2-cell and $G_1, \ldots, G_k$ are the components of the G-expansion of S_f, where for $1 \leq i \leq k$, G_i is embedded in the disk with one ring R_i, then $\sum_{i=1}^{k} w(G_i, \{R_i\}) \leq s(|f|) - c(f)$.*

The main result of [7] bounds the weight of graphs embedded in the disk with one ring.

Theorem 3.4. *Let G be a graph of girth at least 5 embedded in the disk with one ring R of length $l \geq 5$. If G is R-critical, then*

- *$w(G, \{R\}) \leq s(l-3) + s(5)$, and furthermore,*
- *if R does not satisfy (E1), then $w(G, \{R\}) \leq s(l-4) + 2s(5)$,*
- *if (G, R) is not very exceptional, then $w(G, \{R\}) \leq s(l-5) + 5s(5)$, and*
- *if (G, R) is not exceptional, then $w(G, \{R\}) \leq s(l-5) - 5s(5)$.*

We will also need the following property of critical graphs.

Lemma 3.5. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, and assume that G is $\mathcal{R}$-critical. Let C be a non-facial cycle in G bounding an open disk $\Delta \subseteq \hat{\Sigma}$ disjoint from the rings, and let G' be the graph consisting of the vertices and edges of G drawn in the closure of Δ. Then G' may be regarded as graph embedded in the disk with one ring C, and as such it is C-critical.*

This together with Theorem 3.4 implies that property (I9) holds for all embedded critical graphs without contractible (≤ 4)-cycles. Lemma 3.5 is a special case of the following result.

Lemma 3.6. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, and assume that G is $\mathcal{R}$-critical. Let J be a subgraph of G and S be a subset of faces of J satisfying (1). Let G' be an element of the G-expansion of S and $\mathcal{R}'$ its natural rings. If G' is not equal to the union of the rings in $\mathcal{R}'$, then G' is $\mathcal{R}'$-critical.*

Proof. Consider any edge $e' \in E(G')$ that does not belong to any of the rings in $\mathcal{R}'$. By the definition of G-expansion, there is a unique edge $e \in E(G)$ corresponding to e'. Since G is $\mathcal{R}$-critical, there exists a precoloring ψ of $\mathcal{R}$ that does not extend to a 3-coloring of G, but extends to a 3-coloring ϕ of $G - e$. We define a precoloring ψ' of $\mathcal{R}'$. Each ring vertex $v' \in V(G')$ corresponds to a unique vertex $v \in V(G)$. If v' is a weak vertex ring, then v is a weak vertex ring and we set $\psi'(v') = \psi(v)$. Otherwise, we set $\psi'(v') = \phi(v)$. Observe that ϕ corresponds to a 3-coloring of $G' - e'$ that extends ψ'. If ψ' extends to a 3-coloring ϕ' of G', then define ϕ_1 in the following way. If $v \in V(G)$ corresponds to no vertex of G', then $\phi_1(v) = \phi(v)$. If $v \in V(G)$ corresponds to at least one vertex $v' \in V(G')$, then $\phi_1(v) = \phi'(v')$. Note that if v corresponds to several vertices of G', then all these vertices belong to facial rings of G' and all of them have color $\phi(v)$, thus the definition does not depend on which of these vertices we choose. Observe that ϕ_1 is a 3-coloring of G extending ψ, which is a contradiction. Therefore, ψ' extends to a 3-coloring of $G' - e'$, but not to a 3-coloring of G'. Since this holds for every choice of e', it follows that G' is $\mathcal{R}'$-critical. □

Similarly, one can prove the following:

Lemma 3.7. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, and assume that G is $\mathcal{R}$-critical. Let c be a simple closed curve in Σ intersecting G in a set X of vertices. Let Σ_0 be one of the surfaces obtained from Σ by cutting along c. Let us split the vertices of G along c, let G' be the part of the resulting graph embedded in Σ_0, let X' be the set of vertices of G' corresponding to the vertices of X and let $\mathcal{R}' \subseteq \mathcal{R}$ be the the rings of G that are contained in Σ_0. Let Δ be an open disk or a disjoint union of two open disks disjoint from Σ_0 such that the boundary of Δ is equal to the cuff(s) of Σ_0 corresponding to c. Let $\Sigma' = \Sigma_0 \cup \Delta$. Let Y consist of all vertices of X' that are not incident with a cuff in Σ'. For each $y \in Y$, choose an open disk $\Delta_y \subset \Delta$ such that the closures of the disks are pairwise disjoint and the boundary of Δ_y intersects G' exactly in y. Let $\Sigma'' = \Sigma' \setminus \bigcup_{y \in Y} \Delta_y$ and $\mathcal{R}'' = \mathcal{R}' \cup Y$, where the elements of Y are considered to be non-weak vertex rings of the embedding of G' in Σ''. If G' is not equal to the union of the rings in $\mathcal{R}''$, then G' is $\mathcal{R}''$-critical.*

In particular, if G' is a component of a $\mathcal{R}$-critical graph, $\mathcal{R}'$ are the rings contained in G' and G' is not equal to the union of $\mathcal{R}'$, then G' is $\mathcal{R}'$-critical.

4 (≤ 4)-cycles on a cylinder

The most technically difficult part of the proof of Theorem 1.3 is dealing with long cylindrical subgraphs of the considered graph. We work out the details of this situation in the following two sections. We start with the case of a graph embedded in the cylinder with rings of length at most four. We will need the following result on graphs embedded in the disk with a ring of length at most twelve, which follows from the results of Thomassen [15].

Theorem 4.1. *Let G be a graph of girth 5 embedded in the disk with a ring R such that $|R| \leq 12$. If G is R-critical and R is an induced cycle, then*

(a) $|R| \geq 9$ and $G - V(R)$ is a tree with at most $|R| - 8$ vertices, or

(b) $|R| \geq 10$ and $G - V(R)$ is a connected graph with at most $|R| - 5$ vertices containing exactly one cycle, and the length of this cycle is 5, or

(c) $|R| = 12$ and every second vertex of R has degree two and is contained in a facial 5-cycle.

A graph H embedded in the cylinder with (vertex-disjoint) facial rings C_1 and C_2 of length at most 4 is *basic* if every contractible cycle in H has length at least five, H is $\{C_1, C_2\}$-critical, and one of the following holds:

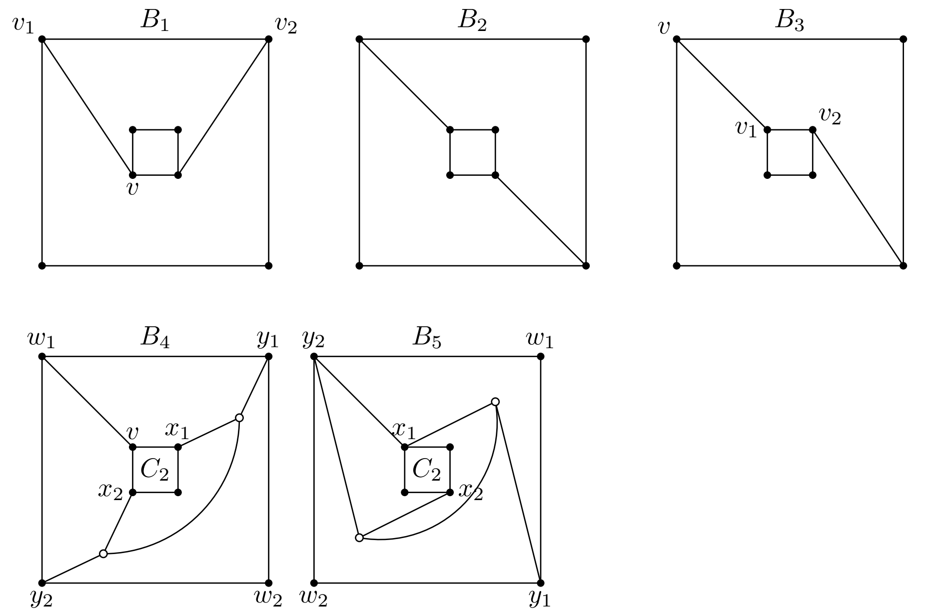

Figure 2: Maximal basic graphs.

- H contains a triangle, or
- H is not 2-connected, or
- the distance between C_1 and C_2 is one and $|V(H) \setminus V(C_1 \cup C_2)| \leq 2$.

Consider a basic 2-connected triangle-free graph. We can cut the graph along a shortest path between C_1 and C_2, resulting in a graph embedded in a disk bounded by a cycle C of length 10. Note that the resulting graph is C-critical by Lemma 3.6. A straightforward case analysis using Theorem 4.1 shows that every 2-connected triangle-free basic graph is a subgraph of one of the graphs drawn in Figure 2.

Observe furthermore that these graphs have the following properties.

Let C_1 and C_2 be the rings of a triangle-free 2-connected basic graph H. There exists a 3-coloring ψ of C_1, vertices $v_1, v_2 \in V(C_2)$ and colors $c_1 \neq c_2$ such that if ϕ is a 3-coloring of $C_1 \cup C_2$ matching ψ on C_1 and satisfying $\phi(v_i) \neq c_i$ for $i \in \{1, 2\}$, then ϕ extends to a 3-coloring of H.

(2)

The colorings for (2) are indicated in Figure 3.

Let C_1 and C_2 be the rings of a triangle-free 2-connected basic graph H, let v_1 and v_2 be two distinct vertices of C_1 and let $c_1 \neq c_2$ be two colors. There exists a vertex $v \in V(C_2)$ and a color c such that every 3-coloring ψ of C_2 such that $\psi(v) \neq c$ extends to a 3-coloring ϕ of H satisfying $\phi(v_1) \neq c_1$ and $\phi(v_2) \neq c_2$.

(3)

Proof. Let us label the triangle-free 2-connected basic graphs and their vertices as in Figure 2. If H is B_1 or B_3 and v_1 and v_2 are as depicted, then set $c = c_2$

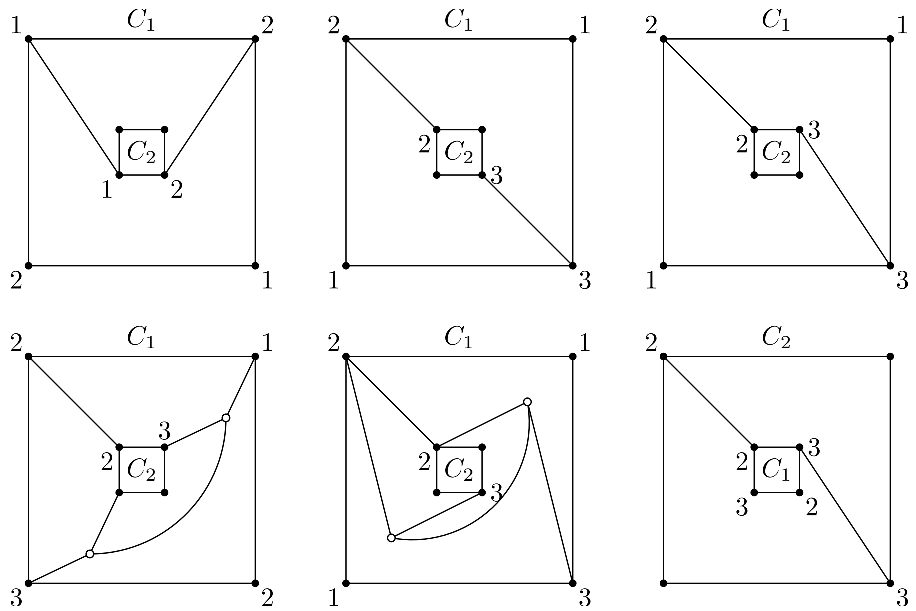

Figure 3: Colorings of basic graphs from (2).

and let v be the vertex indicated in the figure. If $H = B_4$, then let c be the unique color distinct from c_1 and c_2 and let v be the vertex indicated in the figure.

Consider any 3-coloring ψ of C_2. For each vertex $w \in V(C_1)$, let $L'_\psi(w) \subseteq \{1,2,3\}$ be the list consisting of all colors not used by ψ on the neighbors of w. Let $L_\psi(w) = L'_\psi(w)$ if $w \neq v_i$ for $i \in \{1,2\}$, and $L_\psi(w) = L'_\psi(w) \setminus \{c_i\}$ if $w = v_i$.

Suppose first that H is B_1, B_2 or B_3. Note that the sum of the sizes of the lists L_ψ is at least 8 and each of the lists has size at least one. Therefore, C_1 can be colored from the lists given by L_ψ, unless C_1 contains two adjacent vertices with the same list of size one. That is only possible if H is B_1 or B_3 and v_1 and v_2 are as depicted in Figure 2. However, then we can choose the vertex v as indicated in the figure and set $c = c_2$. This ensures that c_2 belongs to $L_\psi(v_1) \setminus L_\psi(v_2)$, and thus $L_\psi(v_1) \neq L_\psi(v_2)$. Therefore, ψ extends to a 3-coloring ϕ of H satisfying $\phi(v_1) \neq c_1$ and $\phi(v_2) \neq c_2$.

Suppose now that H is B_4 or B_5. Let us first consider the case that $\psi(x_1) \neq \psi(x_2)$. If $H = B_4$, then we can by symmetry assume that $\{v_1, v_2\} \neq \{w_1, y_1\}$, as otherwise we can swap the labels x_1 with x_2 and y_1 with y_2. Let $L(w) = L_\psi(v)$ for $w \in V(C_1) \setminus \{y_1\}$ and $L(y_1) = L_\psi(y_1) \setminus \{\psi(x_2)\}$ and observe that any coloring of C_1 from lists given by L extends to a 3-coloring ϕ of H matching ψ on C_2 and satisfying $\phi(v_1) \neq c_1$ and $\phi(v_2) \neq c_2$. Again, the sum of the sizes of the lists L is at least 8 and each of the lists has size at least one, thus such a coloring exists unless C_1 contains two adjacent vertices with the same list of size one. This is not possible, since if $H = B_4$, then $\{v_1, v_2\} \neq \{w_1, y_1\}$.

Finally, let us consider the case that $\psi(x_1) = \psi(x_2)$. If there exists a coloring ψ' of C_1 from lists L_ψ such that $\psi'(y_1) \neq \psi'(y_2)$, then the union of ψ and ψ'

extends to a 3-coloring ϕ of H, which clearly satisfies $\phi(v_1) \neq c_1$ and $\phi(v_2) \neq c_2$. Let us find such a coloring ψ'. If C_1 contains a vertex $w \notin \{y_1, y_2\}$ such that $|L_\psi(w)| = 3$, then it suffices to color the vertices of $Y = V(C_1) \setminus \{w\}$ by pairwise distinct colors from their lists and then color w differently from its neighbors. Such a coloring of Y always exists, since $\sum_{y \in Y} |L_\psi(y)| \geq 6$, all the lists have size at least one and at most three and if all of them have size two, then v_1 and v_2 belong to Y and $L_\psi(v_1) = \{1, 2, 3\} \setminus \{c_1\}$ is different from $L_\psi(v_2) = \{1, 2, 3\} \setminus \{c_2\}$. Therefore, we can assume that all vertices in $V(C_1) \setminus \{y_1, y_2\}$ have lists of size at most two.

Suppose that either $H = B_5$, or $H = B_4$ and $|L_\psi(w_1)| = 2$. In this case $|L_\psi(w_1)| = |L_\psi(w_2)| = 2$ and by symmetry, we can assume that $|L_\psi(y_1)| = 3$ and $|L_\psi(y_2)| \geq 2$. Let us choose a color $\psi'(w_1) = \psi'(w_2) \in L_\psi(w_1) \cap L_\psi(w_2)$ and then color y_1 and y_2 by distinct colors from $L(y_1) \setminus \{\psi'(w_1)\}$ and $L(y_2) \setminus \{\psi'(w_1)\}$, respectively.

Therefore, we can assume that $H = B_4$ and $|L_\psi(w_1)| = 1$. Note that $|L_\psi(w_2)| = 2$ and $|L_\psi(y_1)| = |L_\psi(y_2)| = 3$. By symmetry between v_1 and v_2, we can assume that $v_1 = w_1$ and $v_2 = w_2$. The coloring ψ' exists unless $L_\psi(w_2) = \{1, 2, 3\} \setminus L_\psi(w_1)$. However, this is prevented by the choice of c. $\square$

For a 4-cycle $C = x_1x_2x_3x_4$, the *type* of its 3-coloring λ is the set of the vertices x_i of C such that $\lambda(x_i) \neq \lambda(x_{i+2})$ (where $x_5 = x_1$ and $x_6 = x_2$). Note that the type of λ is $\emptyset$, $\{x_1, x_3\}$ or $\{x_2, x_4\}$. In (2), any coloring of the same type as ψ has the same property, possibly with different colors c_1 and c_2.

Let G and H be graphs with common rings $\{C_1, C_2\}$. We say that H *subsumes* G if every precoloring of $C_1 \cup C_2$ that extends to a 3-coloring of H also extends to a 3-coloring of G.

Lemma 4.2. *Let G be a graph embedded in the cylinder with facial rings $\{R_1, R_2\}$ of length at most 4. If every cycle of length at most 4 in G is non-contractible, then there exists a basic graph H with rings $\{R_1, R_2\}$ that subsumes G. Furthermore, either $H = G$ or $|V(H)| + |E(H)| < |V(G)| + |E(G)|$.*

Proof. Suppose for a contradiction that G is a counterexample such that $|V(G)| + |E(G)|$ is minimal. It follows that G is $\{R_1, R_2\}$-critical, 2-connected and triangle-free, and in particular $|R_1| = |R_2| = 4$. Let $R_1 = a_1a_2a_3a_4$ and $R_2 = b_1b_2b_3b_4$, where the labels are assigned in the clockwise order. Since G is triangle-free and all 4-cycles are non-contractible, it follows that every internal vertex has at most one neighbor in each of the rings.

Suppose that G contains a 5-face $C = v_1v_2v_3v_4v_5$ such that all its vertices are internal and have degree three. For $1 \leq i \leq 5$, let x_i be the neighbor of v_i different from v_{i-1} and v_{i+1} (where $v_0 = v_5$ and $v_6 = v_1$). Observe that if $x_1 = x_3$, then $x_2 \neq x_4$, thus by symmetry assume that $x_1 \neq x_3$. Let $G' = (G - V(C)) + x_1x_3$. Suppose that K' is a cycle of length at most 4 in G' that contains the edge x_1x_3. Then G contains a cycle K of length at most 7 obtained from K' by replacing x_1x_3 by $x_1v_1v_2v_3x_3$. Since v_1 and v_2 have neighbors on the opposite sides of this path, K does not bound a face. By Theorem 4.1, we conclude that K and K' are non-contractible. Therefore, all (≤ 4)-cycles in G' are non-contractible. Furthermore, every precoloring of R_1 and R_2 that extends to a 3-coloring of G' also extends to a 3-coloring of G (the 3-coloring of G' assigns different colors to x_1 and x_3, thus it can be extended to C). Thus, G' subsumes G, and consequently it contradicts the minimality of G. We conclude that

every 5*-face in* G *is incident with a ring vertex or a vertex of degree at least* 4. (4)

It follows that the distance between R_1 and R_2 is at least two: otherwise, if say a_1 is adjacent to b_1, then apply Theorem 4.1 to the graph obtained from G by cutting open along the walk $a_1a_2 \ldots a_1b_1b_2 \ldots b_1$. Outcome (b) is excluded by (4), thus $G - V(R_1 \cup R_2)$ would have at most two vertices and G would be basic.

Suppose that G contains a face $C = v_1v_2 \ldots v_k$ of length $k \geq 7$. We may assume that v_1 is an internal vertex. Let G' be the graph obtained from G by identifying v_1 with v_3 to a vertex v. Consider a cycle $K' \subseteq G'$ of length at most 4 that does not appear in G. Such a cycle corresponds to a cycle K in G of length at most 6, obtained by replacing v by $v_1v_2v_3$. Since v_1 is an internal vertex, v_2 cannot be a ring vertex of degree two. It follows that K does not bound a face and it is non-contractible by Theorem 4.1. Therefore, all (≤ 4)-cycles in G' are non-contractible. Furthermore, every 3-coloring of G' extends to a 3-coloring of G, and we obtain a contradiction with the minimality of G. Therefore, each face of G has length at most 6.

Suppose that G contains a face $C = v_1v_2 \ldots v_6$ of length 6. We can assume that v_1 is an internal vertex. If v_3 or v_5 is an internal vertex, then let G' be the graph obtained from G by identifying v_1, v_3 and v_5 to a single vertex. As in the previous paragraph, we obtain a contradiction. It follows that v_3 and v_5 are ring vertices, and by a symmetrical argument, two of v_2, v_4 and v_6 are ring vertices. If v_2 is internal, then since the distance between R_1 and R_2 is at least two, we can assume that $V(R_1) = \{v_3, v_4, v_5, v_6\}$, and thus v_3 and v_6 are adjacent. In this situation, we consider the graph obtained from G by identifying v_1 with v_5 and v_2 with v_4 (which is isomorphic to $G - \{v_4, v_5\}$, and thus contains no contractible (≤ 4)-cycles), and again obtain a contradiction with the minimality of G. Thus v_2 is not internal, and by symmetry, v_6 is not internal either. Therefore, v_4 is internal and v_2 and v_6 are ring vertices. Since the distance between R_1 and R_2 is at least two, we may assume that $v_2 = a_2$, $v_3 = a_3$, $v_5 = b_4$ and $v_6 = b_1$. We apply Theorem 4.1 to the 10-cycle $B = a_1a_2v_1b_1b_2b_3b_4v_4a_3a_4$. The case (b) is excluded by (4), thus either B is not induced or (a) holds. If B is not induced, then its chord joins v_1 with v_4. However, then the precolorings ψ of the rings that do not extend to 3-colorings of G satisfy $\psi(a_2) = \psi(b_4)$, and we can set H to be the graph consisting of R_1, R_2 and the edge between a_2 and b_4. Therefore, B is an induced cycle and $G - V(B)$ is a tree F with at most two vertices. If F has only one vertex w, then w cannot be adjacent to both v_1 and v_4, hence one of these vertices has degree two, which is a contradiction. If $V(F) = \{x, y\}$, then since v_1 and v_4 have degree at least three, we can assume that x is adjacent to v_1 and a_4 and y is adjacent to b_2 and v_4. However, by identifying v_1 with b_4 and v_4 with a_2, we obtain a graph isomorphic to the graph B_5 of Figure 2, which subsumes G. Therefore,

all internal faces of G *have length* 5. (5)

Together with Lemma 3.5 and Theorem 4.1, this implies that

G contains no contractible cycles of length 6 or 7, and the disk bounded by any contractible 8-cycle K of G consists of two 5-faces separated by a chord of K. (6)

Suppose that G contains a 4-cycle $C = v_1v_2v_3v_4$ different from R_1 and R_2. By the assumptions, C is non-contractible; for $i \in \{1, 2\}$, let G_i be the subgraph of G drawn between R_i and C and let d_i be the distance between R_i and C.

Let us first consider the case that $d_i \geq 1$ and G_i is not basic for some $i \in \{1, 2\}$. By the minimality of G, there exists a basic graph G'_i that subsumes G_i (considered to be embedded in a cylinder with rings R_i and C) such that $|V(G'_i)| + |E(G'_i)| < |V(G_i)| + |E(G_i)|$. Let $G' = G'_i \cup G_{3-i}$ and observe that G' subsumes G and $|V(G')| + |E(G')| < |V(G)| + |E(G)|$. Note that every contractible cycle in G' has length at least five, since neither G'_i nor G_{3-i} contains a contractible (≤ 4)-cycle and G_{3-i} is triangle-free. Therefore, by the minimality of G, there exists a basic graph H which subsumes G' and $|V(H)|+|E(H)| \leq |V(G')|+|E(G')|$. However, then H also subsumes G, which is a contradiction.

We conclude that if $d_i \geq 1$, then G_i is a basic graph, and since it is 2-connected and triangle-free, it follows that $d_i = 1$. Let us choose the labels of R_1 and R_2 and the cycle C so that d_1 is as small as possible. In particular, $d_1 \leq d_2$. Let us discuss the possible cases:

- $d_1 = d_2 = 0$: Since the distance between R_1 and R_2 is at least two, we conclude that $|V(R_1) \cap V(C)| = |V(R_1) \cap V(C)| = 1$. We can assume that $v_1 = a_1$ and $v_3 = b_3$. By Theorem 4.1, the open disks bounded by the closed walks $a_1v_2v_3v_4a_1a_4a_3a_2$ and $b_3b_4b_1b_2b_3v_4a_1v_2$ contain no vertices, and since v_2 and v_4 have degree at least three, we may assume that v_2 is adjacent to a_4 and v_4 to b_2. However, then G contains a triangle $a_1v_2a_4$, which is a contradiction.

- $d_1 = 0$, $d_2 = 1$: We may assume that $a_1 = v_1$. Since G is triangle-free, (6) implies that $|V(C) \cap V(R_1)| = 1$ and $a_3v_3 \in E(G)$. Since $d_2 = 1$, G_2 is a basic graph, and by (5), we conclude that G_2 is isomorphic to B_4 or B_5 from Figure 2. Let $w_1w_2 = G_2 - V(C \cup R_2)$. Up to symmetry, there are two cases to consider:

  - b_3 is adjacent to v_3. Since v_2 and v_4 have degree at least three, we can assume that w_1 is adjacent to v_4 and b_2 and w_2 is adjacent to v_2 and b_4. In this case, we let H be the graph consisting of R_1, R_2 and a vertex z, with a_1 adjacent to b_2 and z to b_2, b_3 and a_3.
  - b_3 is adjacent to v_4. Since v_2 has degree at least three, we can assume that w_1 is adjacent to b_1 and v_4 and w_2 is adjacent to b_3 and v_2. We let H be the graph consisting of R_1, R_2, adjacent vertices z_1 and z_2, and edges a_1b_3, a_1z_1, b_1z_1, b_3z_2 and a_3z_2.

- $d_1 = 1$, $d_2 = 1$: By the choice of C, G does not contain a 4-cycle distinct from R_1 and R_2 that intersects one of them. Additionally, all internal faces of G have length 5 and G_1 and G_2 are basic graphs, hence we can assume that a_1 is adjacent to v_1 and $G_1 - V(R_1 \cup C) = w_1w_2$ with w_1 adjacent to a_4 and v_4 and w_2 adjacent to a_2 and v_2, and G_2 is isomorphic to G_1. Since v_3 has degree at least three, v_1 cannot have a neighbor in R_2, thus there are up to symmetry two possible cases:

- b_1 is adjacent to v_2, $G_2 - V(R_2 \cup C) = w_3w_4$, and $w_3v_1, w_3b_2, w_4v_3, w_4b_4 \in E(G)$. Then, let H be the graph consisting of R_1, R_2 and the edge a_1b_2.
- b_1 is adjacent to v_3, $G_2 - V(R_2 \cup C) = w_3w_4$, and $w_3v_2, w_3b_2, w_4v_4, w_4b_4 \in E(G)$. But then every precoloring of R_1 and R_2 extends to a 3-coloring of G, contrary to the assumption that G is $\{R_1, R_2\}$-critical.

Therefore,

R_1 and R_2 are the only 4-cycles in G. (7)

Suppose that G has a face $C = v_1v_2v_3v_4v_5$ such that $v_2, \ldots, v_5$ are internal vertices of degree three. For $2 \le i \le 5$, let x_i be the neighbor of v_i that is not incident with C. By (7), the vertices x_i are distinct. If at least one of x_3 and x_4 is internal, then let G' be the graph obtained from $G - \{v_2, \ldots, v_5\} + x_2x_5$ by identifying x_3 with x_4 to a new vertex x. Observe that every 3-coloring of G' extends to a 3-coloring of G. Furthermore, suppose that K' is a cycle of length at most 4 in G' that does not appear in G, and let K be the corresponding cycle in G obtained by replacing x_2x_5 by $x_2v_2v_1v_5x_5$ or x by $x_3v_3v_4x_4$ or both. If $|K| \le 7$, then since K cannot bound a face, Theorem 4.1 implies that K and K' are non-contractible. If $|K| \ge 8$, then K contains both $x_2v_2v_1v_5x_5$ and $x_3v_3v_4x_4$, and since $|K'| \le 4$ and G is embedded in the cylinder, it follows that x_4 is adjacent to x_5 or x_3 is adjacent to x_2. This is excluded by (7). Therefore, G' is a smaller counterexample than G, which is a contradiction. Let us now consider the case that both x_3 and x_4 are ring vertices. Here, we exclude the possibility that x_3 and x_4 belong to different rings: If that were the case, then we can assume that $x_3 = a_1$ and $x_4 = b_1$. Since all internal faces of G have length 5, it follows that x_4 and x_5 have a common neighbor v. We apply Theorem 4.1 to the disk bounded by the closed walk $a_1a_2a_3a_4a_1vb_1b_2b_3b_4b_1v$ of length 12. By (4), the case (b) is excluded. Since v has degree at least three, a_1vb_1 cannot be incident with two 5-faces and the case (c) is excluded as well. Therefore, $G - V(R_1 \cup R_2) - \{v\}$ is a tree with four vertices v_2, v_3, v_4 and v_5. By (7), v is not equal to x_2, x_5 or v_2, hence two of these vertices belong to the same ring. Since G is triangle-free, (7) implies that no internal vertex has two neighbors in the same ring, thus we can assume that $v_2 \in V(R_1)$ and $x_2, x_5 \in V(R_2)$. However, the path $x_2v_2v_3v_4v_5x_5$ together with a subpath of R_2 forms a cycle that separates v_2 from R_1, which contradicts the assumption that G is embedded in the cylinder. Therefore,

if $C = v_1v_2v_3v_4v_5$ is a face such that $v_2, \ldots, v_5$ are internal vertices of degree three, then for some $i \in \{1, 2\}$, both v_3 and v_4 have a neighbor in R_i. (8)

Let us now assign charge to vertices and faces of G as follows: each face f gets the initial charge $|f| - 4$ and each vertex v gets the initial charge $\deg(v) - 4$. The sum of the initial charges is -8. Let us redistribute the charge: each 5-face sends $1/3$ to each incident vertex v such that v is internal and has degree three. Furthermore, for each ring vertex w of degree two, if there exists a face $f = v_1v_2v_3v_4v_5$ such all vertices incident with f except for v_1 are internal of degree three and if v_3v_4 is incident with the same face as w, then w sends $1/3$ to f. Note that after this procedure, all vertices and faces have non-negative charge, with the following exceptions: the ring vertices of degree two have charge at most $-7/3$ and the ring vertices of degree three have charge -1. For $i \in \{1, 2\}$, let c_i be the sum of the charges of the vertices of R_i, together with the charges of the faces that share an edge with R_i (such a face cannot share an edge with R_{3-i}, since the distance between R_1 and R_2 is at least two and all internal faces have length 5). Note that $c_1 + c_2 \leq -8$, and we may assume that $c_1 \leq -4$.

For $i \in \{1, 2, 3, 4\}$, let f_i denote the face sharing the edge a_ia_{i+1} with R_1. If a vertex a_i has degree three, then let x_i denote its internal neighbor. Since G is 2-connected, at most two vertices of R_1 have degree two. Let us discuss several cases.

- R_1 contains two vertices of degree two: Since all faces have length 5 and G is triangle-free, these two vertices are non-adjacent, say a_2 and a_4. Similarly, since G does not contain a 4-cycle different from R_1 and R_2, both a_1 and a_3 have degree at least four, and since the sum of the charges of the vertices of R_1 is at most -4, we conclude that $\deg(a_1) = \deg(a_3) = 4$. Let $f_2 = a_1a_2a_3x'_3x'_1$ and $f_4 = a_1a_4a_3x''_3x''_1$. Note that both f_2 and f_4 send charge to at most two vertices, hence their final charge is $1/3$, and since $c_1 \leq -4$, it follows that the charge of a_2 and a_4 is $-7/3$. Therefore, the vertices x'_1, x''_1, x'_3, x''_3 and their neighbors distinct from a_1 and a_3 are internal vertices of degree three. However, these vertices form an 8-cycle, contradicting the criticality of G.

- R_1 contains one vertex of degree two, say a_2, and a_1, a_3 and a_4 have degree three: by (5), x_1 is adjacent to x_3, x_1 and x_4 have a common neighbor x_{41} and x_3 and x_4 have a common neighbor x_{43}. Suppose that x_1 and x_3 have degree three. The path $x_{41}x_1x_3x_{43}$ is a part of a boundary of a 5-face f; let y be the fifth vertex of f. Then $x_{41}x_4x_{43}y$ is a 4-cycle, contradicting (7). Therefore, we may assume that x_1 has degree greater than three. This implies that a_2 does not send any charge and its final charge is -2. Furthermore, f_2 has charge at least $2/3$ and f_4 has charge at least $1/3$, and thus $c_1 = -4$. Furthermore, x_3, x_4, x_{41} and x_{43} are internal and have degree three.

- R_1 contains one vertex of degree two, say a_2, and at least one vertex of R_1 has degree at least four: note that the sum of the charges of a_2 and f_2 is at least -2. It follows that exactly one vertex of R_1 has degree four, two vertices have degree three, and $c_1 = -4$.

- R_1 contains no vertices of degree two. Since $c_1 \leq -4$, it follows that all vertices of R_1 have degree three and all internal vertices of the faces sharing an edge with R_1 have degree three. But then G contains an 8-cycle of internal vertices of degree 3, contradicting the criticality of G.

We conclude that $c_1 = -4$, and by symmetry, $c_2 = -4$. It follows that all charges that are not counted in c_1 and c_2 are equal to zero. Let us now go over the possible cases for the neighborhood of R_1 again, keeping the notation established in the previous paragraph:

- R_1 contains one vertex of degree two, say a_2, and a_1, a_3 and a_4 have degree three: Since all internal vertices have zero charge, x_1 has degree exactly four. Let y_1, y_{41} and y_{43} be the neighbors of x_1, x_{41} and x_{43}, respectively, not incident with f_2, f_3 and f_4. By (5), y_{43} is adjacent to y_1 and to y_{41}, and the vertices y_1 and y_{41} have a common neighbor z distinct from y_{43}. By (7), we have $R_2 = y_1 y_{43} y_{41} z$. However, then we can set H to be the graph consisting of R_1, R_2 and a vertex w, with edges $a_4 y_{41}$, wy_1, wa_1 and wa_4.

- R_1 contains one vertex of degree two, say a_2, one vertex of degree four and two of degree three. Let a_i be the vertex of degree four and x'_i and x''_i its internal neighbors. Since $c_1 = -4$, all internal vertices incident with the faces f_2, f_3 and f_4 have degree three, and by (5) they form a path P with ends x'_i and x''_i. Furthermore, x'_i and x''_i have adjacent neighbors y'_i and y''_i. We let G' consist of $G - V(P)$ and a new vertex w adjacent to y'_i, y''_i and a_i, and observe that every 3-coloring of G' extends to a 3-coloring of G. This contradicts the minimality of G.

□

A finite graph G is an (e_1, e_2)-*chain* if either G is the complete graph on four vertices and e_1 and e_2 form a matching in G; or, there exists an (e_1, u_1u_2)-chain H and G consists of $H - u_1u_2$, vertices y_1, y_2 and u'_2 and edges y_1y_2, $u_2u'_2$, u_1y_1, u_1y_2, u'_2y_1 and u'_2y_2, where $e_2 = y_1y_2$. Let us note that each (e_1, e_2)-chain is a planar graph with chromatic number 4 containing exactly four triangles (two incident with each of e_1 and e_2), and all other faces of G have length 5. The graph G can be embedded in the Klein bottle by putting crosscaps on the edges e_1 and e_2; we call such an embedding *canonical*. Note that no cycle of length less than 5 is contractible in a canonical embedding of G. Thomas and Walls [12] proved the following:

Theorem 4.3. *If G is a 4-critical graph embedded in the Klein bottle so that no cycle of length at most 4 is contractible, then G is a canonical embedding of an (e_1, e_2)-chain, for some edges $e_1, e_2 \in E(G)$.*

For the torus, Thomassen [13] showed that the situation is even simpler.

Theorem 4.4. *If G is embedded in the torus so that no cycle of length at most 4 is contractible, then G is 3-colorable.*

Also, the results of Aksionov [2] imply the following:

Theorem 4.5. *There exists no R-critical graph G embedded in the cylinder with one ring R (and the other cuff disjoint from G) such that R is a cycle of length at most four and every cycle of length at most four in G is non-contractible.*

Let us now give a description of $\{R_1, R_2\}$-critical graphs on a cylinder, where each of R_1 and R_2 is either a vertex ring or a triangle:

Lemma 4.6. *Let G be an $\{R_1, R_2\}$-critical graph embedded in the cylinder, where each of R_1 and R_2 is either a vertex ring or a facial ring of length three. If every cycle of length at most 4 in G is non-contractible, then one of the following holds:*

- *G consists of R_1, R_2 and an edge between them, or*

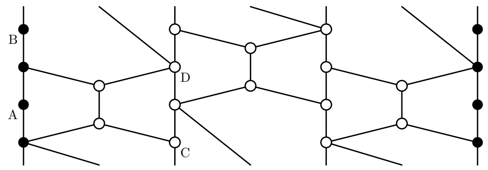

Figure 4: Arbitrarily large critical graph with rings of length four.

- *R_1 and R_2 are triangles and G consists of R_1, R_2 and two edges between them, or*
- *R_1 and R_2 are triangles and G consists of R_1, R_2 and two adjacent vertices of degree three, each having a neighbor in R_1 and in R_2.*

Proof. By Theorem 4.5, we have that G is connected. We may assume that at least one of R_1 and R_2 is a triangle, since if both R_1 and R_2 are vertex rings, we can add a triangle containing R_2 to G. If the distance between R_1 and R_2 is at most two, then G satisfies one of the conclusions of the lemma by Lemma 3.5 applied to the closed walk tracing the rings and the shortest path between them and by Theorem 4.1 (in the cases (b) and (c), the corresponding graphs are not $\{R_1, R_2\}$-critical). Therefore, assume that the distance between R_1 and R_2 is at least three.

Since G is $\{R_1, R_2\}$-critical, there exists a precoloring ψ of $R_1 \cup R_2$ that does not extend to a 3-coloring of G. We identify the vertices of R_1 and R_2 to which ψ assigns the same color and we obtain a graph G' embedded in the torus or in the Klein bottle (in the latter case, we can assume that both R_1 and R_2 are triangles). Note that G' has no loops, since R_1 and R_2 are not adjacent. Observe also that G' contains no contractible (≤ 4)-cycle. Since G' is not 3-colorable, Theorems 4.3 and 4.4 imply that G' is embedded in the Klein bottle and contains a canonical embedding of an (e_1, e_2)-chain as a subgraph. Therefore, G' contains a separating non-contractible 4-cycle C. The subgraph of G corresponding to C contains at least two paths joining R_1 and R_2. However, this implies that the distance between R_1 and R_2 is at most two, which is a contradiction. □

Finally, we give a similar result for $\{R_1, R_2\}$-critical graphs, where each of R_1 and R_2 has length at most four. A *broken chain* is a graph obtained from an (e_1, e_2)-chain by removing the edges e_1 and e_2, see Figure 4 for an illustration (the top of the picture is identified with the bottom, giving an embedding in the cylinder). Note that in any 3-coloring of the graph depicted in Figure 4, if A and B have different colors, then the colors of C and D differ as well. Repeating this observation, we conclude that if the colors of A and B differ, then the colors of the corresponding vertices of the rightmost cycle differ as well. Consequently, this gives an example of an $\{R_1, R_2\}$-critical graph embedded in the cylinder, where R_1 and R_2 are arbitrarily distant 4-cycles.

Dvořák and Lidický [8] gave a complete list of such $\{R_1, R_2\}$-critical graphs embedded in a cylinder without contractible (≤ 4)-cycles (other than broken chains, there are only finitely many). However, their proof is computer assisted. In this paper, we give a much weaker bound on the size of the graphs, which however suffices for our purposes. We start with the case that there are many (≤ 4)-cycles separating R_1 from R_2.

Lemma 4.7. *Let G be an $\{R_1, R_2\}$-critical graph embedded in the cylinder Σ, where each of R_1 and R_2 is either a vertex ring or a facial ring of length at most four. Suppose that every cycle of length at most 4 in G is non-contractible. If G contains at least 34 cycles of length at most 4, then G is a broken chain.*

Proof. The graph G is connected by Theorem 4.5. Let C_1 and C_2 be distinct cycles of length at most 4 in G. We claim that C_1 bounds a closed disk in $\hat{\Sigma}$ that contains C_2. Indeed, otherwise each of the open disks in $\hat{\Sigma}$ bounded by C_1 contains a vertex of C_2, and we conclude that the set $X = V(C_1) \cap V(C_2)$ has size two. But then there exist three disjoint paths of length at most two between the vertices of X, and one of the (≤ 4)-cycles formed by these paths is contractible in Σ, contradicting the assumptions.

We write $C_1 < C_2$ if the closed disk bounded by C_1 in $\Sigma + \hat{R}_2$ contains C_2. Note that $<$ is a linear ordering of the cycles of length at most four in G. Let $K_1, K_2, \ldots, K_m$ be the list of all cycles of length at most four in G sorted according to this ordering. Suppose that K_i and K_j are triangles for some $i < j$. By Theorem 4.1, if $V(K_i) \cap V(K_j) \neq \emptyset$, then $j = i + 1$. If K_i and K_j are vertex-disjoint, then Lemma 4.6 implies that $j \leq i + 3$.

For any $i < j$, if $V(K_i) \cap V(K_j) \neq \emptyset$, then by Theorem 4.1 the area between K_i and K_j consists either of one face or of two 5-faces, thus either $j = i + 1$, or $j = i + 2$ and K_{i+1} is a triangle. In particular, K_i and K_{i+3} are vertex-disjoint.

For $i < j$, let G_{ij} be the subgraph of G drawn between K_i and K_j. Note that if K_i and K_j are vertex-disjoint, then G_{ij} is $\{K_i, K_j\}$-critical. In that case Lemma 4.2 implies that G_{ij} is subsumed by a $\{K_i, K_j\}$-critical basic graph H_{ij}. If K_i and K_j are not vertex-disjoint, then we define $H_{ij} = G_{ij}$.

Consider indices $i < j < k$ and a graph $B \in \{G_{ij}, H_{ij}\}$, and suppose that $B \cup H_{jk}$ contains a contractible cycle C of length at most 4. By the definition of a basic graph, $C \not\subseteq B$ and $C \not\subseteq H_{jk}$; hence, C has length 4 and $C = v_1v_2v_3v_4$, where $v_2, v_4 \in V(K_j)$, $v_1 \in V(B) \setminus V(K_j)$ and $v_3 \in V(H_{jk}) \setminus V(K_j)$. Furthermore, v_2 and v_4 must be consecutive vertices of K_j, thus both B and H_{jk} contain a triangle incident with an edge of K_j. Therefore, we have the following.

For any $i < j < k$ and a graph $B \in \{G_{ij}, H_{ij}\}$, if $B \cup H_{jk}$ contains a contractible cycle of length at most 4, then both B and H_{jk} contain a triangle with an edge in K_j.

(9)

Let U be the set of indices i such that either there exists a triangle K_t with $t \geq \max(1, i-1)$, or there exists j such that $i < j \leq m$ and H_{ij} contains a cutvertex or a triangle. Let L be the set of indices j such that either there exists a triangle K_t with $t \leq \min(m, j+1)$, or there exists i such that $1 \leq i < j$ and H_{ij} contains a cutvertex or a triangle. Suppose that $a \in L$ and $b \in U$ satisfy $b \geq a + 6$. If there exists $t \leq \min(m, a+1)$ such that K_t is a triangle, then let $G_1 = G_{1a}$. Otherwise, there exists $k < a$ such that H_{ka} contains a cutvertex or a triangle, and we set $G_1 = G_{1k} \cup H_{ka}$. Similarly, if there exists $t \geq \max(1, b-1)$ such that K_t is a triangle, then let $G_2 = G_{bm}$, otherwise let $G_2 = H_{bl} \cup G_{lm}$ for $l > b$ such that H_{bl} contains a cutvertex or a triangle. Let $G' = G_1 \cup G_{ab} \cup G_2$. Note that G' contains no contractible (≤ 4)-cycles by (9), the definition of U and L and the choice of G_1 and G_2. Furthermore, observe that G' contains triangles or cutvertices T_1 and T_2 separated by at least three (≤ 4)-cycles belonging to G_{ab}. By Lemma 4.6, every precoloring of $T_1 \cup T_2$ extends to a 3-coloring of the subgraph of G' between T_1 and T_2. Theorem 4.5 implies that every precoloring of R_i extends to a 3-coloring of the subgraph of G' between R_i and T_i, for $i \in \{1, 2\}$. Therefore, every precoloring of $R_1 \cup R_2$ extends to a 3-coloring of G'. However, G' subsumes G, hence every precoloring of $R_1 \cup R_2$ also extends to a 3-coloring of G. This contradicts the criticality of G. We conclude that if $a \in L$ and $b \in U$, then $b \leq a + 5$. Let X be the set of indices i such that $i \in U$ and $i + 2 \in L$. Observe that if X is nonempty, then $\max X - \min X \leq 7$. Let us also remark that if K_i is a triangle, then $j \in X$ for $i - 3 \leq j \leq i + 1$.

Since $m \geq 34$, there exist indices $1 \leq i_1 < i_2 < \ldots < i_8 \leq m$ such that $i_{j+1} = i_j + 2$ for $1 \leq j \leq 4$ and for $5 \leq j \leq 8$, $i_5 \geq i_4 + 9$ and $i_1, i_2, i_3, i_5, i_6, i_7 \notin X$. Note that the definition of X ensures that $G_{i_1 i_4}$ and $G_{i_5 i_8}$ are triangle-free (and consequently, K_{i_j} and $K_{i_{j+1}}$ are vertex-disjoint for $1 \leq j \leq 7$) and that $H_{i_j i_{j+1}}$ is 2-connected and triangle-free for $1 \leq j \leq 3$ and $5 \leq j \leq 7$.

Combining (2) and (3) shows that there exists a precoloring ψ of K_{i_4}, a vertex $v \in V(K_{i_2})$ and a color c such that every precoloring ϕ_2 of $K_{i_2} \cup K_{i_4}$ that matches ψ on K_{i_4} and satisfies $\phi_2(v) \neq c$ extends to a 3-coloring of $H_{i_2 i_3} \cup H_{i_3 i_4}$. Furthermore, an inspection of the basic 2-connected triangle-free graphs shows that that every 3-coloring of K_{i_1} extends to a 3-coloring of $H_{i_1 i_2}$ that assigns v a color different from c. It follows that every precoloring ϕ of $K_{i_1} \cup K_{i_4}$ that matches ψ on K_{i_4} extends to a 3-coloring of $H_{i_1 i_4}$, and thus also to a 3-coloring of $G_{i_1 i_4}$. In fact, it is sufficient to assume that ϕ has the same type S_1 on K_{i_4} as ψ; thus, every precoloring of $K_{i_1} \cup K_{i_4}$ whose type on K_{i_4} is S_1 extends to a 3-coloring of $G_{i_1 i_4}$. Symmetrically, there exists a type S_2 such that every precoloring of $K_{i_5} \cup K_{i_8}$ whose type on K_{i_5} is S_2 extends to a 3-coloring of $G_{i_5 i_8}$.

Let $G' = G_{i_4 i_5}$ with rings $L_1 = K_{i_4} = a_1 a_2 a_3 a_4$ and $L_2 = K_{i_5} = b_1 b_2 b_3 b_4$. Since $i_5 \geq i_4 + 9$, the distance between L_1 and L_2 is at least three. Let G'' be the graph obtained from the embedding of G' in the cylinder in the following way: Cap the holes of the cylinder by disks. If $S_1 = \{a_i, a_{i+2}\}$ for some $i \in \{1, 2\}$, then add the edge $a_i a_{i+2}$ to the face bounded by L_1 and add a crosscap to the middle of this edge. If $S_1 = \emptyset$, then identify a_1 with a_3 to a vertex a_{13} and a_2 with a_4 to a vertex a_{24}. Observe that at most two vertices of L_1 are incident with a (≤ 4)-cycle distinct from L_1 in G', and if there are two such vertices, then they are adjacent. By symmetry, we can assume that L_1 is the only (≤ 4)-cycle incident with at least one of a_2 and a_3. We add a crosscap on the edge $a_{13} a_{24}$ and draw the edges from a_{13} to the neighbors of a_3 and the edges from a_{24} to the neighbors of a_2 through the crosscap. Transform L_2 in the same way according to S_2. Note that G'' is embedded in the Klein bottle and it has no loops.

Consider a cycle C of length at most 4 in G''. Since the distance between L_1 and L_2 is at least three, we may assume that C does not contain any of the vertices $b_1, \ldots, b_4$, b_{13} or b_{24}. Let us first consider the case that $S_1 = \{a_i, a_{i+2}\}$ for some $i \in \{1, 2\}$. If C does not contain the edge $a_i a_{i+2}$, then C is non-contractible in G, and thus it separates the crosscaps in G''. If C contains the edge $a_i a_{i+1}$, then C is one-sided. Suppose now that $S_1 = \emptyset$; as in the construction of G'', we assume that L_1 is the only (≤ 4)-cycle incident with at least one of a_2 and a_3 in G'. If C contains the edge $a_{13} a_{24}$, then C corresponds to a (≤ 4)-cycle in G' containing one of the edges of L_1, which necessarily must be a_1a_4; hence, no other edge of C passes through the crosscap and C is one-sided. If C contains neither a_{13} nor a_{24}, then C is non-contractible in G and separates the crosscaps in G''. If C contained both a_{13} and a_{24}, but not the edge $a_{13}a_{24}$, then since a_2 and a_3 are not incident with (≤ 4)-cycles in G', we conclude that a_1a_4 is incident with two triangles in G'. However, then either one of the triangles or the 4-cycle contained in their union is contractible in G, which is a contradiction. It remains to consider the case that C contains exactly one of a_{13} and a_{24}. By symmetry, assume that C contains a_{13}. Let e'_1 and e'_2 be the edges incident with a_{13} in C, and let e_1 and e_2 be the corresponding edges in G. Since no (≤ 4)-cycle different from L_1 is incident with a_3, we may assume that e_1 is incident with a_1. If e_2 is incident with a_3, then C is one-sided. If e_2 is incident with a_1, then C separates the crosscaps. We conclude that every (≤ 4)-cycle in G'' is non-contractible.

If G'' is 3-colorable, then the corresponding 3-coloring of G' has type S_1 on L_1 and type S_2 on L_2. It follows that every precoloring of $K_{i_1} \cup K_{i_8}$ extends to a 3-coloring of the subgraph $G_{i_1 i_8}$, contradicting the criticality of G.

Therefore, G'' is not 3-colorable and it contains a 4-critical subgraph F. By Theorem 4.3, F is a (x_1x_2, y_1y_2)-chain, for some vertices $x_1, x_2, y_1, y_2 \in V(G'')$, and its embedding derived from the embedding of G'' is canonical. Suppose that $S_1 = \emptyset$ and that L_1 is the only (≤ 4)-cycle in G' incident with a_2 and a_3. By symmetry, we can assume that $x_1 = a_{13}$ and x_1x_2 corresponds to an edge a_3v in G'. Since x_1x_2 is incident with two triangles $x_1x_2v_1$ and $x_1x_2v_2$ in F, but a_3 is not incident with a triangle, we have $a_1v_1, a_1v_2 \in E(G')$. Let us remark that $x_2 \neq a_{24}$, as otherwise we would similarly have $a_4v_1, a_4v_2 \in E(G')$ and at least one of the cycles $a_1a_4v_1$, $a_1a_4v_2$ and $a_1v_1a_4v_2$ would be contractible in G. Hence, both v_1 and v_2 are adjacent to the vertex v in G'. Since the 4-cycle $a_1v_1vv_2$ is non-contractible in G', using Theorem 4.1 we can assume that $a_1a_2a_3vv_1$ and $a_1a_4a_3vv_2$ are faces of G' and a_2 and a_4 have degree two. On the other hand, if $S_1 = \{a_i, a_{i+2}\}$ for some $i \in \{1, 2\}$, then one of x_1x_2, y_1y_2 is equal to a_ia_{i+2}, and since this edge is incident with two triangles in F, it follows that L_1 is a subgraph of F. A symmetrical claim holds at L_2. Theorem 4.1 implies that every face of F not incident with x_1x_2 and y_1y_2 is also a face of G'. Let us recall that F is a (x_1x_2, y_1y_2)-chain, and consequently observe that in all the cases, G' is a broken chain.

Choose the labeling of L_1 and L_2 so that a_1 and b_1 are vertices of degree four in G'. Observe that a precoloring ψ of $L_1 \cup L_2$ extends to a 3-coloring of G' if and only if $\psi(a_1) \neq \psi(a_3)$ or $\psi(b_1) \neq \psi(b_3)$. Let X_1 be the subgraph of G drawn between R_1 and K_{i_4} and let X_2 be the subgraph of G drawn between R_1 and K_{i_5}. Consider a precoloring ϕ of $R_1 \cup K_{i_5}$ that does not extend to a 3-coloring of X_2. Let X'_1 be the graph obtained from X_1 in the following way: first, we add the edge a_1a_3 and put a crosscap on it. If R_1 is a vertex ring or a triangle, then we paste a crosscap over the cuff incident with R_1. If R_1 is a 4-cycle, then we either add an edge between two of its vertices or identify its opposite vertices according to the type of ϕ on R_1 and put a crosscap in the appropriate place, using the same rules as in the construction of G''. Note that X'_1 is embedded in the Klein bottle so that all contractible cycles have length at least five. If X'_1 is 3-colorable, then its 3-coloring corresponds to a 3-coloring of X_1 that matches ϕ on R_1 and assigns a_1 and a_3 different colors. Hence, this coloring extends to a 3-coloring of X_2 that matches ϕ on $R_1 \cup K_{i_5}$, which is a contradiction.

Therefore, X_1' is not 3-colorable, and by Theorem 4.3, X_1' contains a canonical embedding of an (e_1, e_2)-chain F_1, for some vertices $e_1, e_2 \in E(X_1')$. Since F_1 contains four one-sided triangles, it follows that $|R_1| = 4$. However, as in the analysis of G', we conclude that then X_1 is a broken chain. By symmetry, the subgraph of G drawn between K_{i_5} and R_2 is a broken chain as well. This implies that G is a broken chain. □

The case of a cylinder with two rings of length at most four is now easy to handle using Theorem 3.3, thanks to the bound on the size of a subgraph that captures (≤ 4)-cycles given by Lemma 4.7. We will need the following observation.

Lemma 4.8. *Let G be an $\mathcal{R}$-critical graph embedded in a surface Σ with rings $\mathcal{R}$ so that every ($\leq$4)-cycle is non-contractible, let G' be another $\mathcal{R}$-critical graph embedded in Σ with rings $\mathcal{R}$ and let $X \subset F(G)$ and $\{(J_f, S_f) : f \in F(G')\}$ be a cover of G by faces of G'. Let f be an open 2-cell face of G' and let $G_1, \ldots, G_k$ be the components of the G-expansion of S_f, where for $1 \leq i \leq k$, G_i is embedded in the disk with one ring R_i. In this situation, $\sum_{i=1}^k w(G_i, \{R_i\}) \leq s(|f|) + \mathrm{el}(f)$.*

Proof. By Theorem 3.4 and Lemma 3.5, we have

$$\sum_{i=1}^{k} w(G_i, \{R_i\}) \leq \sum_{i=1}^{k} s(|R_i|).$$

Note that we have $s(x) + s(y) \leq s(x + y) \leq s(x) + y$ for every $x, y \geq 5$; hence,

$$\sum_{i=1}^{k} s(|R_i|) \leq s\left(\sum_{i=1}^{k} |R_i|\right) = s(|f| + \mathrm{el}(f)) \leq s(|f|) + \mathrm{el}(f).$$

□

Let cyl be a function satisfying the following for all non-negative integers x and y:

- $\mathrm{cyl}(0, 0) = 0$
- $\mathrm{cyl}(x, y) = \mathrm{cyl}(y, x)$
- if $x > 0$, then $\mathrm{cyl}(x, y) \geq \mathrm{cyl}(0, y) + x + 13$
- if $x, y > 1$, then $\mathrm{cyl}(x, y) \geq \mathrm{cyl}(1, x) + \mathrm{cyl}(1, y) + 19$
- for any non-negative integer $y' < y$, we have

  $$\mathrm{cyl}(x, y) \geq \mathrm{cyl}(x, y') + s(y - y' + 8) \geq \mathrm{cyl}(x, y') + 1$$

- $\mathrm{cyl}(x, y) \geq s(x + y + 11)$
- if $x \geq 4$, then $\mathrm{cyl}(x, y) \geq 886$
- $2\mathrm{cyl}(6, 7) \leq \mathrm{cyl}(7, 7)$
- if $x \leq 4$ and $5 \leq y \leq 6$, then

  $$\mathrm{cyl}(x, y) \geq (2/3 + 52\epsilon)(x + y) + 20(40 + 5\mathrm{cyl}(4, 4)/s(5) + 692)/3$$

- if $x \le 7$, then $\mathrm{cyl}(x, 7) \ge 3/2(x + 7) + 20(60 + 5\mathrm{cyl}(6, 6)/s(5) + 692)/3$
- if $x, y \ge 5$, then $\mathrm{cyl}(x, y) \ge \mathrm{cyl}(4, x) + \mathrm{cyl}(4, y) + \mathrm{cyl}(4, 4)$

We are now ready to prove the main result of this section.

Theorem 4.9. *Let G be a graph embedded in the cylinder with rings R_1 and R_2 of length at most four, where R_2 is a facial ring. Suppose that every (≤ 4)-cycle in G is non-contractible. If G is $\{R_1, R_2\}$-critical and not a broken chain, then $w(G, \{R_1, R_2\}) \le \mathrm{cyl}(|R_1|, |R_2|)$.*

Proof. We proceed by induction, and assume that the claim holds for all graphs with fewer than $|E(G)|$ edges. By Lemma 4.6, we can assume that $|R_2| = 4$. By Theorem 4.5, G is connected. By Lemma 3.1, G satisfies (I0), (I1) and (I2). Furthermore, we already observed that every critical graph satisfies (I9), and (I6) and (I8) hold trivially.

If (I3) is false, then G contains a cutvertex v. Observe that v separates R_1 from R_2. Cut the cylinder along a non-contractible simple closed curve intersecting G only in v, and let G_1 and G_2 be the pieces of G drawn in the resulting cylinders, where $R_1 \subseteq G_1$ and $R_2 \subseteq G_2$. If $G_1 = R_1$, then note that R_1 is a facial ring, as v is a cutvertex. The graph G_2 is $\{v, R_2\}$-critical, hence by induction $w(G_2, \{v, R_2\}) \le \mathrm{cyl}(1, |R_2|)$. But then $w(G, \{R_1, R_2\}) \le w(G_2, \{v, R_2\}) + |R_1| \le \mathrm{cyl}(1, |R_2|) + |R_1| \le \mathrm{cyl}(|R_1|, |R_2|)$ as required.

On the other hand, if $G_1 \ne R_1$, then G_1 is $\{R_1, v\}$-critical. By Lemma 4.6, we have $|R_1| \ge 1$. Furthermore, if $|R_1| \le 3$, then G_1 consists of R_1 and an edge between v and a vertex w of R_1. If that is the case, then G_2 is $\{v, R_2\}$-critical, where v is taken as a weak vertex ring: consider any edge e of G_2 not belonging to R_2. Since G is $\{R_1, R_2\}$-critical, there exists a precoloring ϕ of R_1 and R_2 that extends to a coloring of $G - e$, but not to G. Let ψ be the precoloring of v and R_2 such that that $\psi(v) = \phi(w)$ and ψ matches ϕ on R_2. Note that ψ extends to a coloring of $G_2 - e$, but not to G_2. Since the choice of e was arbitrary, this shows that G_2 is $\{v, R_2\}$-critical with v weak. By induction, we have $w(G_2, \{v, R_2\}) \le \mathrm{cyl}(0, |R_2|)$. Hence, if f is the cuff face of v in G_2, then $w(G, \{R_1, R_2\}) \le \mathrm{cyl}(0, |R_2|) + s(|f| + 2 + |R_1|) - s(|f|) \le \mathrm{cyl}(0, |R_2|) + 2 + |R_1| \le \mathrm{cyl}(|R_1|, |R_2|)$.

Finally, if $|R_1| = 4$, then we can assume that v is not a ring vertex (let us recall that $|R_2| = 4$), and thus G_i is $\{R_i, v\}$-critical for $i \in \{1, 2\}$ (v is taken as a non-weak vertex ring). Let f_1 and f_2 be the cuff faces of G_1 and G_2, respectively, incident with v. By induction, we have $w(G, \{R_1, R_2\}) \le 2\mathrm{cyl}(1, 4) + s(|f_1| + |f_2|) - s(|f_1|) - s(|f_2|) \le 2\mathrm{cyl}(1, 4) + 8 \le \mathrm{cyl}(4, 4)$. Therefore, we can assume that (I3) holds.

If (I5) is false, then since (I3) holds, we can assume that the two vertices r_1 and r_2 belong to a ring of length four, say to R_2. If the internal face incident with r_1r_2 has length five, then r_1r_2 is a part of a triangle T separating R_1 from R_2. By applying induction to the subgraph of G drawn between R_1 and T, we conclude that $w(G, \{R_1, R_2\}) \le \mathrm{cyl}(|R_1|, 3) + s(5) < \mathrm{cyl}(|R_1|, |R_2|)$. Otherwise, we apply induction to the graph obtained by contracting the edge r_1r_2, and obtain $w(G, \{R_1, R_2\}) \le \mathrm{cyl}(|R_1|, 3) + 1 \le \mathrm{cyl}(|R_1|, |R_2|)$. Hence, assume that (I5) holds.

Suppose now that the distance between R_1 and R_2 is at most four. If R_1 is a vertex ring, then add a triangle R_1' forming a boundary of the cuff incident with

R_1 and note that this operation does not decrease the weight of G; otherwise, let $R_1' = R_1$. Next, we use Lemma 3.5 to the closed walk consisting of R_1', R_2 and the shortest path between R_1 and R_2 traversed twice. By Theorem 3.4, we have $w(G, \{R_1, R_2\}) \le s((|R_1| + 3) + |R_2| + 8) \le \mathrm{cyl}(|R_1|, |R_2|)$. Therefore, assume that (I7) holds.

Consider a path P of length at most four with both ends being ring vertices. By the previous paragraph, both ends of P belong to the same ring R. By (I5), P has length at least three. Since G is embedded in the cylinder, there exists a subpath Q of R such that $P \cup Q$ is a contractible cycle. Note that $|P \cup Q| \le |P| + 3 \le 7$, and by Theorem 4.1, $P \cup Q$ bounds a face. Therefore, G is well-behaved and satisfies (I4).

Let M be the subgraph of G consisting of edges incident with (≤ 4)-cycles. Since G is not a broken chain, Lemma 4.7 implies that $|E(M)| \le 132$. Note that M captures (≤ 4)-cycles of G. If the assumptions of Theorem 3.3 are not satisfied, then $w(G, \{R_1, R_2\}) \le (2/3 + 26\epsilon)\ell(\{R_1, R_2\}) + 20|E(M)|/3 < 886 \le \mathrm{cyl}(|R_1|, |R_2|)$. Therefore, assume the contrary.

Then, there exists an $\{R_1, R_2\}$-critical graph G' embedded in the cylinder with rings R_1 and R_2 such that $|E(G')| < |E(G)|$, satisfying conditions (a)–(e) of Theorem 3.3. By (b), all (≤ 4)-cycles in G' are non-contractible. By Theorem 4.5, G' is connected, and thus all its faces are open 2-cell. Let $X \subset F(G)$ and $\{(J_f, S_f) : f \in F(G')\}$ be the cover of G by faces of G' as in (d). For $f \in F(G')$, let $G_1^f, \ldots, G_{k_f}^f$ be the components of the G-expansion of S_f. Since Σ_f is a disk and all surfaces of the G-expansion of S_f are fragments of Σ_f, it follows that for $1 \le i \le k_f$, G_i^f is embedded in the disk with one ring R_i^f. By the definition of a cover of G by faces of G', we have

$$w(G, \{R_1, R_2\}) = \sum_{f \in F(G)} w(f) = \sum_{f \in X} w(f) + \sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \{R_i^f\}).$$

Suppose first that G' is 2-connected, and thus its embedding is closed 2-cell. By Theorem 3.3(c), G' has a face of length at least 6, hence G' is not a broken chain. Therefore, by induction we have $w(G', \{R_1, R_2\}) \le \mathrm{cyl}(|R_1|, |R_2|)$. Since each internal face of G' is closed 2-cell, Theorem 3.3(e) implies that

$$\sum_{i=1}^{k_f} w(G_i^f, \{R_i^f\}) \le s(|f|) - c(f)$$

for every $f \in F(G')$, and consequently (using Theorem 3.3(d) in the last inequality), we have

$$\begin{aligned}
\sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \{R_i^f\}) &\le \sum_{f \in F(G')} s(|f|) - c(f) \\
&= w(G', \{R_1, R_2\}) - \sum_{f \in F(G')} c(f) \\
&\le w(G', \{R_1, R_2\}) - |X| s(6).
\end{aligned}$$

Putting the inequalities together, we obtain

$$\begin{aligned} w(G, \{R_1, R_2\}) &\le w(G', \{R_1, R_2\}) + \left(\sum_{f\in X} w(f)\right) - |X|s(6) \\ &= w(G', \{R_1, R_2\}) \le \mathrm{cyl}(|R_1|, |R_2|), \end{aligned}$$

since the face in X (if any) has length 6 by the definition of a cover of G by faces of G'.

Finally, let us consider the case that G' is not 2-connected. Let v be a cutvertex in G' and let G_1 and G_2 be the subgraphs of G' intersecting in v such that $G' = G_1 \cup G_2$, $R_1 \subseteq G_1$ and $R_2 \subseteq G_2$. Similarly to the analysis of the property (I3) for G, we show that $|R_1| \ge 1$, if $|R_1| \le 3$, then $w(G', \{R_1, R_2\}) \le \mathrm{cyl}(0, |R_2|) + 2 + |R_1| \le \mathrm{cyl}(|R_1|, |R_2|) - 11$, and if $|R_1| = 4$, then $w(G', \{R_1, R_2\}) \le 2\mathrm{cyl}(1, 4) + 8 \le \mathrm{cyl}(4, 4) - 11$. By Lemma 4.8, we have

$$\sum_{f\in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \{R_i^f\}) \le w(G', \{R_1, R_2\}) + \sum_{f\in F(G')} \mathrm{el}(f) \le w(G', \{R_1, R_2\}) + 10.$$

Combining the inequalities, we have

$$\begin{aligned} w(G, \{R_1, R_2\}) &\le w(G', \{R_1, R_2\}) + 10 + \sum_{f\in X} w(f) \\ &\le \mathrm{cyl}(|R_1|, |R_2|) - 1 + \sum_{f\in X} w(f) \\ &< \mathrm{cyl}(|R_1|, |R_2|). \end{aligned}$$

□

5 Narrow cylinder

In this section, we consider graphs embedded in the cylinder with two rings of length at most 7. First, let us state an auxiliary result that will also be useful in the case of general surfaces. Consider a graph embedded in a surface Σ. If K_1 and K_2 are two cycles surrounding a cuff C and Δ_1 and Δ_2 are the open disks bounded by K_1 and K_2, respectively, in $\Sigma + \hat{C}$, then we say that K_1 and K_2 are *incomparable* if $\Delta_1 \not\subseteq \Delta_2$ and $\Delta_2 \not\subseteq \Delta_1$.

Lemma 5.1. *Let G be a graph in a surface Σ with rings $\mathcal{R}$, such that G is $\mathcal{R}$-critical and every (≤ 4)-cycle is non-contractible. Let K_0 be a cycle in G of length at most seven surrounding a ring R, let C be the cuff incident with R and let Δ be the closed disk in $\Sigma + \hat{C}$ bounded by K_0. In this situation, at most $10|K_0|$ edges of G drawn outside of Δ are incident with (≤ 7)-cycles surrounding R that are incomparable with K_0.*

Proof. Let X be the set of edges drawn outside of Δ that are incident with (≤ 7)-cycles surrounding R and incomparable with K_0. Initially, we give each edge of X charge 1 and all the edges of K_0 charge 0. Next, we aim to move the charge from X to K_0.

For an edge $x \in X$, choose a (≤ 7)-cycle K surrounding R incomparable with K_0 and containing x. Note that at least one edge of $E(K) \setminus E(K_0)$ is drawn in Δ. Let $K = P_1 \cup P_2$, where P_1 and P_2 are paths intersecting only in their endvertices such that P_1 has ends in K_0 and otherwise is drawn in the interior of Δ. Let $K_0 = P_3 \cup P_4$, where P_3 and P_4 are paths sharing endvertices with P_1 and P_2 and the cycle $P_1 \cup P_3$ surrounds R. For $1 \leq i \leq 4$, let m_i be the length of P_i. Since all (≤ 4)-cycles are non-contractible, we have $m_2 + m_3 \geq 5$ and $m_1 + m_4 \geq 5$. Since $m_1 + m_2 + m_3 + m_4 = |K_0| + |K| \leq 14$, it follows that $m_2 + m_3 \leq 9$ and $m_1 + m_4 \leq 9$. Note that the closed walk $P_2 \cup P_3$ is contractible. Let K' be the symmetric difference of P_2 and P_3. Note that K' is a union of a contractible cycles, and since every contractible cycle has length at least five, it follows that K' has only one component. We distribute the charge of x among the edges of $K' \cap K_0$ evenly.

Let us consider the case that K' does not bound a face of G. By Theorem 4.1, it follows that $|K'| \geq 8$ and at most one vertex is contained in the open disk bounded by K'. Since $m_2 + m_3 \leq 9$, we have that P_2 and P_3 are edge-disjoint and $K' = P_2 \cup P_3$. Since $m_1 \geq 1$, we have $m_2 \leq 6$ and thus $m_3 \geq 2$. Suppose first that only one edge e is drawn in the open disk bounded by K'. Let f_1 and f_2 be the faces of G in this disk, and assume that f_1 shares no edge with P_3. In this situation, we say that the charge sent from the edges of $f_1 \cap K$ to the edges of P_3 passes through e. Note that if $m_3 = 2$, then $m_1 = 1$, $m_2 = 6$, $|K'| = 8$ and $|f_1| = |f_2| = 5$, thus in this case 4 units of charge pass through e; otherwise, at most 5 units of charge pass through e.

Suppose now that a vertex is contained in the open disk bounded by K', and let f_1, f_2 and f_3 be the faces of G in this disk. Note that $|f_1| = |f_2| = |f_3| = 5$ and $m_3 \geq 3$. For $1 \leq i < j \leq 3$, let e_{ij} be the common edge of f_i and f_j. Let us first consider the case that two of these faces, say f_2 and f_3, share an edge with P_3. We let the amount of charge of the edges of $f_1 \cap K$ proportional to $|E(f_i) \cap E(P_3)|$ pass through e_{1i}, for $i \in \{2, 3\}$. The other possible case is that only one face, say f_3, shares edges with P_3; in this situation, the charge of $f_1 \cap K$ passes through e_{13} and the charge of $f_2 \cap K$ passes through e_{23}.

Let us now analyze the final charge of the edges of K_0. Consider $e \in K_0$, let f be the face of G incident with e that is not a subset of Δ and let $m = |E(f) \cap E(K_0)|$. If $|f| \geq 10$, then no charge is sent to e. If $7 \leq |f| \leq 9$, then no charge passes through edges of f, and thus e has charge at most 8. Similarly, if $|f| = 6$ and $m = 1$, then no charge passes through edges of f, and thus e has charge at most 5. If $|f| = 6$ and $m \geq 2$, then at most four units pass through each edge of $E(f) \setminus E(K_0)$ and each of these edges additionally sends one unit of charge. This charge is evenly divided between the edges of $E(f) \cap E(K_0)$, hence e has charge at most 10. If $|f| = 5$ and $m = 1$, then at most one unit passes through each edge of $E(f) \setminus E(K_0)$, and e has charge at most 8. If $|f| = 5$ and $m \geq 2$, then at most five units of charge pass through each edge of $E(f) \setminus E(K_0)$ and each of these edges additionally sends one unit of charge. This charge is evenly divided between the edges of $E(f) \cap E(K_0)$, and thus e has charge at most 9.

Therefore, each edge of K_0 has charge at most 10, and $|X| \leq 10|K_0|$. □

We will also need several related claims regarding cycles near a ring.

Lemma 5.2. *Let G be a graph embedded in the cylinder with rings R_1 and R_2, such that G is $\{R_1, R_2\}$-critical and every (≤ 4)-cycle is non-contractible.*

If $|R_1| = 4$, then at most 93 edges of G are incident with a non-contractible (≤ 7)-cycle that shares a vertex with R_1.

Proof. Let C_1 be the cuff incident with R_1. Let K_0 be a non-contractible (≤ 7)-cycle that shares a vertex with R_1 such that the closed disk Δ bounded by K_0 in $\Sigma + \hat{C}_1$ is as large as possible. Observe that every edge of G incident with a non-contractible (≤ 7)-cycle that shares a vertex with R_1 is either drawn in Δ or is incident with a non-contractible (≤ 7)-cycle that is incomparable with K_0. Theorem 4.1 implies that at most 23 edges of G are drawn in Δ. Together with Lemma 5.1, the claim follows. $\square$

If R is a 6-cycle and C is a (≤ 6)-cycle in a graph G, then we say that C is *bound* to R if either $|V(C) \cap V(R)| \geq 3$ or G contains edges cr and $c'r'$ with $c, c' \in V(C) \setminus V(R)$ and $r, r' \in V(R) \setminus V(C)$ such that r and r' are non-adjacent.

Lemma 5.3. *Let G be a graph embedded in the cylinder with rings R_1 and R_2, such that every (≤ 4)-cycle is non-contractible. Suppose that $|R_1| = 6$ and R_1 is an induced cycle, and let $X \subseteq E(G)$ be the set of edges incident with non-contractible (≤ 6)-cycles bound to R_1. If G is $\{R_1, R_2\}$-critical, then $|X| \leq 346$.*

Proof. Let C be a non-contractible (≤ 6)-cycle in G that is bound to R_1. If $|V(C) \cap V(R_1)| \geq 3$, then for every edge $e \in E(C) \setminus E(R_1)$, there exists a contractible cycle K in $C \cup R_1$ of length at most 8 that contains e, sharing at least $|K| - 4$ edges with R_1.

If G contains edges cr and $c'r'$ with $c, c' \in V(C) \setminus V(R)$ and $r, r' \in V(R) \setminus V(C)$ such that r and r' are non-adjacent, then G contains two contractible cycles K_1 and K_2 bounding disjoint open disks, such that $E(C) \subset E(K_1) \cup E(K_2)$ and $|K_1| + |K_2| = |R_1| + |C| + 4 \leq 16$. Furthermore, for $i \in \{1, 2\}$, the cycle K_i shares at least $\max(2, |K_i| - 7)$ edges with R_1. Since $|K_{3-i}| \geq 5$, we have $|K_i| \leq 11$.

By Theorem 4.1, we conclude that each edge $e \in X \setminus E(R_1)$ satisfies at least one of the following claims:

1. e is incident with a face f_1 of length at most 11 sharing at least two edges with R_1, or
2. e is incident with a face f_1 of length at most 8 sharing an edge with R_1, or
3. there exist faces f_1 and f_2 sharing an edge such that e is incident with f_2, f_1 shares $k \in \{2, 3, 4\}$ edges with R_1 and $|f_1| + |f_2| \leq 9 + k$, or
4. there exist faces f_1, f_2 and f_3 of length five such that e is incident with f_3, f_1 shares at least 3 edges with R_1 and f_2 shares an edge both with f_1 and with f_3, or
5. there exist faces f_1, f_1' and f_2 such that e is incident with f_2, f_2 shares an edge with both f_1 and f_1', each of f_1 and f_1' shares $k \in \{1, 2\}$ edges with R_1, $|f_1| = |f_1'| = 5$ and $|f_2| = 3 + 2k$, or
6. there exist faces f_1 and f_2 with $|f_1|, |f_2| \leq 6$ such that e is incident with f_2, f_1 shares at least $|f_1| - 3$ edges with R_1, and there exists a path P consisting of at most four internal vertices of degree three which joins a vertex incident with f_1 with a vertex incident with f_2 and not incident with e.

In each of the cases, we say that the edges of $E(f_1 \cup f_1') \cap E(R_1)$ *see* the edge e. For each $e \in X \setminus E(R_1)$, let $n(e)$ denote the number of edges of R_1 that see it. Consider an edge $e' \in E(R_1)$ and let

$$N(e') = \sum_{e' \text{ sees } e} \frac{1}{n(e)}.$$

Let f_1 be the internal face incident with e'. If $|f_1| > 11$, then $N(e') = 0$. If $9 \le |f_1| \le 11$, then e' sees at most 9 edges by the first claim, which are also seen by another edge of R_1, and thus $N(e') \le 9/2$. Suppose that $|f_1| \le 8$. If f_1 shares $k \in \{2, 3, 4\}$ edges with R_1, then it sees at most $(|f_1| - k)(9 + k - |f_1|)$ edges satisfying the first three claims, and each of those is seen by at least k other edges; hence, these edges contribute at most $(|f_1|-k)(9+k-|f_1|)/k \le 10$ to $N(e')$. Therefore, if $|f_1| \ge 7$, then $N(e') \le 10$.

Suppose that $|f_1| \le 6$ and consider the edges seen by e' and satisfying the last claim. There are at most two choices for the first vertex of P, the second vertex is uniquely determined and there are two choices for each of the third and fourth vertex, giving at most $2 + 2 + 4 + 8 = 16$ possibilities for the last vertex of P and at most 64 edges seen by e'. Since $|f_1|$ shares at least 2 edges with R_1, this contributes at most 32 to $N(e')$.

Finally, if $|f_1| = 5$, then the edges satisfying the fourth and the fifth claim contribute at most $32/3 + 4$ to $N(e')$. Consequently, we have $N(e') \le 170/3$. The sum of $N(e')$ over all edges of R_1 gives an upper bound on $|X \setminus E(R_1)|$, hence $|X| \le 346$. □

Similarly, we can prove the following.

Lemma 5.4. *Let G be a graph embedded in the cylinder with rings R_1 and R_2, such that every (≤ 4)-cycle is non-contractible. Suppose that $|R_1| = 7$ and R_1 is an induced cycle, and let $X \subseteq E(G)$ be the set of edges incident with non-contractible 7-cycles that share at least four vertices with R_1. If G is $\{R_1, R_2\}$-critical, then $|X| \le 35$.*

The first paper of this series [5] together with the results of Gimbel and Thomassen [9] and Aksenov et al. [1] implies implies the following.

Theorem 5.5. *Let G be a graph embedded in the cylinder with one ring R of length at most 7 (the other cuff does not correspond to a ring). Suppose that all (≤ 4)-cycles in G are non-contractible and that G has girth at least $|R| - 3$. If G is R-critical and R is an induced cycle, then $|R| = 6$ and G contains a triangle C such that all vertices of C are internal and have mutually distinct and non-adjacent neighbors in R.*

Finally, we can prove the main result of this section.

Lemma 5.6. *Let G be a graph embedded in the cylinder with rings R_1 and R_2, where $|R_1| \le |R_2|$ and $5 \le |R_2| \le 7$. Suppose that every (≤ 4)-cycle in G is non-contractible. Furthermore, assume that the following conditions hold:*

- *if $|R_1| = 4$, then all other 4-cycles in G are vertex-disjoint with R_1,*
- *if $|R_1| \ge 5$, then G contains no (≤ 4)-cycle,*

- *if $|R_2| = 7$, then G contains no triangle distinct from R_1, and R_2 is an induced cycle, and*
- *if $|R_2| = 6$, then R_2 is an induced cycle and G contains no triangle T such that all vertices of T are internal and they have mutually distinct and non-adjacent neighbors in R_2.*

If G is $\{R_1, R_2\}$-critical, then $w(G, \{R_1, R_2\}) \le \mathrm{cyl}(|R_1|, |R_2|)$.

Proof. By induction, we can assume that the claim holds for all graphs with fewer than $|E(G)|$ edges. By Theorems 4.5 and 5.5, G is connected. Note that G satisfies (I0), (I1), (I2), (I6), (I8) and (I9). The cases that G is not 2-connected or that the distance between R_1 and R_2 is at most four are dealt with in the same way as in the proof of Theorem 4.9, hence assume that (I3) and (I7) hold.

If P is a path of length at most four with both ends being ring vertices, then both ends belong to the same ring R_i for some $i \in \{1, 2\}$. Since G is embedded in the cylinder, there exists a subpath Q of R_i such that $P \cup Q$ is a contractible cycle. Let us consider the case that $|Q| > |P|$, and let Q' be the path with edge set $E(R_i) \setminus E(Q)$. Note that $Q' \cup P$ is a non-contractible cycle shorter than $|R_i|$. We apply induction to the subgraph of G between R_{3-i} and $Q' \cup P$. Furthermore, we use Theorem 3.4 to bound the weight of the subgraph embedded in the disk bounded by $Q \cup P$. We conclude that $w(G, \{R_1, R_2\}) \le \mathrm{cyl}(|R_{3-i}|, |Q' \cup P|) + s(|Q \cup P|) = \mathrm{cyl}(|R_{3-i}|, |Q' \cup P|) + s(|R_i| - |Q' \cup P| + 2|P|) \le \mathrm{cyl}(|R_{3-i}|, |R_i|)$, since $2|P| \le 8$.

Therefore, we can assume that $|Q| \le |P|$ for each such path P. This implies that (I4) holds. Furthermore, $|P \cup Q| \le 8$, and by Theorem 4.1, at most two faces of G are in the disk bounded by $P \cup Q$.

Suppose that (I5) is false, and the ring R_i for some $i \in \{1, 2\}$ contains adjacent vertices r_1 and r_2 of degree two. By (I3), we have $|R_i| \ge 4$, and by the previous paragraph, the internal face incident with r_1r_2 has length at least 6. We apply induction to the graph obtained by contracting the edge r_1r_2, and obtain $w(G, \{R_1, R_2\}) \le \mathrm{cyl}(|R_{3-i}|, |R_i| - 1) + 1 \le \mathrm{cyl}(|R_1|, |R_2|)$. Hence, assume that (I5) holds. Together with the observations from the previous paragraph, this implies that G is well-behaved.

If $|R_1| = |R_2| = 7$ and G contains a non-contractible (≤ 6)-cycle, then by induction we have $w(G, \{R_1, R_2\}) \le 2\mathrm{cyl}(6, 7) \le \mathrm{cyl}(7, 7)$, hence we can assume that if $|R_1| = |R_2| = 7$, then all non-contractible cycles have length at least seven.

Let $k = 6$ if $|R_2| = 7$ and $k = 4$ otherwise. Let M be the subgraph of G containing

- edges incident with non-contractible ($\le k$)-cycles,
- if $|R_1| = 4$, then also include edges incident with non-contractible (≤ 7)-cycles sharing a vertex with R_1,
- if $|R_i| = 6$ for some $i \in \{1, 2\}$, then include all edges of non-contractible ($\le$6)-cycles bound to R_i, and
- if $|R_i| = 7$ for some $i \in \{1, 2\}$, then include all edges of non-contractible 7-cycles that share at least four vertices with R_i.

Let us bound the number of edges of M. Note that if $|R_2| < 7$, then $k = 4$ and by the assumptions, if G contains a $(\leq k)$-cycle, then $|R_1| \leq 4$. Suppose that there exists a non-contractible $(\leq k)$-cycle C, and choose C so that the closed subset Σ' of Σ between R_1 and C is as large as possible. Note that $|R_1| \leq k < |R_2|$. If $|R_1| = k = 4$, then no 4-cycle distinct from R_1 contains a vertex of $V(R_1)$, hence the subgraph G' of G embedded in Σ' is not a broken chain. All non-contractible $(\leq k)$-cycles in G are either drawn in Σ' or are incomparable with C. By Theorem 3.4 (when R_1 and C intersect), Theorem 4.9 (when R_1 and C are vertex-disjoint and $|C| \leq 4$) or by induction (when R_1 and C are vertex-disjoint and $5 \leq |C| \leq 6$), we conclude that the total weight of the internal faces of G' is at most $\max(\mathrm{cyl}(k, k), s(2k)) = \mathrm{cyl}(k, k)$. In particular, at most $5\mathrm{cyl}(k, k)/s(5)$ edges of G are drawn in Σ', and by Lemma 5.1, at most $10k + 5\mathrm{cyl}(k, k)/s(5)$ edges of G are incident with non-contractible $(\leq k)$-cycles. By Lemmas 5.2, 5.3 and 5.4, we have $|E(M)| \leq 10k + 5\mathrm{cyl}(k, k)/s(5) + 692$.

Note that M captures (≤ 4)-cycles of G, and that $(2/3 + 26\epsilon)(|R_1| + |R_2|) + 20(10k+5\mathrm{cyl}(k, k)/s(5)+692)/3 \leq \mathrm{cyl}(|R_1|, |R_2|)$; therefore, we can assume that we can apply Theorem 3.3. Let G' be the $\{R_1, R_2\}$-critical graph embedded in the cylinder with rings R_1 and R_2 such that $|E(G')| < |E(G)|$, satisfying the conditions of Theorem 3.3. In particular, (b) implies that all (≤ 4)-cycles in G' are non-contractible; and furthermore, using the choice of M we have

- if $|R_2| = 7$, then G' contains no triangle distinct from R_1,
- if $|R_1| = 4$, then no 4-cycle in G' distinct from R_1 contains a vertex of $V(R_1)$,
- if $6 \leq |R_i| \leq 7$ for some $i \in \{1, 2\}$, then R_i is an induced cycle in G', and
- if $|R_i| = 6$ for some $i \in \{1, 2\}$, then G' contains no triangle T such that all vertices of R_i are internal and have non-adjacent neighbors in R_i.

These constraints enable us to apply Theorem 5.5 to show that G' is connected. It follows that all its faces are open 2-cell. Furthermore, the assumptions on non-contractible cycles from the statement of Lemma 5.6 are satisfied for G', except that G' can contain non-contractible (≤ 4)-cycles even if $|R_1| \geq 5$.

Let $X \subset F(G)$ and $\{(J_f, S_f) : f \in F(G')\}$ be the cover of G by faces of G' as in Theorem 3.3(d). For $f \in F(G')$, let $G_1^f, \ldots, G_{k_f}^f$ be the components of the G-expansion of S_f, where for $1 \leq i \leq k_f$, G_i^f is embedded in the disk with one ring R_i^f. We have

$$w(G, \{R_1, R_2\}) = \sum_{f \in F(G)} w(f) = \sum_{f \in X} w(f) + \sum_{f \in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \{R_i^f\}).$$

The case that G' is not 2-connected is dealt with in the same way as in the proof of Theorem 4.9; hence, assume that G is 2-connected. If G' does not satisfy the assumptions of Lemma 5.6, then $|R_1| \geq 5$ and G' contains a (≤ 4)-cycle. Let C_1 and C_2 be the (≤ 4)-cycles in G' such that the closed subset $\Sigma' \subseteq \Sigma$ between C_1 and C_2 is as large as possible, and observe that all (≤ 4)-cycles in G' belong to the subgraph G_c of G' embedded in Σ'. By Theorem 3.3(a), if G_c is a broken chain, then it has at most four internal faces. Therefore, Theorem 4.9 implies that the total weight of the internal faces of G_c is at most $\mathrm{cyl}(4, 4)$. Applying induction to the subgraphs of G' between R_1 and C_1 and between R_2 and C_2, we have $w(G', \{R_1, R_2\}) \leq \mathrm{cyl}(4, |R_1|) + \mathrm{cyl}(4, |R_2|) + \mathrm{cyl}(4, 4) \leq \mathrm{cyl}(|R_1|, |R_2|)$.

If G' satisfies the assumptions of Lemma 5.6, then the same inequality $w(G', \{R_1, R_2\}) \leq \mathrm{cyl}(|R_1|, |R_2|)$ follows by induction. Since each face of G' is closed 2-cell, we conclude that $w(G, \{R_1, R_2\}) \leq w(G', \{R_1, R_2\}) \leq \mathrm{cyl}(|R_1|, |R_2|)$ as in the proof of Theorem 4.9. $\square$

6 Graphs on surfaces

Let $\mathrm{gen}(g, t, t_0, t_1)$ be a function defined for non-negative integers g, t, t_0 and t_1 such that $t \geq t_0 + t_1$ as

$$\mathrm{gen}(g, t, t_0, t_1) = 120g + 48t - 4t_1 - 5t_0 - 120.$$

Let $\mathrm{surf}(g, t, t_0, t_1)$ be a function defined for non-negative integers g, t, t_0 and t_1 such that $t \geq t_0 + t_1$ as

- $\mathrm{surf}(g, t, t_0, t_1) = \mathrm{gen}(g, t, t_0, t_1) + 116 - 42t = 8 - 4t_1 - 5t_0$ if $g = 0$ and $t = t_0 + t_1 = 2$,
- $\mathrm{surf}(g, t, t_0, t_1) = \mathrm{gen}(g, t, t_0, t_1) + 114 - 42t = 6t - 4t_1 - 5t_0 - 6$ if $g = 0$, $t \leq 2$ and $t_0 + t_1 < 2$, and
- $\mathrm{surf}(g, t, t_0, t_1) = \mathrm{gen}(g, t, t_0, t_1)$ otherwise.

We will need the following properties of the function surf:

Lemma 6.1. *If g, g', t, t_0, t_1, t'_0, t'_1 are non-negative integers, then the following holds:*

(a) *Assume that if $g = 0$ and $t \leq 2$, then $t_0 + t_1 < t$. If $t \geq 2$, $t'_0 \leq t_0$, $t'_1 \leq t_1$ and $t'_0 + t'_1 \geq t_0 + t_1 - 2$, then $\mathrm{surf}(g, t-1, t'_0, t'_1) < \mathrm{surf}(g, t, t_0, t_1)$.*

(b) *If $g' < g$ and either $g' > 0$ or $t \geq 2$, then $\mathrm{surf}(g', t, t_0, t_1) \leq \mathrm{surf}(g, t, t_0, t_1) - 120(g - g') + 32$.*

(c) *Let g'', t', t'', t''_0 and t''_1 be nonnegative integers satisfying $g = g' + g''$, $t = t' + t''$, $t_0 = t'_0 + t''_0$, $t_1 = t'_1 + t''_1$, either $g'' > 0$ or $t'' \geq 1$, and either $g' > 0$ or $t' \geq 2$. Then, $\mathrm{surf}(g', t', t'_0, t'_1) + \mathrm{surf}(g'', t'', t''_0, t''_1) \leq \mathrm{surf}(g, t, t_0, t_1) - \delta$, where $\delta = 16$ if $g'' = 0$ and $t'' = 1$ and $\delta = 56$ otherwise.*

(d) *If $g \geq 2$, then $\mathrm{surf}(g - 2, t, t_0, t_1) \leq \mathrm{surf}(g, t, t_0, t_1) - 124$*

Proof. Let us consider the claims separately.

(a) If $g = 0$ and $t = 2$, then $\mathrm{surf}(g, t, t_0, t_1) \geq 1$, while $\mathrm{surf}(g, t-1, t'_0, t'_1) = 0$. If $g = 0$ and $t = 3$, then $\mathrm{surf}(g, t, t_0, t_1) \geq 9$ and $\mathrm{surf}(g, t-1, t'_0, t'_1) \leq 6$. Finally, if $g > 0$ or $t > 3$, then $\mathrm{surf}(g, t, t_0, t_1) = \mathrm{gen}(g, t, t_0, t_1)$ and $\mathrm{surf}(g, t-1, t'_0, t'_1) = \mathrm{gen}(g, t-1, t'_0, t'_1)$, and $\mathrm{gen}(g, t, t_0, t_1) - \mathrm{gen}(g, t-1, t'_0, t'_1) = 48 - 5(t_0 - t'_0) - 4(t_0 - t'_0) \geq 48 - 5(t_0 + t_1 - t'_0 - t'_1) \geq 38$.

(b) If $g' > 0$ or $t > 2$, then $\mathrm{surf}(g', t, t_0, t_1) = \mathrm{gen}(g', t, t_0, t_1)$ and we have $\mathrm{surf}(g', t, t_0, t_1) = \mathrm{surf}(g, t, t_0, t_1) - 120(g - g')$. If $g' = 0$ and $t = 2$, then $\mathrm{surf}(g', t, t_0, t_1) - \mathrm{surf}(g, t, t_0, t_1) + 120(g - g') \leq 116 - 42t = 32$.

(c) Suppose first that $g'' = 0$ and $t'' = 1$, i.e., we have $g = g'$ and $t = t' + 1$. If $g > 0$, then $\mathrm{surf}(g, t, t_0, t_1) - \mathrm{surf}(g'', t'', t''_0, t''_1) - \mathrm{surf}(g', t', t'_0, t'_1) = \mathrm{gen}(g, t, t_0, t_1) + (4t''_1 + 5t''_0) - \mathrm{gen}(g', t', t'_0, t'_1) = 48$. If $g = 0$, then $t' \geq 2$ and we have $\mathrm{surf}(g', t', t'_0, t'_1) \leq \mathrm{gen}(g', t', t'_0, t'_1) + 116 - 2 \cdot 42 = \mathrm{gen}(g', t', t'_0, t'_1) + 32$. Hence, $\mathrm{surf}(g, t, t_0, t_1) - \mathrm{surf}(g'', t'', t''_0, t''_1) - \mathrm{surf}(g', t', t'_0, t'_1) \geq \mathrm{gen}(g, t, t_0, t_1) + (4t''_1 + 5t''_0) - (\mathrm{gen}(g', t', t'_0, t'_1) + 32) = 16$. In both cases, the claim follows.

Therefore, we can assume that if $g'' = 0$, then $t'' \geq 2$. Therefore, we have $\mathrm{surf}(g'', t'', t''_0, t''_1) \leq \mathrm{gen}(g'', t'', t''_0, t''_1) + 32$ and $\mathrm{surf}(g', t', t'_0, t'_1) \leq \mathrm{gen}(g', t', t'_0, t'_1) + 32$. It follows that $\mathrm{surf}(g, t, t_0, t_1) - \mathrm{surf}(g', t', t'_0, t'_1) - \mathrm{surf}(g'', t'', t''_0, t''_1) \geq \mathrm{gen}(g, t, t_0, t_1) - \mathrm{gen}(g', t', t'_0, t'_1) - \mathrm{gen}(g'', t'', t''_0, t''_1) - 64 = 120 - 64 = 56$.

(d) We have $\mathrm{surf}(g, t, t_0, t_1) - \mathrm{surf}(g - 2, t, t_0, t_1) \leq \mathrm{gen}(g, t, t_0, t_1) - (\mathrm{gen}(g - 2, t, t_0, t_1) + 116) = 124$.

□

Consider a graph H embedded in a surface Π with rings $\mathcal{Q}$, and let f be an internal face of H. Let us recall that Π_f is the surface corresponding to the face in the natural way, as defined in Section 3. Let a_0 and a_1 be the number of weak and non-weak rings, respectively, that form one of the facial walks of f by themselves. Let a be the number of facial walks of f. We define $\mathrm{surf}(f) = \mathrm{surf}(g(\Pi_f), a, a_0, a_1)$.

Let G_1 be a graph embedded in Σ_1 with rings $\mathcal{R}_1$ and G_2 a graph embedded in Σ_2 with rings $\mathcal{R}_2$. Let $m(G_i)$ denote the number of edges of G_i that are not contained in the boundary of Σ_i. Let us write $(G_1, \Sigma_1, \mathcal{R}_1) \prec (G_2, \Sigma_2, \mathcal{R}_2)$ to denote that the quadruple $(g(\Sigma_1), |\mathcal{R}_1|, m(G_1), |E(G_1)|)$ is lexicographically smaller than $(g(\Sigma_2), |\mathcal{R}_2|, m(G_2), |E(G_2)|)$.

A graph G embedded in a surface Σ with rings $\mathcal{R}$ has *internal girth at least five* if every (≤ 4)-cycle in G is equal to one of the rings. Let $t_0(\mathcal{R})$ and $t_1(\mathcal{R})$ be the number of weak and non-weak vertex rings in $\mathcal{R}$, respectively. In order to prove Theorem 1.3, we show the following more general claim.

Theorem 6.2. *There exists a constant η with the following property. Let G be a graph embedded in a surface Σ with rings $\mathcal{R}$. If G is $\mathcal{R}$-critical and has internal girth at least five, then $w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + \eta\,\mathrm{surf}(g(\Sigma), |\mathcal{R}|, t_0(\mathcal{R}), t_1(\mathcal{R}))$.*

Proof. Let $\eta = 1867 + 67\mathrm{cyl}(7, 7)/s(5)$. We proceed by induction and assume that the claim holds for all graphs G' embedded in surfaces Σ' with rings $\mathcal{R}'$ such that $(G', \Sigma', \mathcal{R}') \prec (G, \Sigma, \mathcal{R})$. Let $g = g(\Sigma)$, $t_0 = t_0(\mathcal{R})$ and $t_1 = t_1(\mathcal{R})$. By Theorem 3.4, the claim holds if $g = 0$ and $|\mathcal{R}| = 1$, hence assume that $g > 0$ or $|\mathcal{R}| > 1$. Similarly, if $g = 0$ and $|\mathcal{R}| = 2$, then we can assume that $t_0 + t_1 \leq 1$ by Lemma 4.6. By Lemma 3.1, Lemma 3.2 and Theorem 4.1, G satisfies (I0), (I1), (I2), (I6) and (I9).

Suppose now that there exists a path P of length at most four with ends in distinct rings $R_1, R_2 \in \mathcal{R}$. By choosing the shortest such path, we can assume that P intersects no other rings. If R_1 or R_2 is a vertex ring, first replace it by a facial ring of length three by adding new vertices and edges in the incident cuff. Let $J = P \cup \bigcup_{R \in \mathcal{R}} R$ and let $S = \{f\}$, where f is the face of J incident with edges of P. Let $\{G'\}$ be the G-expansion of S, let Σ' be the surface in that G' is embedded and let $\mathcal{R}'$ be the natural rings of G'.

Note that $g(\Sigma') = g$, $|\mathcal{R}'| = |\mathcal{R}| - 1$, $\ell(\mathcal{R}') \le \ell(\mathcal{R}) + 14$ and $t_0(\mathcal{R}') + t_1(\mathcal{R}') \ge t_0 + t_1 - 2$. Since $(G', \Sigma', \mathcal{R}') \prec (G, \Sigma, \mathcal{R})$, by induction and by Lemma 6.1(a) we have $w(G, \mathcal{R}) = w(G', \mathcal{R}') \le \eta\mathrm{surf}(g, |\mathcal{R}| - 1, t_0(\mathcal{R}'), t_1(\mathcal{R}')) + \ell(\mathcal{R}) + 14 < \eta\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + \ell(\mathcal{R})$. Therefore, we can assume that no such path exists, and in particular, (I7) holds.

Next, we aim to prove property (I3). For later use, we will consider a more general setting.

Let H be a graph embedded in Π with rings $\mathcal{Q}$ such that at least one internal face of H is not open 2-cell and no face of H is omnipresent. If H is $\mathcal{Q}$-critical, has internal girth at least five and $(H, \Pi, \mathcal{Q}) \preceq (G, \Sigma, \mathcal{R})$, then

$$w(H, \mathcal{Q}) \le \ell(\mathcal{Q}) + \eta\Big(\mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 7 - \sum_{h \in F(H)} \mathrm{surf}(h)\Big). \tag{10}$$

Proof. We prove the claim by induction. Consider for a moment a graph H' of internal girth at least 5 embedded in a surface Π' with rings $\mathcal{Q}'$ with $(H', \Pi', \mathcal{Q}') \prec (H, \Pi, \mathcal{Q})$, such that either $H' = \mathcal{Q}'$ or H' is $\mathcal{Q}'$-critical. We claim that

$$w(H', \mathcal{Q}') \le \ell(\mathcal{Q}') + \eta\Big(\mathrm{surf}(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}')) - \sum_{h \in F(H')} \mathrm{surf}(h)\Big). \tag{11}$$

The claim obviously holds if $H' = \mathcal{Q}'$, since $w(H', \mathcal{Q}') \le \ell(\mathcal{Q}')$ in that case; hence, it suffices to consider the case that H' is $\mathcal{Q}'$-critical. If at least one internal face of H' is not open 2-cell and no face of H' is omnipresent, then this follows by an inductive application of (10) (we could even strengthen the inequality by $7C$). If all internal faces of H' are open 2-cell, then note that $\mathrm{surf}(h) = 0$ for every $h \in F(H')$, and since $(H', \Pi', \mathcal{Q}') \prec (G, \Sigma, \mathcal{R})$, we can apply Theorem 6.2 inductively to obtain (11). Finally, suppose that H' has an omnipresent face f, let $\mathcal{Q}' = \{Q_1, \ldots, Q_t\}$ and for $1 \le i \le t$, let f_i be the boundary walk of f such that f_i and Q_i are contained in a closed disk $\Delta_i \subset \Pi' + \hat{Q}_i$. Since all components of H' are planar and contain only one ring, Lemma 3.2 implies that all internal faces of H' distinct from f are closed 2-cell. Furthermore, each vertex ring forms component of the boundary of f by itself, hence $\mathrm{surf}(f) = \mathrm{surf}(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}'))$. If Q_i is a facial ring, then by applying Theorem 3.4 to the subgraph H'_i of H' embedded in Δ_i, we conclude that the weight of H'_i is at most $s(|Q_i|)$ and that $|f_i| \le |Q_i|$. Note that $s(|Q_i|) - s(|f_i|) \le |Q_i| - |f_i|$. Therefore, we again obtain (11):

$$\begin{aligned} w(H', \mathcal{Q}') &\le |f| + \sum_{i=1}^{t} s(|Q_i|) - s(|f_i|) \le |f| + \sum_{i=1}^{t} |Q_i| - |f_i| \\ &= \ell(\mathcal{Q}') \\ &= \ell(\mathcal{Q}') + \eta\Big(\mathrm{surf}(g(\Pi'), |\mathcal{Q}'|, t_0(\mathcal{Q}'), t_1(\mathcal{Q}')) - \sum_{h \in F(H')} \mathrm{surf}(h)\Big). \end{aligned}$$

Let us now return to the graph H. Since H is $\mathcal{Q}$-critical, Theorem 1.1 implies that no component of H is a planar graph without rings. Let f be a face of H

which is not open 2-cell. Since H has such a face and f is not omnipresent, we have $g(\Pi) > 0$ or $|\mathcal{Q}| > 2$. Let c be a simple closed curve in f infinitesimally close to a facial walk W of f. Cut Π along c and cap the resulting holes by disks (c is always a 2-sided curve). Let Π_1 be the surface obtained this way that contains W, and if c is separating, then let Π_2 be the other surface. Since f is not omnipresent, we can choose W so that either $g(\Pi_1) > 0$ or Π_1 contains at least two rings of $\mathcal{Q}$. Let us discuss several cases:

- *The curve c is separating and H is contained in Π_1.* In this case f has only one facial walk, and since f is not open 2-cell, Π_2 is not the sphere. It follows that $g(\Pi_1) = g(\Pi) - g(\Pi_2) < g(\Pi)$, and thus $(H, \Pi_1, \mathcal{Q}) \prec (H, \Pi, \mathcal{Q})$. Note that the weights of the faces of the embedding of H in Π and in Π_1 are the same, with the exception of f whose weight in Π is $|f|$ and weight in Π_1 is $s(|f|) \geq |f| - 8$. By (11), we have

  $$w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + 8 + \eta\Big(\mathrm{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) + \mathrm{surf}(f) - \sum_{h \in F(H)} \mathrm{surf}(h)\Big).$$

  Note that $\mathrm{surf}(f) = 120g(\Pi_2) - 72$. By Lemma 6.1(b), we conclude that

  $$w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + 8 + \eta\Big(\mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 40 - \sum_{h \in F(H)} \mathrm{surf}(h)\Big).$$

- *The curve c is separating and Π_2 contains a nonempty part H_2 of H.* Let H_1 be the part of H contained in Π_1. Let $\mathcal{Q}_i$ be the subset of $\mathcal{Q}$ belonging to Π_i and f_i the face of H_i corresponding to f, for $i \in \{1, 2\}$. Note that f_1 is an open disk, hence $\mathrm{surf}(f_1) = 0$. Using (11), we get

  $$\begin{aligned} w(H, \mathcal{Q}) \quad \leq \quad & w(f) - w(f_1) - w(f_2) + \ell(\mathcal{Q}_1) + \ell(\mathcal{Q}_2) + \\ & + \eta \sum_{i=1}^{2} \mathrm{surf}(g(\Pi_i), |\mathcal{Q}_i|, t_0(\mathcal{Q}_i), t_1(\mathcal{Q}_i)) + \\ & + \eta\Big(\mathrm{surf}(f) - \mathrm{surf}(f_2) - \sum_{h \in F(H)} \mathrm{surf}(h)\Big). \end{aligned}$$

  Note that $w(f) - w(f_1) - w(f_2) \leq 16$ and $\ell(\mathcal{Q}_1) + \ell(\mathcal{Q}_2) = \ell(\mathcal{Q})$. Also, $\mathrm{surf}(f) - \mathrm{surf}(f_2) \leq 48$, and when $g(\Pi_f) = 0$ and f has only two facial walks, then $\mathrm{surf}(f) - \mathrm{surf}(f_2) \leq 6$.

  By Lemma 6.1(c), we have

  $$\sum_{i=1}^{2} \mathrm{surf}(g(\Pi_i), |\mathcal{Q}_i|, t_0(\mathcal{Q}_i), t_1(\mathcal{Q}_i)) \leq \mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - \delta,$$

  where $\delta = 16$ if $g(\Pi_2) = 0$ and $|\mathcal{Q}_2| = 1$ and $\delta = 56$ otherwise. Note that if $g(\Pi_2) = 0$ and $|\mathcal{Q}_2| = 1$, then $g(\Pi_f) = 0$ and f has only two facial walks. We conclude that $\mathrm{surf}(f) - \mathrm{surf}(f_2) - \delta \leq -8$. Therefore, $w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + 16 + \eta\Big(\mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 8 - \sum_{h \in F(H)} \mathrm{surf}(h)\Big)$.

- *The curve c is not separating.* Let f_1 be the face of H (in the embedding in Π_1) bounded by W and f_2 the other face corresponding to f. Again, note that $\mathrm{surf}(f_1) = 0$. By (11) applied to H embedded in Π_1, we obtain the following for the weight of H in Π:

$$\begin{aligned} w(H, \mathcal{Q}) \quad &\leq \quad w(f) - w(f_1) - w(f_2) + \ell(\mathcal{Q}) + \\ &\quad +\eta \mathrm{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) + \\ &\quad +\eta\Big(\mathrm{surf}(f) - \mathrm{surf}(f_2) - \sum_{h \in F(H)} \mathrm{surf}(h)\Big). \end{aligned}$$

Since c is two-sided, $g(\Pi_1) = g(\Pi) - 2$, and

$$\mathrm{surf}(g(\Pi_1), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) = \mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 124$$

by Lemma 6.1(d). Since $\mathrm{surf}(f) - \mathrm{surf}(f_2) \leq 48$ and $w(f) - w(f_1) - w(f_2) \leq 16$, we have $w(H, \mathcal{Q}) \leq \ell(\mathcal{Q}) + 16 + \eta\Big(\mathrm{surf}(g(\Pi), |\mathcal{Q}|, t_0(\mathcal{Q}), t_1(\mathcal{Q})) - 76 - \sum_{h \in F(H)} \mathrm{surf}(h)\Big)$.

The results of all the subcases imply (10). □

Let H be a graph embedded in Σ with rings $\mathcal{R}$ and let f be an omnipresent face of H. If H is $\mathcal{R}$-critical, has internal girth at least five and no component of H satisfies (E1), (E2) or (E3), then

$$w(H, \mathcal{R}) \leq \ell(\mathcal{R}) - \kappa = \ell(\mathcal{R}) - \kappa + \eta\Big(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - \sum_{h \in F(H)} \mathrm{surf}(h)\Big),$$

where $\kappa = 5 - 5s(5)$ if H has exactly one component not equal to a ring and this component is exceptional, $\kappa = 5 + 5s(5)$ if H has exactly one component not equal to a ring and this component is not exceptional, and $\kappa = 6$ otherwise. (12)

Proof. Since H is $\mathcal{R}$-critical and f is an omnipresent face, each component of H is planar and contains exactly one ring. In particular, all internal faces of H distinct from f are closed 2-cell. For $R \in \mathcal{R}$, let H_R be the component of H containing R. Exactly one boundary walk W of f belongs to H_R. Cutting along W and capping the hole by a disk, we obtain an embedding of H_R in a disk with one ring R. Let f_R be the face of this embedding bounded by W. Note that either $H_R = R$ or H_R is $\{R\}$-critical. If R is a vertex ring, then we have $H_R = R$; hence, every vertex ring in $\mathcal{R}$ forms a facial walk of f, and $\mathrm{surf}(f) = \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1)$. Consequently, $\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) = \sum_{h \in F(H)} \mathrm{surf}(h)$, and it suffices to prove the first inequality of the claim.

Suppose that $H_R \neq R$ for a ring $R \in \mathcal{R}$. Since H_R does not satisfy (E1), (E2) or (E3), Theorem 3.4 implies that $w(H_R, \{R\}) \leq s(|R| - 5) + \alpha$, where $\alpha = 5s(5)$ if H_R satisfies (E4) or (E5) and $\alpha = -5s(5)$ otherwise. Since f_R is a face of H_R and $s(y) - s(x) > 5s(5)$ for every $y > x \geq 5$, we have $|f_R| \leq |R| - 5$. Furthermore, $w(H_R, \{R\}) - w(f_R) \leq s(|R) - 5) + \alpha - s(|f_R|) \leq |R| - |f_R| - 5 + \alpha$.

Summing over all the rings, we obtain

$$\begin{aligned} w(H,\mathcal{R}) &= w(f) + \sum_{R\in\mathcal{R}} (w(H_R,\{R\}) - w(f_R)) \\ &\le |f| + \sum_{R\in\mathcal{R}} (|R| - |f_R|) - \kappa \\ &= \ell(\mathcal{R}) - \kappa, \end{aligned}$$

□

Let H be an $\mathcal{R}$-critical graph embedded in Σ with rings $\mathcal{R}$ so that all internal faces of H are open 2-cell. If H is $\mathcal{R}$-critical, has internal girth at least five, $|E(H)| \le |E(G)|$ and an internal face f of H is not closed 2-cell, then $w(H,\mathcal{R}) \le \ell(\mathcal{R}) + \eta\Big(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 1/2\Big)$.

(13)

Proof. Since f is not closed 2-cell, there exists a vertex v appearing at least twice in the facial walk of f. There exists a simple closed curve c going through the interior of f and joining two of the appearances of v. Cut the surface along c and patch the resulting hole(s) by disk(s). Let v_1 and v_2 be the two vertices to that v is split. For each of v_1 and v_2, if it is not incident with a cuff, drill a new hole next to it in the incident patch.

If c is separating, then let H_1 and H_2 be the resulting graphs embedded in the two surfaces Σ_1 and Σ_2 obtained by this construction; if c is not separating, then let H_1 be the resulting graph embedded in a surface Σ_1. We choose the labels so that $v_1 \in V(H_1)$. If c is two-sided, then let f_1 and f_2 be the faces to that f is split by c, where f_1 is a face of H_1. If c is one-sided, then let f_1 be the face in Σ_1 corresponding to f. Note that $|f_1| + |f_2| = |f|$ in the former case, and thus $w(f) - w(f_1) - w(f_2) \le 16$. In the latter case, we have $w(f) = w(f_1)$.

If c is separating, then for $i \in \{1,2\}$, let $\mathcal{R}_i$ consist the rings of $\mathcal{R} \setminus \{v\}$ contained in Σ_i, and if none of these rings contains v, then also of the vertex ring v_i. Here, v_i is weak if v is an internal vertex, Σ_{3-i} is a cylinder and the ring of G_{3-i} distinct from v_{3-i} is a vertex ring. If c is not separating, then let $\mathcal{R}_1$ consist of the rings of $\mathcal{R} \setminus \{v\}$ as well as those of v_1 and v_2 that are not incident with any of the rings in this set. In this case, we treat v_1 and v_2 as non-weak vertex rings.

Suppose first that c is not separating. Note that H_1 has at most two more rings (of length 1) than H and $g(\Sigma_1) \in \{g-1, g-2\}$ (depending on whether c is one-sided or not), and that H_1 has at least two rings. If H_1 has only one more ring than H, then

$$\begin{aligned} \mathrm{surf}(g(\Sigma_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) &\le \mathrm{surf}(g-1, |\mathcal{R}|+1, t_0, t_1+1) \\ &\le \mathrm{gen}(g-1, |\mathcal{R}|+1, t_0, t_1+1) + 32 \\ &= \mathrm{gen}(g, |\mathcal{R}|, t_0, t_1) - 44 \\ &= \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 44. \end{aligned}$$

Let us now consider the case that H_1 has two more rings than H (i.e., that v is an internal vertex). If $g(\Sigma_1) = 0$ and $|\mathcal{R}_1| = 2$, then note that both rings of H_1 are vertex rings. Lemma 4.6 implies that H_1 has only one edge; but the corresponding edge in H would form a loop, which is a contradiction. Consequently, we have $g(\Sigma_1) > 1$ or $|\mathcal{R}_1| \geq 3$, and

$$\begin{aligned}
\mathrm{surf}(g(\Sigma_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) &\leq \mathrm{surf}(g-1, |\mathcal{R}|+2, t_0, t_1+2) \\
&= \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 32.
\end{aligned}$$

By induction, we can apply Theorem 6.2 inductively for H_1, concluding that $w(H, \mathcal{R}) \leq \ell(\mathcal{R}) + 18 + \eta\Big(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 32\Big)$, and the claim follows.

Next, we consider the case that c is separating. Let us remark that G_i is $\mathcal{R}_i$-critical for $i \in \{1, 2\}$. This is obvious unless v_i is a weak vertex ring. However, in that case Lemma 4.6 implies that G_{3-i} is a single edge, and the $\mathcal{R}_i$-criticality of G_i is argued in the same way as in the proof of Theorem 4.9. Thus, we can apply Theorem 6.2 inductively for G_1 and G_2, and we have

$$\begin{aligned}
w(H, \mathcal{R}) &= w(H_1, \mathcal{R}_1) + w(H_2, \mathcal{R}_2) + w(f) - w(f_1) - w(f_2) \\
&\leq \ell(\mathcal{R}) + 18 + \eta \sum_{i=1}^{2} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i))
\end{aligned}$$

Therefore, it suffices to prove that

$$\sum_{i=1}^{2} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \leq \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 1. \tag{14}$$

Note that if $g(\Sigma_i) = 0$, then $|\mathcal{R}_i| \geq 2$ for $i \in \{1, 2\}$. If $|\mathcal{R}_1| + |\mathcal{R}_2| = |\mathcal{R}| + 1$, we have

$$\begin{aligned}
\sum_{i=1}^{2} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) &\leq \sum_{i=1}^{2} (\mathrm{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + 32) \\
&= \mathrm{gen}(g, |\mathcal{R}|, t_0, t_1) - 12 \\
&= \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 12.
\end{aligned}$$

Therefore, we can assume that $|\mathcal{R}_1| + |\mathcal{R}_2| = |\mathcal{R}| + 2$, i.e., v is an internal vertex. Suppose that for both $i \in \{1, 2\}$, we have $g(\Sigma_i) > 0$ or $|\mathcal{R}_i| > 2$. Then,

$$\begin{aligned}
\sum_{i=1}^{2} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) &= \sum_{i=1}^{2} \mathrm{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \\
&= \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 32.
\end{aligned}$$

and the claim follows.

Hence, we can assume that say $g(\Sigma_1) = 0$ and $|\mathcal{R}_1| = 2$. Then, $\mathcal{R}_1 = \{v_1, R_1\}$ for some ring R_1, $g(\Sigma_2) = g$ and $|\mathcal{R}_2| = |\mathcal{R}|$. Since G_1 is $\mathcal{R}_1$-critical, Lemma 4.6 implies that R_1 is not a weak vertex ring. If R_1 is a vertex ring, then v_2 is a weak vertex ring of $\mathcal{R}_2$ which replaces the non-weak vertex ring R_1. Therefore, $\mathrm{surf}(g(\mathcal{R}_2), |\mathcal{R}_2|, t_0(\mathcal{R}_2), t_1(\mathcal{R}_2)) = \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 1$. Furthermore, $\mathrm{surf}(g(\mathcal{R}_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) = \mathrm{surf}(0, 2, 0, 2) = 0$, and the claim follows.

Finally, consider the case that R_1 is a facial ring. By symmetry, we can assume that if $g(\Sigma_2) = 0$ and $|\mathcal{R}_2| = 2$, then $\mathcal{R}_2$ contains a facial ring. Since $\mathcal{R}_2$

is obtained from $\mathcal{R}$ by replacing a facial ring R_1 by a non-weak vertex ring v_2, we have $\text{surf}(g(\mathcal{R}_2), |\mathcal{R}_2|, t_0(\mathcal{R}_2), t_1(\mathcal{R}_2)) = \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 4$. Furthermore, $\text{surf}(g(\mathcal{R}_1), |\mathcal{R}_1|, t_0(\mathcal{R}_1), t_1(\mathcal{R}_1)) = \text{surf}(0, 2, 0, 1) = 2$. Consequently,

$$\sum_{i=1}^{2} \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \leq \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 2.$$

Therefore, inequality (14) holds. □

By (10), (12) and (13), we can assume that G satisfies (I3).

Suppose that G contains a path P of length at most six joining two distinct vertices u and v of a ring $R \in \mathcal{R}$, such that $V(P) \cap V(R) = \{u, v\}$ and $R \cup P$ contains no contractible cycle. Since the distance between any two rings in G is at least five, all vertices of $V(P) \setminus \{u, v\}$ are internal. Let J be the subgraph of G consisting of P and of the union of the rings, and let S be the set of internal faces of J. Clearly, S and J satisfy (1). Let $\{G_1, \ldots, G_k\}$ be the G-expansion of S, and for $1 \leq i \leq k$, let Σ_i be the surface in that G_i is embedded and let $\mathcal{R}_i$ be the natural rings of G_i. Note that $\sum_{i=1}^{k} t_0(\mathcal{R}_i) = t_0$ and $\sum_{i=1}^{k} t_1(\mathcal{R}_i) = t_1$. Let $r = \left(\sum_{i=1}^{k} |\mathcal{R}_i|\right) - |\mathcal{R}|$ and observe that either $r = 0$ and $k = 1$, or $r = 1$ and $1 \leq k \leq 2$ (depending on whether the curve in $\hat{\Sigma}$ corresponding to a cycle in $R \cup P$ distinct from R is one-sided, two-sided and non-separating or two-sided and separating). Furthermore, $\sum_{i=1}^{k} g(\Sigma_i) = g + 2k - r - 3$.

We claim that $(G_i, \Sigma_i, \mathcal{R}_i) \prec (G, \Sigma, \mathcal{R})$ for $1 \leq i \leq k$. This is clearly the case, unless $g(\Sigma_i) = g$. Then, we have $k = 2$, $r = 1$ and $g(\Sigma_{3-i}) = 0$. Since $R \cup P$ contains no contractible cycle, Σ_{3-i} is not a disk, hence $|\mathcal{R}_{3-i}| \geq 2$ and $|\mathcal{R}_{3-i}| < |\mathcal{R}|$, again implying $(G_i, \Sigma_i, \mathcal{R}_i) \prec (G, \Sigma, \mathcal{R})$.

By induction, we have $w(G_i, \mathcal{R}_i) \leq \ell(\mathcal{R}_i) + \eta \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i))$, for $1 \leq i \leq k$. Since every internal face of G is an internal face of G_i for some $i \in \{1, \ldots, k\}$ and $\sum_{i=1}^{k} \ell(\mathcal{R}_i) \leq \ell(\mathcal{R}) + 12$, we conclude that $w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + 12 + \eta \sum_{i=1}^{k} \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i))$. Note that for $1 \leq i \leq k$, we have that Σ_i is not a disk and $\mathcal{R}_i$ contains at least one facial ring, and thus $\text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \leq \text{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + 30$. Therefore,

$$\begin{aligned}
&\sum_{i=1}^{k} \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \\
&\quad = \sum_{i=1}^{k} (\text{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + 30) \\
&\quad \leq \text{surf}(g, |\mathcal{R}|, t_0, t_1) + 120(2k - r - 3) + 48r - 120(k - 1) + 60 \\
&\quad = \text{surf}(g, |\mathcal{R}|, t_0, t_1) + 120k - 72r - 180 \\
&\quad \leq \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 12.
\end{aligned}$$

The inequality of Theorem 6.2 follows; therefore, we can assume that

if P is a path of length at most six joining two distinct vertices of a ring R, then $R \cup P$ contains a contractible cycle. (15)

Let us note that since $g > 0$ or $|\mathcal{R}| \geq 2$, this contractible cycle is unique.

Consider now a path P of length at most four, such that its ends u and v are ring vertices and all other vertices of P are internal. By (I7), both ends of P belong to the same ring R; let P, P_1 and P_2 be the paths in $R \cup P$ joining u and v. By the previous paragraph, we can assume that $P \cup P_2$ is a contractible cycle. Suppose that the disk bounded by $P \cup P_2$ neither is a face nor consists of two 5-faces. By Theorem 4.1, we have $|P \cup P_2| \geq 9$. Let J, S, G_i, Σ_i and $\mathcal{R}_i$ (for $i \in \{1,2\}$) be defined as in the previous paragraph, where Σ_2 is a disk and $\mathcal{R}_2$ consists of a single ring corresponding to $P \cup P_2$. Since $g(\Sigma_1) = g$, $|\mathcal{R}_1| = |\mathcal{R}|$ and $|E(G_1)| < |E(G)|$, by induction we have $w(G_1, \{\mathcal{R}_1\}) \leq \ell(\mathcal{R}_1) + \eta \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1)$. Note that $\ell(\mathcal{R}_1) = \ell(\mathcal{R}) + |P| - |P_2|$. Furthermore, Theorem 3.4 implies $w(G_2, \{\mathcal{R}_2\}) \leq s(|P| + |P_2|) = |P| + |P_2| - 8$. Therefore, $w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + \eta \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + 2|P| - 8$. Since $|P| \leq 4$, the claim of Theorem 6.2 follows. Therefore, we can assume that the disk bounded by $P \cup P_1$ is either a face or consists of two 5-faces. The same calculation also excludes the possibility that $|P| \leq 2$, since $s(|P| + |P_2|) \leq |P| + |P_2| - 4$ for any P and P_2 such that $|P| + |P_2| \geq 5$. In particular, we can assume that (I4) holds for G.

Suppose that G contains two adjacent vertices r_1 and r_2 of degree two. By (I7), both r_1 and r_2 are incident with the same facial ring R. By (I3), we have $|R| \geq 4$. By (I4), the internal face f incident with $r_1 r_2$ has length at least six. Let G' be the graph obtained from G by contracting $r_1 r_2$, let $\mathcal{R}'$ be the set of rings of G' obtained from $\mathcal{R}$ by contracting edge $r_1 r_2$ in R, and let f' be the face of G' corresponding to f. Observe that G' is $\mathcal{R}'$-critical. Suppose that G' contains a (≤ 4)-cycle C' distinct from the rings. Then G contains a (≤ 5)-cycle C distinct from the rings containing $r_1 r_2$. Since G has internal girth at least 5, we have $|C| = 5$, and we obtain a contradiction with (I4). Therefore, G' has internal girth at least 5. By induction, we have $w(G', \mathcal{R}') = \ell(\mathcal{R}') + \eta \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1)$, and since $\ell(\mathcal{R}) = \ell(\mathcal{R}') + 1$ and $w(f) \leq w(f') + 1$, G satisfies the inequality of Theorem 6.2. Therefore, assume that G satisfies (I5). Together with the previous paragraph, this implies that G is well-behaved.

Suppose that G contains a non-contractible cycle C of length at most seven that does not surround any of the rings. By (I7), C intersects at most one ring, and by (15), C shares at most one vertex with this ring. Let $s = 1$ if C intersects a ring, and $s = 0$ otherwise. Let J be the subgraph of G consisting of C and of the union of the rings, and let S be the set of internal faces of J. Clearly, S and J satisfy (1). Let $\{G_1, \ldots, G_k\}$ be the G-expansion of S, and for $1 \leq i \leq k$, let Σ_i be the surface in that G_i is embedded and let $\mathcal{R}_i$ be the natural rings of G_i. Let $r = \left(\sum_{i=1}^{k} |\mathcal{R}_i|\right) - |\mathcal{R}|$. Note that either $r + s = 1$ and $k = 1$, or $r + s = 2$ and $1 \leq k \leq 2$. Observe that $\sum_{i=1}^{k} g(\Sigma_i) = g - s - r + 2k - 2$. Furthermore, $\sum_{i=1}^{k} t_0(\mathcal{R}_i) + \sum_{i=1}^{k} t_1(\mathcal{R}_i) \geq t_0 + t_1 - s$ and $\sum_{i=1}^{k} \ell(\mathcal{R}_i) \leq \ell(\mathcal{R}) + 14$.

If $g(\Sigma_1) = g$, then $k = 2$ and $g(\Sigma_2) = 0$; furthermore, Σ_2 has at least two cuffs, and if $s = 0$, then it has at least three cuffs, since C does not surround a ring. Thus, if $g(\Sigma_1) = g$, then $r = 2 - s$ and consequently $|\mathcal{R}_1| = |\mathcal{R}| + r - |\mathcal{R}_2| = |\mathcal{R}| + 2 - s - |\mathcal{R}_2| < |\mathcal{R}|$. The same argument can be applied to Σ_2 if $k = 2$, hence $(G_i, \Sigma_i, \mathcal{R}_i) \prec (G, \Sigma, \mathcal{R})$ for $1 \leq i \leq k$.

By induction, we conclude that

$$w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + 14 + \eta \sum_{i=1}^{k} \mathrm{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)).$$

For $1 \leq i \leq k$, let $\delta_i = 72$ if $g(\Sigma_i) = 0$ and $|\mathcal{R}_i| = 1$, let $\delta_i = 30$ if $g(\Sigma_i) = 0$ and $|\mathcal{R}_i| = 2$, and let $\delta_i = 0$ otherwise, and note that since $\mathcal{R}_i$ contains a facial ring, we have $\text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), r_1(\mathcal{R}_i)) = \text{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), r_1(\mathcal{R}_i)) + \delta_i$.

If $k = 2$, then recall that since C does not surround a ring, we have either $g(\Sigma_i) > 0$ or $|\mathcal{R}_i| \geq 3 - s$ for $i \in \{1, 2\}$; hence, $\delta_1 + \delta_2 \leq 30s$.

If $k = 1$, then note that G is not embedded in the projective plane with no rings (Thomassen [13] proved that every projective planar graph of girth at least five is 3-colorable); hence, if $s = 0$, then either $g(\Sigma_1) > 0$, or $|\mathcal{R}_1| \geq 2$. Consequently, we have $\delta_1 \leq 30 + 42s$.

Combining the inequalities, we obtain $\sum_{i=1}^{k} \delta_i \leq 60 + 43s - 30k$, and

$$\begin{aligned}
&\sum_{i=1}^{k} \text{surf}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) \\
&\quad = \sum_{i=1}^{k} \text{gen}(g(\Sigma_i), |\mathcal{R}_i|, t_0(\mathcal{R}_i), t_1(\mathcal{R}_i)) + \delta_i \\
&\quad \leq \text{surf}(g, |\mathcal{R}|, t_0, t_1) + 120(2k - r - s - 2) + 48r - 120(k-1) + 5s + \sum_{i=1}^{k} \delta_i \\
&\quad = \text{surf}(g, |\mathcal{R}|, t_0, t_1) + 90k - 72(r+s) - 60 \\
&\quad \leq \text{surf}(g, |\mathcal{R}|, t_0, t_1) - 24.
\end{aligned}$$

This implies the inequality of Theorem 6.2. Therefore, assume that every non-contractible cycle of length at most 7 surrounds a ring. In particular, G satisfies (I8).

For each ring $R \in \mathcal{R}$, let M_R be the set of all edges incident with cycles of G of length at most 7 that surround R, and let C_R be such a cycle chosen so that the part Σ_R of Σ between R and C_R is as large as possible. By Lemma 5.1, at most 70 edges of M_R are drawn outside of Σ_R. Let K_R be a (≤ 7)-cycle in $G \cap \Sigma_R$ chosen so that the part Σ'_R of Σ between R and K_R is as small as possible. Analogically to Lemma 5.1, we see that at most 70 edges of $M_R \cap \Sigma_R$ are drawn outside of Σ'_R. We claim that at most $5\text{cyl}(7,7)/s(5)$ edges of G are drawn in Σ'_R: When K_R and C_R are vertex-disjoint, this follows from Lemma 5.6. When K_R intersects C_R, this is implied by Lemma 3.5 and Theorem 3.4, since $\text{cyl}(7,7) > s(14)$. We conclude that $|M_R| \leq 140 + 5\text{cyl}(7,7)/s(5)$.

Let M consist of all facial rings of length at most four and of all non-contractible cycles in G of length at most 7. Observe that $M = \bigcup_{R \in \mathcal{R}} M_R$, and thus $|E(M)| \leq (140 + 5\text{cyl}(7,7)/s(5))|\mathcal{R}|$. Note that M captures all (≤ 4)-cycles in G. If $w(G, \mathcal{R}) \leq 8g + 8|\mathcal{R}| + (2/3 + 26\epsilon)\ell(\mathcal{R}) + 20|E(M)|/3 - 16$, then $w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + \eta\text{surf}(g, |\mathcal{R}|, t_0, t_1)$ by the choice of η, and Theorem 6.2 is true. Therefore, assume that this is not the case, and thus the assumptions of Theorem 3.3 are satisfied.

Let G' be the $\mathcal{R}$-critical graph embedded in Σ such that $|E(G')| < |E(G)|$, satisfying the conditions of Theorem 3.3. In particular, (b) together with the choice of M implies that G' has internal girth at least five. Let $X \subset F(G)$ and $\{(J_f, S_f) : f \in F(G')\}$ be the cover of G by faces of G' as in Theorem 3.3(d). For $f \in F(G')$, let $\{G_1^f, \ldots, G_{k_f}^f\}$ be the G-expansion of S_f and for $1 \leq i \leq k_f$, let Σ_i^f be the surface in that G_i^f is embedded and let $\mathcal{R}_i^f$ denote the natural rings of G_i^f. We have

$$w(G,\mathcal{R}) = \sum_{f\in F(G)} w(f) = \sum_{f\in X} w(f) + \sum_{f\in F(G')} \sum_{i=1}^{k_f} w(G_i^f, \mathcal{R}_i^f). \tag{16}$$

Consider a face $f \in F(G')$. We have $g(\Sigma_f) \le g$. If $g(\Sigma_f) = g$, then every component of G' is planar, and since G' is $\mathcal{R}$-critical, each component of G' contains at least one ring of $\mathcal{R}$; consequently, f has at most $|\mathcal{R}|$ facial walks and Σ_f has at most $|\mathcal{R}|$ cuffs. Since the surfaces embedding the components of the G-expansion of S_f are fragments of Σ_f, we have $(G_i^f, \Sigma_i^f, \mathcal{R}_i^f) \prec (G, \Sigma, \mathcal{R})$ for $1 \le i \le k_f$: otherwise, we would have $m(G_i^f) = m(G)$, hence by the definition of G-expansion, the boundary of S_f would have to be equal to the union of rings in $\mathcal{R}$, contrary to the definition of a cover of G by faces of G'.

Therefore, we can apply Theorem 6.2 inductively for G_i^f and we get $w(G_i^f, \mathcal{R}_i^f) \le \ell(\mathcal{R}_i^f) + \eta\mathrm{surf}(g(\Sigma_i^f), |\mathcal{R}_i^f|, t_0(\mathcal{R}_i^f), t_1(\mathcal{R}_i^f))$. Observe that since $\{\Sigma_1^f, \ldots, \Sigma_{k_f}^f\}$ are fragments of Σ_f, we have

$$\sum_{i=1}^{k_f} \mathrm{surf}(g(\Sigma_i^f), |\mathcal{R}_i^f|, t_0(\mathcal{R}_i^f), t_1(\mathcal{R}_i^f)) \le \mathrm{surf}(f),$$

and we obtain

$$\sum_{i=1}^{k_f} w(G_i^f, \mathcal{R}_i^f) \le |f| + \mathrm{el}(f) + \eta\mathrm{surf}(f). \tag{17}$$

In case that f is open 2-cell, all fragments of f are disks and we can use Theorem 3.4 instead of Theorem 6.2, getting the stronger inequality $w(G_i^f, \mathcal{R}_i^f) \le s(\ell(\mathcal{R}_i^f))$ for $1 \le i \le k_f$. Summing these inequalities, we can strengthen (17) to

$$\sum_{i=1}^{k_f} w(G_i^f, \mathcal{R}_i^f) \le w(f) + \mathrm{el}(f) + \eta\mathrm{surf}(f). \tag{18}$$

The inequalities (16), (18) and Theorem 3.3(d) imply that

$$\begin{aligned} w(G,\mathcal{R}) &\le |X|s(6) + \sum_{f\in F(G')} (w(f) + \mathrm{el}(f) + \eta\mathrm{surf}(f)) \\ &\le w(G',\mathcal{R}) + s(6) + 10 + \eta \sum_{f\in F(G')} \mathrm{surf}(f). \end{aligned}$$

If G' has a face that is neither open 2-cell nor omnipresent, then (10) implies that

$$w(G',\mathcal{R}) \le \ell(\mathcal{R}) + \eta\Big(\mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) - 7 - \sum_{f\in F(G')} \mathrm{surf}(f)\Big),$$

and consequently G satisfies the outcome of Theorem 6.2. Therefore, we can assume that all internal faces of G' are either open 2-cell or omnipresent. Similarly, using (13) we can assume that if no face of G' is omnipresent, then all of them are closed 2-cell.

Suppose first that G has no omnipresent face. Using (16) and Theorem 3.3(d) and (e) and applying Theorem 6.2 inductively for G', we have

$$\begin{aligned} w(G, \mathcal{R}) &\leq |X|s(6) + \sum_{f \in F(G')} w(f) - c(f) \\ &= w(G', \mathcal{R}) + |X|s(6) - \sum_{f \in F(G')} c(f) \\ &\leq w(G', \mathcal{R}) \leq \ell(\mathcal{R}) + \eta \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1). \end{aligned}$$

It remains to consider the case that G' has an omnipresent face h. Then, every component of G is a plane graph with one ring, and by Lemma 3.2, we conclude that every internal face of G different from h is closed 2-cell and G' satisfies (I6). By Theorem 3.3(d), we have $c(h) \neq -\infty$, hence no component of G' satisfies (E1), (E2) or (E3). By (16) and Theorem 3.3(d) and (e) and by (18), we have

$$\begin{aligned} w(G, \mathcal{R}) &\leq |X|s(6) + \sum_{f \in F(G'), f \neq h} (w(f) - c(f)) + \sum_{i=1}^{k_h} w(G_i^h, \mathcal{R}_i^h) \\ &= w(G', \mathcal{R}) + |X|s(6) + (c(h) - w(h)) - \sum_{f \in F(G')} c(f) + \sum_{i=1}^{k_h} w(G_i^h, \mathcal{R}_i^h) \\ &\leq w(G', \mathcal{R}) + c(h) - w(h) + \sum_{i=1}^{k_h} w(G_i^h, \mathcal{R}_i^h) \\ &\leq w(G', \mathcal{R}) + c(h) + \mathrm{el}(h) + \eta \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) \end{aligned}$$

We use (12) to bound the weight of G'. We obtain (κ is defined as in (12))

$$w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + \eta \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1) + c(h) + \mathrm{el}(h) - \kappa.$$

By Theorem 3.3(d), we have $\mathrm{el}(h) \leq 5$. By the definition of κ and of the contribution of h, it follows that $c(h) + \mathrm{el}(h) \leq \kappa$. Therefore,

$$w(G, \mathcal{R}) \leq \ell(\mathcal{R}) + \eta \mathrm{surf}(g, |\mathcal{R}|, t_0, t_1)$$

as required. □

Let us remark that Theorem 6.2 implies the special case of Theorem 1.5 for graphs with no 4-cycles, by considering the triangles to be rings (we need to first split their vertices so that they become vertex-disjoint, then drill holes in them). Furthermore, Theorem 1.3 follows as a special case when the set of rings is empty.

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