Szemerédi's regularity lemma. Let $\varepsilon > 0$.

A partition $(V_0, V_1, \ldots, V_k)$ of $V(G)$ is $\varepsilon$-regular if

(i) $|V_0| \leq \varepsilon |V(G)|$

(ii) $|V_1| = |V_2| = \cdots = |V_k|$

(iii) all but at most $\varepsilon k^2$ pairs $(V_i, V_j)$ are $\varepsilon$-regular ($1 \leq i, j \leq k$)

Let $A, B \subseteq V(G)$ be disjoint. We say that $(A, B)$ is $\varepsilon$-regular in $G$ if for all $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon |A|$ and $|Y| \geq \varepsilon |B|$ we have

$$|d(A, B) - d(X, Y)| \leq \varepsilon$$

where

$$d(C, D) = \frac{|<C, D>|}{|C| \cdot |D|}$$

$$<C, D> = \{e; e \text{ has one end in } C, \text{ the other in } D\}$$
Theorem (Szemerédi's regularity lemma) \( \forall \varepsilon > 0 \forall \text{ integer } m \exists k \leq m \forall \text{ graph } G \) on \( n \geq m \) vertices \( \exists k \) with \( m \leq k \leq M \) and an \( \varepsilon \)-regular partition \((V_0, V_1, V_2, \ldots, V_k)\) of \( V(G) \).

Preparatory for proof: For \( A, B \subseteq V(G) \) disjoint define \( q(A, B) := \frac{K_{A,B}}{n^2 |A||B|} \), where \( n := |V(G)| \).

For a partition \( P = (C_1, C_2, \ldots, C_k) \) let \( q(P) := \sum_{\{C_i, C_j\} \subseteq P} q(C_i, C_j) = \sum_{i < j} q(C_i, C_j) \).

For a partition \( P = (C_0, C_1, \ldots, C_k) \) with an exceptional set \( C_0 \)

\( q(P) := \sum_{c \in C_0} q(\{c\}, \{c\}) + \sum_{c \in C_0} \sum_{c' \in C \setminus \{c\}} q(\{c\}, \{c'\}) \).
Erdős-Stone Theorem

Turán's Thm

\[ T_{r-1}(n) = \text{complete (r-1)-partite graph with color classes as close to each other in size as possible} \]

\[ T_r(n) \]

\[ \left\lfloor \frac{n}{r} \right\rfloor \text{ or } \left\lfloor \frac{n}{r-1} \right\rfloor \]

Thm (Turán) If \( G \) has no \( \mathbb{T}_r \) subgraph, then

\[ |E(G)| \leq |E(T_{r-1}(n))| \]

with equality if and only if \( G \cong T_{r-1}(n) \).

Pick \( G \) with no \( \mathbb{T}_r \) subgraph with \( |E(G)| \) maximum.

If \( x \circ y = \cdots = \cdots = \circ y \)

\( \deg(x) = \deg(y) \). WLOG \( G \) not complete multipartite.

So \( \exists x, y, z \)

\[ \deg(x) = \deg(y) = \deg(z) \]. Delete \( x, z \), replicate \( y \) twice.
\[ |E(G)| \geq t_{r-1}(n) + \varepsilon n^2 \]

then \( G \) has a \( K^r \), a \( - \) subgraph.

Equivalently, the hypothesis can be stated as

\[ |E(G)| \geq \left( \frac{r-2}{r-1} + \varepsilon \right) \frac{n^2}{2} \]

Def. \( \text{ex}(n, H) := \max \{ |E(G)| : \chi(G) = n, G \text{ has no } H \text{-subgraph} \} \)

So \( t_{r-1}(n) = \text{ex}(n, K_r) \)

Corollary \( \lim_{n \to \infty} \frac{\text{ex}(n, H)}{n^2} = \frac{2(H) - 2}{2(H) - 1} \)
Proof. If $\chi(H) = r$, then $H \leq K^r_{\infty}$ for some $\infty$.

So, it follows from Erdős--Stone.

Conversely, $H \neq T_{r-1}(n)$, because $\chi(T_{r-1}(n)) = r-1$.

Thus $\text{ex}(n, H) \geq \text{ex}(n, T_{r-1}(n))$, and so

$\lim_{n \to \infty} \frac{\text{ex}(n, H)}{(n)} \geq \lim_{n \to \infty} \frac{T_{r-1}(n)}{C_2} = \frac{r-2}{r-1}$. \qed
Proof of Erdős-Stone: Let $r$, $s$ and $\theta > 0$ be given. Want to show $\exists n_0$ s.t. if $|V(G)| \geq n_0$, then $|E(G)| \geq \left(\frac{1}{2} \frac{r-1}{r-2} + \theta\right)n^2 \Rightarrow K_r \subseteq G$.

Let $\varepsilon = \varepsilon(r, s, \theta)$ and $m = m(r) \text{ TBD}$. By Steiner's regularity lemma, $\exists M$ s.t. a graph on $\geq m$ vertices has an $\varepsilon$-regular partition. Let $(V_0, V_1, \ldots, V_s)$ be such an $\varepsilon$-regular partition. Let $R$ be "regularity graph" defined by

$$V(R) = \{1, 2, \ldots, s\}$$

$$i \sim j \text{ in } R \text{ if } (V_i, V_j) \text{ is } \varepsilon \text{-regular and }$$

the density $d(V_i, V_j) \geq \theta$.

Claim: $\varepsilon, m$ can be chosen so that $R$ has a $K_r$ subgraph.

Pf: If not, then by Turán's theorem $|E(R)| \leq \frac{1}{2} \frac{r-2}{r-1} \varepsilon^2$.
\[ |E(G)| \leq \varepsilon n^2 + k \frac{1}{2} \left( \frac{n}{k} \right)^2 + |E(R)| \cdot \left( \frac{n}{k} \right)^2 + \]

- Edge incident with \( V_0 \)
- Edges in \( V_i \) for some \( i \)
- \( V_i - V_j \) edge for \( ij \in E(R) \)
- \( V_i - V_j \) edge
- \( (V_i, V_j) \) und \( E \)-regular
- \( d(V_i, V_j) \leq \gamma \)

\[ \leq \frac{1}{2} \frac{r-2}{r-1} n^2 + \varepsilon n^2 + \frac{1}{k} \frac{n^2}{k} + \varepsilon n^2 + \frac{\varepsilon}{k} n^2 \leq \]

\[ \leq \frac{1}{2} \frac{r-2}{r-1} n^2 + \left( 2\varepsilon + \frac{1}{2m} + \frac{k}{2} \right) n^2 \]

\[ \leq \delta/2 \]

So on choosing \( \varepsilon, m \) s.t.
\[ 4\varepsilon + \frac{1}{m} < \gamma \]
we get a contradiction.
Next we show that if $\varepsilon$ is sufficiently small, then $K^r_d \subseteq G$. We may assume that $\{v_2, \ldots, v_r\}$ is a clique in $R$.

It suffices to show $\exists v_1, v_2, \ldots, v_k \in V_i$, distinct, and sets $V_i' \subseteq V_i$, such that

{} See $\{v_1, \ldots, v_k\}$ complete to $V_i'$ for all $i = 2, \ldots, r$

$|V_i'| \geq \text{const} \times |V_i|$
Lemma Let \( Y \subseteq V_2 \) with \( |Y| \geq \varepsilon |V_2| \). Then all but \( \varepsilon |V_1| \) vertices of \( V_1 \) have \( (\delta-\varepsilon) |Y| \) neighbors in \( Y \).

Proof. Let \( X = \{ v \in V_1 : v \text{ is adjacent to } \leq (\varepsilon-\varepsilon) |Y| \text{ vertices in } Y \} \).

Then \( |X \cap Y| < |X| (\delta-\varepsilon) |Y| \). Then \( d(X, Y) < \delta-\varepsilon \), and so \( |X| < \varepsilon |V_1| \), as desired.

By lemma, all but \( (\varepsilon-\varepsilon) |V_1| \) vertices of \( V_1 \) have \( (\delta-\varepsilon) |Y| \) neighbors in \( Y \) for \( Y = V_2, V_3, \ldots, V_{r-1} \). Pick \( v_i \in V_1 \) with this property and let \( W_i \subseteq V_i \) be such that
\((W_i) \subseteq (\delta - \varepsilon) |V_i|\) and \(v_i\) is complete to \(W_i\) for all \(i = 1, 2, \ldots, r\). Repeat the same argument but with \(V_2 \rightarrow V_r\) replaced by \(W_2 \rightarrow W_r\). We find \(v_2 \in V_1 - \{v_1\}\) and \(Z_i \subseteq W_i\) such that \(|Z_i| = (\delta - \varepsilon) |W_i|\) and \(v_2\) is complete to \(Z_i\) for all \(i = 2, 3, \ldots, r\). After \(r\) iterations we will end up with the vertices \(v_1, v_2, \ldots, v_r\) and sets \(V_2', V_3', \ldots, V_r'\).

\[\square\]

Homework: Prove the triangle removal lemma

Theorem: \(\forall \varepsilon > 0 \exists \delta > 0\) such that if \(G\) is a graph on \(n\) vertices with at most \(\delta n^2\) triangles

then \(\exists F \subseteq E(G)\) such that \(|F| \leq \varepsilon n^2\) and \(G\backslash F\) is triangle-free. 
\[o(n^3)\) triangles \(\Rightarrow o(n^5)\) edges destroy all triangles.
Corollary: \[ \lim_{n \to \infty} \frac{\text{ex}(H, n)}{\binom{n}{2}} = \frac{\chi(H) - 2}{\chi(H) - 1} \]

Definition: The upper density of an (infinite) graph \( G \) is

\[ \limsup \left\{ \frac{|E(H)|}{\binom{|V(H)|}{2}} \mid H \subseteq G, H \text{ finite} \right\} \]

Corollary: The upper density of an infinite graph is

\[ 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1. \]

Proof: Let \( G_r \) be the complete \( r \)-partite graph with \( \infty \) vertices in each class. Then \( G_r \) has upper density \( \chi_r \cdot 1 - \frac{1}{r} \).
Let $c$ be the upper density of $G$, and suppose that $c > 1 - \frac{1}{r-1}$ for some $r \geq 2$. Then $\exists \varepsilon > 0$ such that for $k = 1, 2, \ldots$, $G$ has a subgraph $H_k$ on $n_k^c$ vertices such that $n_k \to \infty$ and

$$|E(H_k)| \geq \frac{1}{2} \left( 1 - \frac{1}{r-1} + \varepsilon \right) n_k^c$$

By Erdős-Stone each $H_k$ has a $K_{k^c}$ subgraph, where $k_k \to \infty$. So

$$\alpha = \lim_{k \to \infty} \frac{|E(H_k)|}{\binom{n_k}{2}} = 1 - \frac{1}{r}.$$