PFAFFIAN ORIENTATIONS OF GRAPHS

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joint work with

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Graphs have vertices
Graphs have vertices and edges

A perfect matching consists of independent edges saturating all the vertices
Graphs have vertices and edges

A perfect matching consists of independent edges saturating all the vertices
The Pfaffian of a skew symmetric matrix $A$

$$\text{Pf}(A)=\sum \text{sign}\begin{pmatrix} 1 & 2 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & \ldots & i_n & j_n \end{pmatrix} a^{i_1j_1} a^{i_2j_2} \ldots a^{i_nj_n}$$

the summation over all partitions $\{\{i_1,j_1\},\{i_2,j_2\}, \ldots,\{i_n,j_n\}\}$ of $[2n]$ into unordered pairs.

**Lemma** $\text{Pf}^2(A)=\det(A)$

In particular, $\text{Pf}(A)$ can be computed efficiently.
The Pfaffian of a skew symmetric matrix $A$

$$\text{Pf}(A) = \sum \text{sign} \begin{pmatrix} 1 & 2 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & \ldots & i_n & j_n \end{pmatrix} a_{i_1j_1} a_{i_2j_2} \ldots a_{i_nj_n}$$

the summation over all partitions $\{\{i_1,j_1\},\{i_2,j_2\},\ldots,\{i_n,j_n\}\}$ of $[2n]$ into unordered pairs.

Now let $A$ be a skew adjacency matrix of a graph $G$. Order the pairs $(i_k,j_k)$ to make sure $a_{i_kj_k} = 1$. Then

$$\text{Pf}(A) = \sum \text{sgn}_D(M)$$

$D(M$)
The Pfaffian of a skew symmetric matrix \( A \)

\[
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DEF An orientation \( D \) of a graph \( G \) is Pfaffian if \( \text{sgn}_D(M) = \text{sgn}_D(M') \) for every two perfect matchings \( M, M' \).

In that case the number of perfect matchings can be efficiently calculated.
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**DEF** An orientation \( D \) of a graph \( G \) is **Pfaffian** if 
\( \text{sgn}_D(M) = \text{sgn}_D(M') \) for every two perfect matchings \( M, M' \).

**Equivalently:** Every even cycle \( C \) such that \( G \setminus V(C) \) has a perfect matching ("central cycle") has an odd number of edges directed in either direction (is "oddly oriented").

**Equivalently:** Either \( G \) has no perfect matching, or for some perfect matching \( M \), every \( M \)-alternating cycle is oddly oriented.
Example:
Example:
Example:

Oddly oriented
Example:
Example:

Not oddly oriented
The number of perfect matchings is $\eta^{N(1+o(1))}$. 
THM (Kasteleyn 1963) Every planar graph has a Pfaffian orientation.

PROOF Orient $G$ so that $\forall$ cycle $C$: $C$ is clockwise odd $\iff$ $C$ encloses even number of vertices of $G$. 
Six equivalent problems
Let $A=(a_{i,j})_{i,j=1,...,n}$ be a 0,1-matrix.

$$\det(A)=\sum \sgn(\sigma) \, a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$$

$$\per(A)=\sum a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$$

**PROBLEM 1 (Polya 1913)** Given a square 0,1-matrix $A$, does there exist a 0,1,-1-matrix $B$ obtained from $A$ by changing some of the 1’s to -1’s in such a way that

$$\per A = \det B?$$
PROBLEM 1 (Polya 1913) Given a square $0,1$-matrix $A$, does there exist a $0,1,-1$-matrix $B$ obtained from $A$ by changing some of the $1$’s to $-1$’s in such a way that $\text{per } A = \det B$?

EXAMPLE Not true for $J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

PROOF

$$\sum_{\sigma} \text{sgn}(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} b_{3\sigma(3)}$$
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**EXAMPLE** Not true for $J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

**PROOF**

$$1 = \prod_{\sigma} \text{sgn}(\sigma) \ b_{1\sigma(1)} b_{2\sigma(2)} b_{3\sigma(3)}$$
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PROOF

$$1 = \prod_{\sigma} \text{sgn}(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} b_{3\sigma(3)} = (-1)^3 b_{11}^2 b_{12}^2 b_{13}^2 \cdots b_{33}^2 = -1$$
PROBLEM 2 Given a bipartite graph, does it have a Pfaffian orientation?

PROBLEM 3 Given a directed graph, does it have no even directed cycle?

PROBLEM 3’ Given a directed graph, is there a function \( w: E(D) \rightarrow \mathbb{Z} \) such that no directed cycle has even total weight?
A real $n \times n$ matrix $A$ is sign-nonsingular if every real $n \times n$ matrix $B$ with the same sign pattern is nonsingular.

**PROBLEM 4.** Given a square matrix, is it sign-nonsingular?

**Application to sign-solvability.** Given $Ax=b$, is the sign pattern of $x$ uniquely determined by the sign patterns of $A$ and $x$?
AN APPLICATION

Economic model of a banana trade:

$S$ supply of bananas  \quad $D$ demand for bananas

$p$ unit price of bananas  \quad $t$ people’s taste for bananas

Then

(1) \quad \frac{\partial S}{\partial p} > 0, \quad \frac{\partial D}{\partial p} < 0, \quad \frac{\partial D}{\partial t} > 0

Equilibrium equations and (1) imply that as people’s taste for bananas increases, so do the price and supply(=demand). The general question leads to sign-solvability.
THEOREM. It is NP-hard to decide if a hypergraph is bipartite. It is NP-hard to decide if a hypergraph is minimally non-bipartite.

THEOREM (Seymour) If a hypergraph is minimally non-bipartite, then \(|E| \geq |V|\).

PROBLEM 5. Given a hypergraph \((V,E)\) with \(|V| = |E|\) is it minimally non-bipartite?
Matrices to bipartite graphs to digraphs

Polya matrix \( \text{iff} \) Pfaffian orientation \( \text{iff} \) \( \exists w : E(D) \rightarrow \mathbb{Z} \)
no even cycle
Characterizing bipartite Pfaffian graphs

**THEOREM** (Little 1975) A bipartite graph $G$ has a Pfaffian orientation $\Leftrightarrow G$ has no $K_{3,3}$ matching minor.

$G$ is a matching minor of $H$ if $G$ can be obtained from a central subgraph of $H$ by bicontracting.
WMA every edge belongs to a perfect matching

A cut $C$ in $G$ is **tight** if $|C \cap M| = 1 \ \forall$ perfect matching $M$.

Tight cut decomposition:

Bipartite graphs with no tight cut are **braces**, nonbipartite are called **bricks**.
The Heawood graph:
THEOREM (McCuaig; Robertson, Seymour, RT) A brace has a Pfaffian orientation ⇔ it either is isomorphic to the Heawood graph, or can be obtained by repeatedly $C_4$-summing, starting from planar braces.

\[
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array}
\]

COROLLARY. There is an $O(n^2)$ algorithm to solve the six problems mentioned earlier.
Pfaffian orientations in general graphs
Theorem (Kasteleyn): Every planar graph is Pfaffian.

Theorem (Norine) A graph is Pfaffian if and only if it can be drawn in the plane (possibly with crossings) so that every perfect matching intersects itself an even number of times.
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Theorem (Norine) A graph is Pfaffian if and only if it can be drawn in the plane (possibly with crossings) so that every perfect matching intersects itself an even number of times.
Proof:

$S \subseteq E(G)$ is a Pfaffian marking of a drawing of $G$ in the plane if for every perfect matching $M$ of $G$ the parity of self intersections of $M$ is equal to the parity of $|M \cap S|$.

**Theorem:** For a graph $G$, the following are equivalent:

1. $G$ is Pfaffian
2. Some drawing of $G$ in the plane has a Pfaffian marking
3. Every drawing of $G$ in the plane has a Pfaffian marking
4. There exists a drawing of $G$ in the plane such that every perfect matching intersects itself even number of times.
\[
sign\begin{pmatrix}
1 & 2 & \ldots & 2n-1 & 2n \\
i_1 & j_1 & \ldots & i_n & j_n \\
\end{pmatrix} = \\
= \text{sign}(\prod_{k>l}(i_k-i_l)(j_k-j_l)\prod_{k}(j_k-i_l)\prod_{k}(j_k-j_l)) \times \\
\times \text{sign}(\prod_{k}(j_k-i_l)) \\

\text{sign}((i_k-i_l)(i_k-j_l)(j_k-i_l)(j_k-j_l)) = -1 \\

\text{if and only if edges } k \text{ and } l \text{ intersect.}
In a standard drawing of a graph with a Pfaffian orientation, the set of backward edges is a Pfaffian marking.

If $S$ is a Pfaffian marking of a standard drawing of a graph then the orientation in which backward edges are exactly the edges of $S$ is Pfaffian.
Changing The Drawing

Event 1

Event 2

Event 3
Changing The Drawing
Changing The Drawing
Norine proved a more general theorem about $T$-joins. Corollaries:

- **Theorem (Hannani, Tutte)** If $G$ can be drawn in the plane in such a way that every two edges cross even number of times, then $G$ is planar.

- **Theorem (Kleitman)**: Let $G=K_{2j+1}$ or $G=K_{2j+1,2k+1}$. Then the parity of the total number of crossings of non-adjacent edges is independent of the choice of the drawing of $G$ in the plane.

- Purely combinatorial reformulation of Turan’s brickyard problem (the problem of estimating the crossing number of a complete bipartite graph).
A labeled graph $G$ is $k$-Pfaffian if there exist orientations $D_1, D_2, \ldots, D_k$ of $G$ and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, such that for every perfect matching $M$ of $G$

$$\sum_{i=1}^{k} \alpha_i \text{sgn}_{D_i}(M) = 1.$$ 

**Theorem (Galluccio, Loebl; Tesler, 1999):** Every graph that can be embedded in the surface of genus $g$ is $4^g$-Pfaffian.

**Theorem (Norine)** Every 3-Pfaffian graph is Pfaffian.

**Theorem (Norine)** A graph is 4-Pfaffian if and only if it can be drawn on the torus (possibly with crossings) so that every perfect matching intersects itself an even number of times.

**Theorem (Norine)** Every 5-Pfaffian graph is 4-Pfaffian.
A labeled graph $G$ is $k$-Pfaffian if there exist orientations $D_1, D_2, \ldots, D_k$ of $G$ and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, such that for every perfect matching $M$ of $G$

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**Theorem (Gallucio, Loebl; Tesler, 1999):** Every graph that can be embedded in the surface of genus $g$ is $4^g$-Pfaffian.

**Conjecture:** For a graph $G$ and integer $g \geq 0$ TFAE:
1. There exists a drawing of $G$ on an orientable surface of genus $g$ such that every perfect matching intersects itself an even number of times.
2. $G$ is $4^g$-Pfaffian.
3. $G$ is $(4^{g+1} - 1)$-Pfaffian.
Pfaffian Labelings and Signs of Edge Colorings
A graph $G$ is $k$-edge-choosable if for every set system
$\{S_e: e \in E(G)\}$ such that $|S_e| = k$ there exists a proper
edge-coloring $c$ with $c(e) \in S_e$ for every $e \in E(G)$.

**List Edge Coloring Conjecture:** Every $k$-edge
colorable graph is $k$-edge-choosable.

**THM (Ellingham, Goddyn, based on Alon, Tarsi)** True for $k$-regular planar graphs

**THM (Norine, RT, based on Alon, Tarsi)** True for $k$-regular Pfaffian graphs

The proof uses “signs” of edge-colorings, and in a sense the method works only for Pfaffian graphs
A graph $G$ is \textit{k-edge-choosable} if for every set system $\{S_e: e \in E(G)\}$ such that $|S_e| = k$ there exists a proper edge-coloring $c$ with $c(e) \in S_e$ for every $e \in E(G)$.

\textbf{List Edge Coloring Conjecture:} Every $k$-edge colorable graph is $k$-edge-choosable.

\textbf{THM (Ellingham,Goddyn, based on Alon,Tarsi)}
True for $k$-regular planar graphs

\textbf{THM (Norine,RT, based on Alon,Tarsi)}
True for $k$-regular Pfaffian graphs

\textbf{CONJECTURE} Every 2-connected 3-regular Pfaffian graph is 3-edge-colorable
A graph $G$ is $k$-edge-choosable if for every set system 
$\{S_e : e \in E(G)\}$ such that $|S_e| = k$ there exists a proper
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True for $k$-regular Pfaffian graphs

**CONJECTURE** Every 2-connected 3-regular Pfaffian
graph is 3-edge-colorable (Implies the 4-color theorem)
Let $\Gamma$ be an Abelian group, we assume $1, -1 \in \Gamma$.

Let $G$ be a graph. $V(G) = \{1, 2, \ldots, 2n\}$.

$l: E(G) \to \Gamma$ is a Pfaffian labeling of $G$ if for every perfect matching $M = \{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_n, j_n\}\}$, $i_k < j_k$ we have

$$\prod_{e \in M} l(e) = \text{sign} \begin{pmatrix} 1 & 2 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & \ldots & i_n & j_n \end{pmatrix}.$$ 

The previous theorem holds more generally for graphs that admit a Pfaffian labeling, but those are not much different from Pfaffian graphs.
THEOREM (Little 1975) A bipartite graph $G$ has a Pfaffian orientation $\iff G$ has no $K_{3,3}$ matching minor.

$G$ is a matching minor of $H$ if $G$ can be obtained from a central subgraph of $H$ by bicontracting.

THEOREM (Fischer, Little) A near-bipartite graph $G$ has a Pfaffian orientation $\iff G$ has no matching minor isomorphic to $K_{3,3}$, $\Gamma_1$, or $\Gamma_2$.

For general graphs need to add infinitely many graphs.
Basic classes of Pfaffian graphs:

- Planar graphs
- Graphs which have “even-faced” embeddings in the Klein bottle

Is there a decomposition theorem?

Obstacle: dense Pfaffian bricks
Dense Pfaffian Bricks
Dense Pfaffian Bricks
Dense Pfaffian Bricks
Dense Pfaffian Bricks
Dense Pfaffian Bricks
Dense Pfaffian Bricks

2n-2 vertices
$(n^2 + 5n - 12)/2$ edges
$K_n$ subgraph
The **tightness** of a cut \( C \) in a graph \( G \) is the maximum of \(|M \cap C|\) over all perfect matchings \( M \) of \( G \).
Tightness leads to the notion of matching-width, analogous to tree-width.

**THM (Norine, RT)** Can test in poly time if a graph of bounded matching-width is Pfaffian

**CONJECTURE** Huge matching-width $\Rightarrow$ large grid matching minor
Bipartite Pfaffian graphs are well-understood, and their characterization solves other problems. General Pfaffian graphs are not, but there are interesting connections to other areas.