A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges. An $H$ minor is a minor isomorphic to $H$. 
THEOREM (Tutte) Every 3-connected simple graph can be obtained from a wheel by repeatedly adding edges (between nonadjacent vertices) and splitting vertices.
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SEYMOUR’S SPLITTER THM Let $H \not= K_4$ and $G \not= \text{wheel}$ be simple 3-connected, $H \leq_m G$. Then $G$ can be obtained from $H$ by repeatedly adding edges (between nonadjacent vertices) and splitting vertices.
KURATOWSKI’S THEOREM. A graph is planar \iff it has no $K_5$ or $K_{3,3}$ minor.
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COROLLARY. A simple 3-connected graph $G$ has no $K_{3,3}$ minor ⇔ $G$ is planar or $G \cong K_5$. 
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COROLLARY. A simple 3-connected graph $G$ has no $K_{3,3}$ minor ⇔ $G$ is planar or $G \cong K_5$.

PROOF of ⇒. We may assume $G$ is nonplanar. By Kuratowski’s theorem $G$ has a $K_5$ minor. By Seymour’s theorem $G$ can be obtained from $K_5$ as stated. Now $G \cong K_5$, for otherwise $G$ has a $K_{3,3}$ minor.
THEOREM (Wagner) A graph has no $K_5$ minor $\iff$ it can be obtained by means of 0-, 1-, 2-, and 3-sums from planar graphs and $V_8$. 
A graph $G$ is **internally 4-connected (I4C)** if it is simple, 3-connected, has at least five vertices and for every separation $(A, B)$ of order 3, one of $A$, $B$ has at most 3 edges.
A graph $G$ is internally 4-connected (I4C) if it is simple, 3-connected, has at least five vertices and for every separation $(A, B)$ of order 3, one of $A$, $B$ has at most 3 edges.
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**THM (Johnson, RT)** Except for eight well-defined families, an I4C graph $G$ can be “built” from an I4C minor of itself similarly as in Seymour’s theorem. The intermediate graphs are allowed to have one “violation” of I4C, but the next graph in the sequence “repairs” this violation.
LADDERS
Violating vertex, edge, pair
Violating vertex, edge, pair
SPECIAL ADDITION
SPECIAL ADDITION

e


v
SPECIAL ADDITION

\[ e \rightarrow v \]

SPECIAL SPLIT

\[ e \rightarrow e \]
THM Johnson, RT If $H \leq_m G$, $H$ is not $K_{3,3}, K_5$, cube or octahedron, $G$ is not a ladder or biwheel, then $\exists$ sequence $J_0 = H, J_1, \ldots, J_k = G$

- each $J_i$ is I4C except possibly for one violating edge
- no edge is violating in $J_i$ and $J_{i+1}$
- $J_i$ is obtained from $J_{i-1}$ by (special) addition or (special) split
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THM Johnson, RT The minimal nonplanar I4C graphs other than $K_{3,3}, K_5$ are: $K_6^=, \overline{C}_7, K_{3,3} + \text{deg 4 vertex}, V_8, \text{cube+diagonal}$. 
Application to Negami’s conjecture.

A graph $K$ is a cover of a graph $H$ if there exists an onto mapping $p : V(K) \rightarrow V(H)$ such that for every $v \in V(K)$ the neighbors of $v$ in $K$ are mapped bijectively onto the neighbors of $p(v)$ in $H$. 
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NEGAMI’S CONJECTURE. A connected graph has a planar cover $\iff$ it is projective planar.
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THM (Hliněný, RT) Modulo obvious constructions, there are at most 16 counterexamples to Negami’s conjecture.
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REMARK. It suffices to show that $K_{1,2,2,2}$ has no planar cover.
Robertson’s Theorem

THM An I4C graph $G$ has no $V_8$ minor $\iff$

1. $G$ is planar, or
2. $G \setminus X$ is edgeless for some $X \subseteq V(G)$, $|X| \leq 4$, or
3. $G \setminus u \setminus v$ is a cycle for some $u, v \in V(G)$, or
4. $G \cong L(K_{3,3})$, or
5. $|V(G)| \leq 7$
ROBERTSON’S THEOREM

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(4) $G \cong L(K_{3,3})$, or

(5) $|V(G)| \leq 7$

PROOF Let $G$ be nonplanar, I4C, no $V_8$ minor. We know $G \geq_m K_6^=, \overline{C}_7, K_{3,3} + \text{deg 4 vertex}, V_8, \text{or cube+diag}$. 
THEOREM An $I4C$ graph has no octahedron minor $\iff$

(1) $G$ is a Möbius ladder, or

(2) $G$ is isomorphic to a minor of Petersen,

The last graph has all possible triads with feet in the 5-element independent set.
A cubic graph is **cyclically 5-connected** \((C_5C)\) if it is simple, 3-connected, \(\not\equiv K_4\), and for every set \(F \subseteq E(G)\) of size at most 4, at most 1 component of \(G \setminus F\) has cycles.

**Biladders**
Cyclically 5-connected graphs
THEOREM (Robertson, Seymour, RT) Let $G$ be a C5C cubic graph that is not a biladder, and let $H$ be a C5C minor of $G$. Then $G$ can be obtained from $H$ by repeatedly applying the operations of

(i) adding a handle

(ii) adding a pentagon.
THEOREM (Robertson, Seymour, RT) A C5C cubic graph $G$ has no Petersen minor if and only if it is

(i) apex ($G \setminus v$ planar for some $v$), or

(ii) doublecross (2 crossings on the same region), or

(iii) has a “hamburger structure”, or

(iv) has a “hose structure”.

Structure of graphs with no $K_6$ minor is not known.
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**THEOREM (Mader)** If $G$ has $n$ vertices and no $K_6$-minor, then $G$ has at most $4n - 10$ edges.
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**THEOREM (Mader)** If $G$ has $n$ vertices and no $K_6$-minor, then $G$ has at most $4n - 10$ edges.

**JORGENSEN’S CONJECTURE** Every 6-connected graph with no $K_6$-minor is apex ($\equiv$planar + one vertex).
EXTREMAL PROBLEMS
For small $t$:
No $K_t$ minor $\Rightarrow$ at most $(t - 2)n - \binom{t-1}{2}$ edges
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No $K_2$ minor $\Rightarrow$ at most 0 edges
No $K_4$ minor $\Rightarrow$ at most $2n - 3$ edges
For small $t$:
No $K_t$ minor $\Rightarrow$ at most $(t - 2)n - \binom{t-1}{2}$ edges
No $K_2$ minor $\Rightarrow$ at most 0 edges
No $K_4$ minor $\Rightarrow$ at most $2n - 3$ edges
No $K_5$ minor $\Rightarrow$ at most $3n - 6$ edges
For small $t$:
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No $K_5$ minor $\Rightarrow$ at most $3n - 6$ edges

No $K_6$ minor $\Rightarrow$ at most $4n - 10$ edges

No $K_7$ minor $\Rightarrow$ at most $5n - 15$ edges
For small $t$:

No $K_t$ minor $\Rightarrow$ at most $(t - 2)n - \binom{t-1}{2}$ edges

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No $K_4$ minor $\Rightarrow$ at most $2n - 3$ edges

No $K_5$ minor $\Rightarrow$ at most $3n - 6$ edges

No $K_6$ minor $\Rightarrow$ at most $4n - 10$ edges

No $K_7$ minor $\Rightarrow$ at most $5n - 15$ edges

No $K_8$ minor $\not\Rightarrow$ at most $6n - 21$ edges
For small $t$:
No $K_t$ minor ⇒ at most $(t - 2)n - \binom{t-1}{2}$ edges
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No $K_8$ minor ⇒ at most $6n - 20$ edges
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No $K_8$ minor $\not\Rightarrow$ at most $6n - 21$ edges

No $K_8$ minor $\Rightarrow$ at most $6n - 20$ edges

No $K_9$ minor $\Rightarrow$ at most $7n - 27$ edges??
For small $t$:

No $K_t$ minor $\Rightarrow$ at most $(t - 2)n - \left(\frac{t-1}{2}\right)$ edges

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**THM Thomason**

No $K_t$ minor $\Rightarrow$ at most $(0.319 + o(1))t\sqrt{\log tn}$ edges
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No $K_t$ minor $\Rightarrow$ at most $(0.319 + o(1))t\sqrt{\log tn}$ edges

**CONJECTURE** $\forall t \exists N$ if $G$ is $(t - 2)$-connected and $|G| > N$, then $|E(G)| \leq (t - 2)n - \binom{t-1}{2}$. 
THM Jorgensen No $K_{4,4}$ minor $\Rightarrow \leq 4n - 8$ edges.
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THEOREM (Mader, conjectured by Dirac)
Every graph on \( n \) vertices and at least \( 3n - 5 \) edges has a \( K_5 \) subdivision.
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Every graph on $n$ vertices and at least $3n - 5$ edges has a $K_5$ subdivision.

CONJECTURE (Kelmans, Seymour)
Every 5-connected nonplanar graph has a $K_5$ subdivision.
THM Jorgensen No $K_{4,4}$ minor $\Rightarrow \leq 4n - 8$ edges.

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Every graph on $n$ vertices and at least $3n - 5$ edges has a $K_5$ subdivision.

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Every $5$-connected nonplanar graph has a $K_5$ subdivision.

Implies Mader’s theorem (Kezdy, McGuiness)