\( \chi'(G) = \min \{ k \mid G \text{ has a } k \text{-edge-coloring} \} \)

\[ \Delta(G) \leq \chi'(G) \]

**Thm (Vizing)** For every simple graph \( G \)

\[ \chi'(G) \leq \Delta(G) + 1 \]

**Example**

\[ |E(G)| = 3k \]
\[ \Delta(G) = 2k \]
\[ \chi'(G) = 3k \]

**Proof:** By induction on \( |E(G)| \). By induction we can color all but one edge of \( G \) using \( \Delta(G) + 1 \) colors.

Let \( xy \) be the uncolored edge. For every \( v \in V(G) \), there is a "missing color" at \( v \) - a color not used by any edge incident with \( v \).
Construct this for as long as $t_1, \ldots , t_k$ are distinct and the edges $x y_i$ exist.

Case 1  $t_k$ is missing at $x$. 

Color $x y_i$ using $t_i$ for $i = 1, 2, \ldots , k$.

Case 2  $t_k = t_j$ for some $j = 1, 2, \ldots , k-1$.

Let $H$ be the subgraph of $G$ consisting of edges colored $\neq$ or $t_k$. 
Notice that if we swap $x$ and $t_k$ in any component of $H$ we get a valid edge-coloring. Either $(x$ and $y_j)$ or $(x$ and $y_k)$ are not in the same component of $H$.

Case 2a $x, y_j$ not in the same component

Swap $A, t_k = t_j$ in the component of $H$ containing $y_j$. Color $x y_j$ using $t$, color $x y_i$ using $t_i$ for $i = 1, 2, \ldots, j - 1$.

Case 2b Analogous (replace $j$ by $k$)
\[ \Sigma_{\text{in}} = \{a, b, \ldots\} \quad \quad \Sigma_{\text{out}} \]

\[ a \rightarrow a_1, \; a_2, \; \text{or} \; a_i, \ldots \]

\[ b \rightarrow b_1, \; b_2, \; \text{or} \; b_i, \ldots \]

\( a, b \) are confoundable if \( a_j = b_i \) for some \( i \).

\underline{Example} \quad \Sigma = \{a, b, c, d, e\}

The graph indicates confoundable pairs.

\( a \) or \( c \)

Any sequence of \( a \)'s and \( c \)'s of length \( t \)

A family of \( 2^t \) pairwise unconfoundable sequences of length \( t \).
Two sequences \((x_i \rightarrow y_i)\), \((y_i \rightarrow y_i)\) are confoundable if \(\forall i=1, 2, \ldots, t\) either \(x_i=y_i\) or \(x_i, y_i\) are confoundable.

A better family: Notice that

\[ab, bd, ca, de, ec\]

are pairwise unconfoundable.

\[aa, bd, cb, de, ec\]

Take any sequence of these of length \(t/2\). That will give a collection of pairwise unconfoundable words of size \(5^{t/2} = (2 \log 5)^{t/2} = 2^{(\frac{t}{2} \log 5)}\).

Def. \(K, L\) \(\Box\) \(L\) vertex-set \(V(K) \times V(L)\)

\((k_1, l_1) \sim (k_2, l_2)\) if \((k_1, l_1) \not\in (k_2, l_2)\)

\(k_1 = k_2\) or \(k_1 \sim k_2\) in \(K\), \(l_1 = l_2\) or \(l_1 \sim l_2\) in \(L\)
Example: $K_2 \square K_2$

Example: $C_5 \square C_5$
Observations:

1. \( \chi(G_1 \boxtimes G_2) \geq \chi(G_1) \chi(G_2) \)

2. \( \chi(C_5 \boxtimes C_5) = 5 \)

Let \( \chi(G) = \chi(G^c) = \min \# \text{ cliques covering the vertices of } G \)

3. \( \chi(G_1) \chi(G_2) \geq \chi(G_1 \boxtimes G_2) \)

Proof:

Let \( K_{r} \rightarrow K_{r} \) be a cover by cliques of \( G_1 \)

\( L_{r} \rightarrow L_{r} \)

Take \( \{K_i \times L_j : 1 \leq i \leq r, 1 \leq j \leq \lambda \} \)

Let \( G^{t} := G \boxtimes G \boxtimes \ldots \boxtimes G \) \( t \) times.

The Shannon capacity of \( G \) is defined as

\[
\lim_{t \to \infty} \frac{1}{t} \log \chi(G^t).
\]
By (1), \[ \alpha(G^tt) \geq \alpha(G^t) \alpha(G^t) \]

\[ \log \alpha(G^tt) \geq \log \alpha(G^t) + \log \alpha(G^t) \]

By Fekete's lemma, \(\lim_{t \to \infty} \frac{\log \alpha(G^t)}{t} \) exists and is equal to \(\sup_{t \geq 1} \frac{\log \alpha(G^t)}{t}\).

If \(\alpha(G) = \nu(G)\), then

\[ \alpha(G) \leq \alpha(G^t) \]

\[ (\nu(G))^t \geq \nu(G^t) \geq \alpha(G^t) \geq \alpha(G) \]

So the Shannon capacity of \(G\) is \(\log \alpha(G)\).

What are the minimal graphs that do not satisfy \(\alpha(G) = \nu(G)\)? What is known about what graphs satisfy \(\alpha(H) = \nu(H)\) for every induced subgraph \(H\)? These are precisely perfect graphs.
Theorem (Lovász) The Shannon capacity of $G$ is $(\log_S)^{1/2}$.

Don't know $S_c$ of $G$. 