

# BEYOND GROTZSCH'S THEOREM

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joint work with

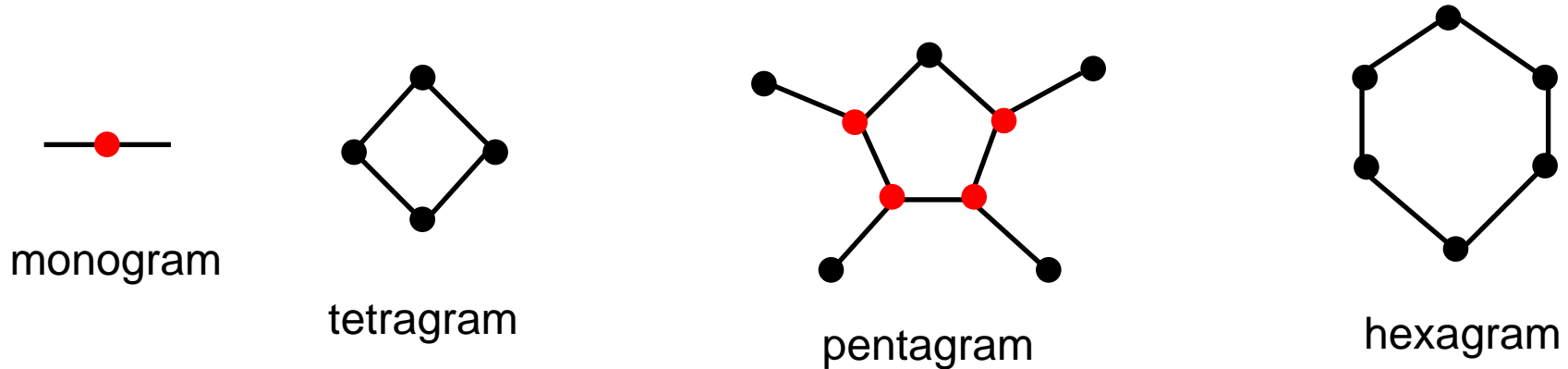
Zdenek Dvorak and K. Kawarabayashi

and

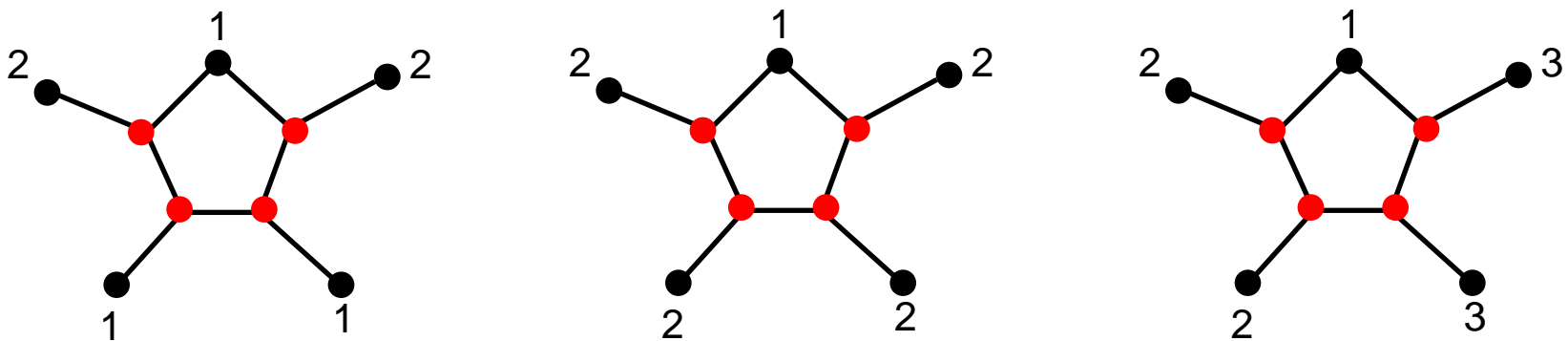
Zdenek Dvorak and Daniel Kral

**THM (Grotzsch)**  $\forall$  triangle-free planar graph is 3-colorable

**PROOF** Reducible configurations:

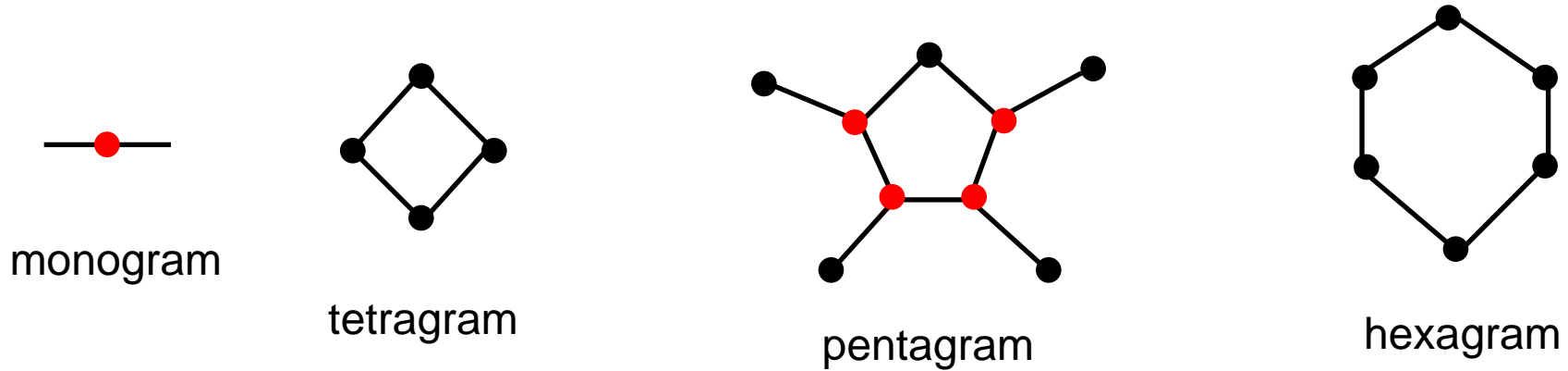


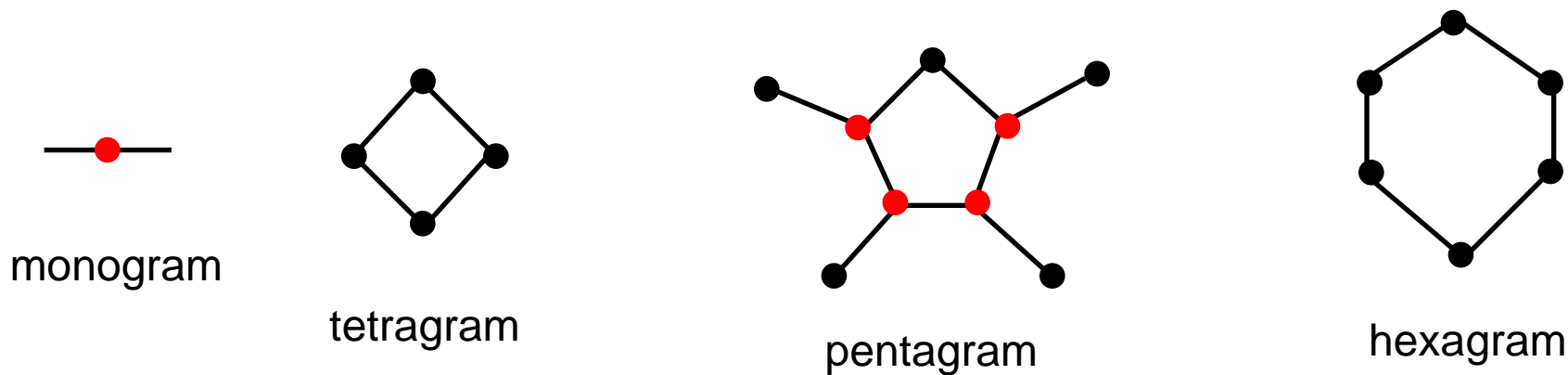
Non-extendable pentagram colorings:



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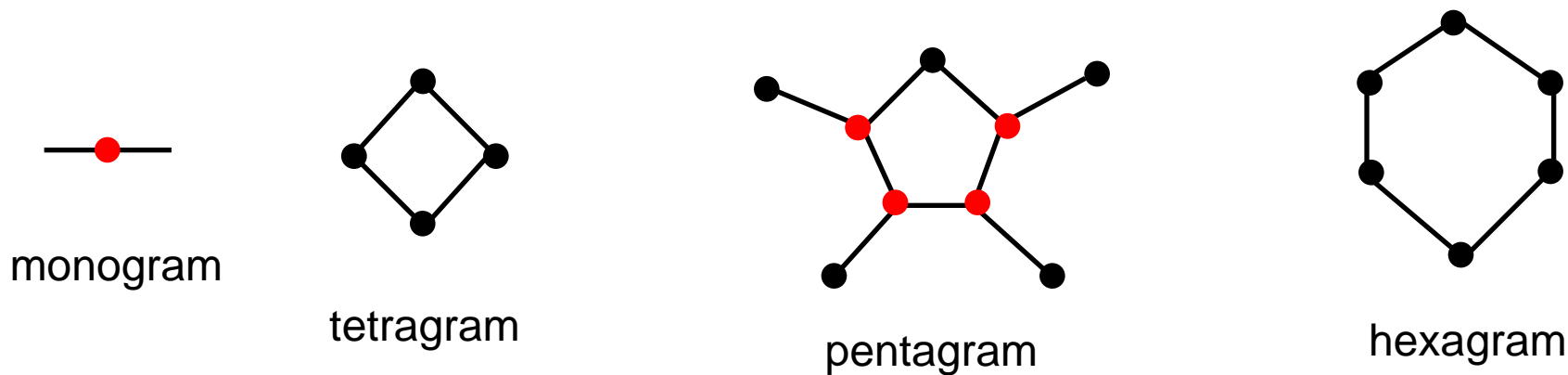


**LEMMA 1**  $G$  connected triangle-free planar,  $C$  outer cycle,  $|C| \leq 6$ ,  $G \neq C \Rightarrow \exists$  multigram s.t. red vertices are not in  $C$

**PF**

$$\text{ch}(v) = \begin{cases} \deg(v) - 4 & v \text{ not in } C \\ -1/3 & v \in C, \deg 2 \\ 0 & \text{o.w.} \end{cases} \quad \text{ch}(f) = \begin{cases} |f| - 4 & f \text{ internal} \\ 0 & f \text{ outer} \end{cases}$$

$$\begin{aligned} \sum \text{charges} &= \sum (d(v) - 4) + \sum (|f| - 4) - t_2/3 + \sum_{v \in C} (4 - d(v)) - (l - 4) \\ &\leq 2E - 4V + 2E - 4F - t_2/3 + 2t_2 + t_3 - l + 4 \leq 2t_2/3 - 4 < 0. \end{aligned}$$



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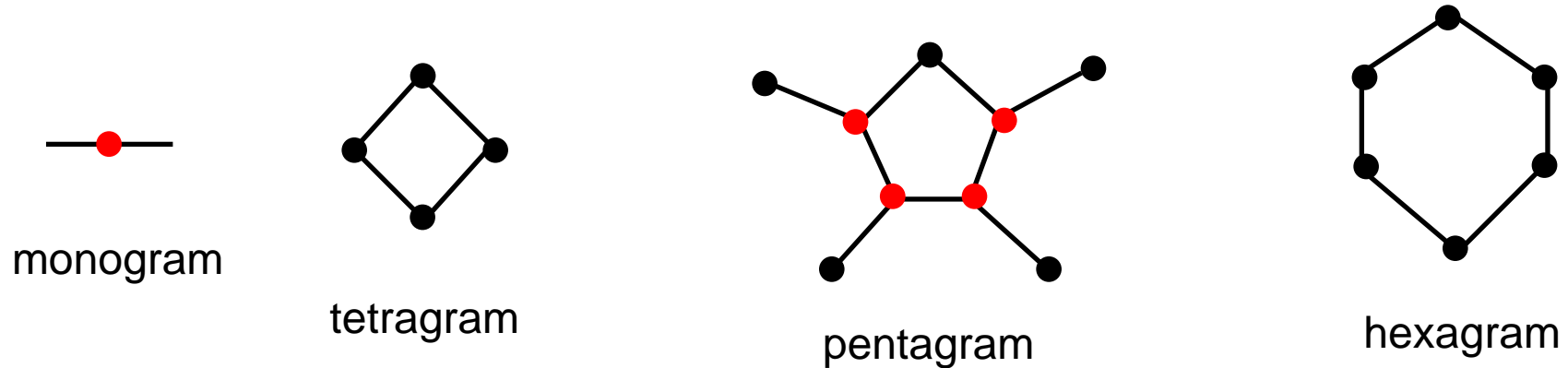
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**LEMMA 2** Every triangle-free planar graph has a safe multigram (reduction does not create triangles)

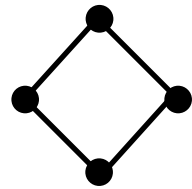
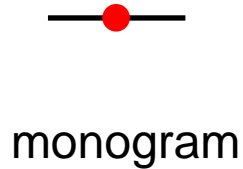
**PF** Pick a separating  $\leq 6$ -cycle  $C$  with smallest inside, or facial  $\leq 6$ -cycle if no separating cycle. Apply Lemma 1 to  $C$  and its interior. Q.E.D.

**PF of Grotzsch:** multigram  $\Rightarrow$  smaller graph  $\Rightarrow$  induction

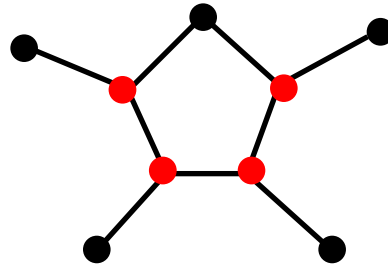
# LINEAR-TIME ALGORITHM

**A problem:** which vertices of a  $C_4$  to identify?

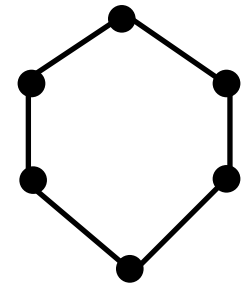
**Kowalik:** An  $O(n \log n)$  algorithm



tetragram



pentagram



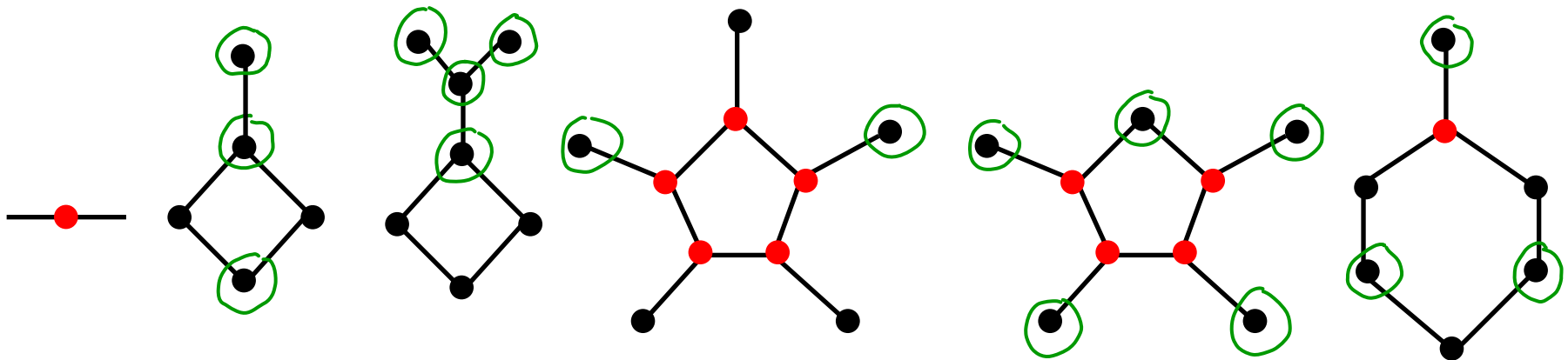
hexagram

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## LINEAR-TIME ALGORITHM

**LEMMA 2** Every triangle-free planar graph has a safe multigram (reduction does not create triangles)

**LEMMA 3** Every triangle-free planar graph has a secure multigram (safe+marked vertices have bounded degree)



**ALGORITHM** Find secure multigram and reduce in constant time to a smaller graph; recurse

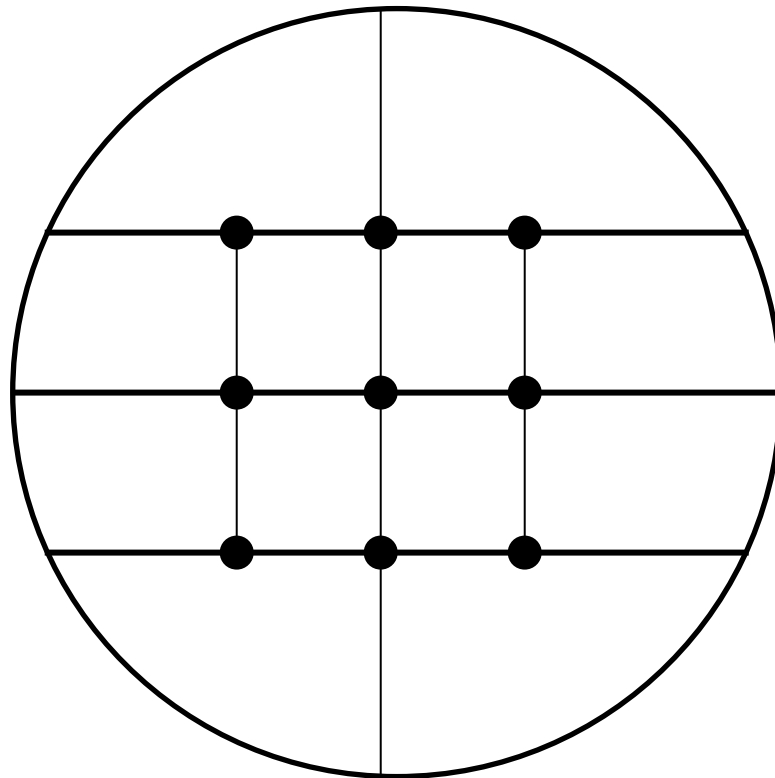
# PART II

Triangle-free graphs on surfaces

**THM (Thomassen)** Every 4-critical graph of girth  $\geq 5$  in  $\Sigma$  has at most  $f(g(\Sigma))$  vertices.

Not true for graphs of girth  $\geq 4$ :

**THM (Youngs)** For every non-bipartite quadrangulation  $G$  of the projective plane  $\chi(G)=4$ .



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**COR** Every 4-critical graph of girth  $\geq 5$  in  $\Sigma$  has at most  $cg(\Sigma)$  vertices.

**APPLICATION** For every surface  $\Sigma$  there is a linear-time algorithm to test whether a given triangle-free graph in  $\Sigma$  is 3-colorable.

**COR (Kawarabayashi, Thomassen)**  $\forall \Sigma \forall$  triangle-free  $G \hookrightarrow \Sigma$  all but  $cg(\Sigma)$  vertices of  $G$  can be 3-colored.

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**MAIN LEMMA**  $G$  plane, girth  $\geq 5$ , outer cycle  $C$ .

If  $G$  is  $C$ -critical and  $|C| \geq 12$ , then

$$\sum (|f|-8)^+ \leq |C|-12 \quad (\text{sum over all finite faces})$$

$G$  is  $C$ -critical if for every proper subgraph  $H$  containing  $C$  some 3-coloring of  $C$  extends to  $H$  but not to  $G$

**PF**  $\exists$  face of negative charge  $\Rightarrow$  reducible configuration.

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# Open problems

Structure of 3-colorable triangle-free toroidal graphs?

Fix a proper minor-closed class  $\mathcal{F}$ . Can  $\chi(G)$  be approximated to within an additive error of 10 in poly-time for  $G \in \mathcal{F}$ ?

HAPPY BIRTHDAY, CARSTEN



