Application to algebra

A ring is \((R, +, 0, 1)\), where \(R\) is an abelian group with identity 0
- is associative and 1 is an identity
- \((a + b) c = ac + bc, c(a + b) = ca + cb\)

Examples (i) \(\mathbb{Z}\)
(ii) Matrices over any ring

The commutator in a ring is \([a, b] = ab - ba\). More generally,
\[
[a_1, a_2, \ldots, a_k] = \sum \sigma \in S_k \cdot a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(k)}
\]

If \([a_1, a_2, \ldots, a_k] = 0\) for all \(a_i \in R\), then \(R\) is said to satisfy the \(k\)th polynomial identity.

Theorem (Amitsur, Levi-Zab) The ring of \(k \times k\) matrices over a commutative ring satisfies the \(2k\)th polynomial identity. In other words, if \(A_1, A_2, \ldots, A_{2k}\) are \(k \times k\) matrices with entries in a commutative ring \(R\), then \([A_1, A_2, \ldots, A_{2k}] = 0\).

Proof. Since \([A_1, \ldots, A_{2k}]\) is linear in each variable, it suffices to prove the claim for matrices \(E_{ij}\) of the form \(E_{ij}\) (each entry 0, except for \(a_{ij} = 1\)). So must show
\[
[E_{ij_1}, E_{ij_2}, \ldots, E_{ij_{2k}}] = 0
\]

Define digraph \(D\) with directed multigraph with vertex-set \(\{1, 2, \ldots, k\}\) and edges \((i, j), (j, i)\). Then \([E_{ij_1}, E_{ij_2}, \ldots, E_{ij_{2k}}] = 0\) if

- \(E_{ij_1} E_{ij_2} = 0\) if \(i, j \in [1, k]\)
- \(E(i, j_i) E(i, j) = 0\) if \(i, j \notin [1, k]\)

is an Euler trail in \(D\).
Define a directed multigraph $D$ with $V(D) = \{1, 2, \ldots, k\}$, $E(D) = \{e_1, \ldots, e_{2k}\}$, where $e_i = (i, i+1)$. Then $E_1(e_i) \cdot E_2(e_i) \cdot \cdots \cdot E_{2k}(e_i) \neq 0 \iff \sigma(e_1) \cdot \sigma(e_2) \cdot \cdots \cdot \sigma(e_{2k})$

is an Euler trail in $D$. So

$$
\begin{bmatrix}
E_{ij} & E_{ij} & \cdots & E_{ij}
\end{bmatrix} = \begin{bmatrix}
E_{1i} & E_{2i} & \cdots & E_{2k_i}
\end{bmatrix} = \sum_{\sigma} \text{sgn}(\sigma) E_{1\sigma(e_1)}E_{2\sigma(e_2)}E_{2k\sigma(e_{2k})}
$$

$$
= \sum_{\sigma} \text{sgn}(\sigma) \sum_{W} \text{sgn}(W)
$$

last summation over all Euler trails, where $\text{sgn}(W)$ is the sign of the permutation determined by $W$. Further,

$$
= \sum_{x, y \in V(D)} \sum_{Euler\, trail\, x \to y} \text{sgn}(W)
$$

So need

**Lemma** Let $D$ be a directed multigraph with $|E(D)| \geq 2|V(D)|$, and let $x, y \in V(D)$. Then

$$
E(D, x, y) := \sum_{W} \text{sgn}(W) = 0,
$$

summation over all Euler trails from $x$ to $y$.

Proof: $n = |V(D)|$, $m = |E(D)|$. What are isolated vertices.

Get a new graph with $n + m + 1 - 2n = m + 1 - n$ nodes & $m + m + 1 - 2n + 1 = 2(m + 1 - n)$ edges.

Moreover, $|E(D', x, y)| = |E(D, x, y)|$. So suffices to prove for $x = y$ & $m = 2n$ & no isolated nodes.
By induction. WFA $D$ has an Euler tour, or else $E(D, t, e) = 0$. So vertex $v$.

**Reduction 0.** Parallel edge $\Rightarrow E(D, t, e) = 0$.

**Reduction 1.** Vertex of degree 2 other than $v$.

Delete $b$ to go by induction.

**Reduction 2.** Loop at a vertex of degree $\neq 2$.

If none of the reductions, then either every vertex has $\deg^+(v)$, or $\deg^+(v) = 1$, $\deg^+(w) = 6$ for some $w$, and $\deg^+(u) = 4$ for all other $u$.

$\Rightarrow \exists 2$ adjacent roots of $\deg^+ = 2$ (exercise)
\[ \varepsilon(D, z, t) = \varepsilon(D, z, t) + \varepsilon(D_2, z, t) - \varepsilon(D_6, z, t) - \varepsilon(D_7, z, t) \]