Theorem (Erdős & Szekeres) For every integer n there exists an integer \( N \) such that among any \( N \) points in the plane in general position (no 3 on a line) there exist \( n \) points that form a convex \( n \)-gon.

Lemma Out of 5 points in the plane in general position some four of them form a convex 4-gon.

Proof. Take the convex hull of the 5 points. What it's a triangle.

Proof of Theorem. Let \( N = r_4(n, 5) \). That is, if we color 4-tuples on \( \{5, \ldots, N\} \) red and blue, then either there is a set \( A \subseteq \{5, \ldots, N\} \) of size \( n \) such that every 4-tuple in \( A \) is red, or there is a set \( B \subseteq \{5, \ldots, N\} \)
of site 5 such that every 4-tuple in \( B \) is blue.

This \( N \) exists by Ramsey's theorem. Let \( X = \mathbb{R}^2 \) be set of \( N \) points in general position. Color 4-tuples of \( X \) red if they form a convex 4-gon

\[ \begin{cases} \text{red} & \text{if they form a convex 4-gon} \\ \text{blue} & \text{otherwise} \end{cases} \]

By lemma we must get a set \( A \) as above.

By Caratheodory's theorem, if some member of \( A \) is in the convex hull of the others, then it is in the convex hull of 3 other points \( \Rightarrow \) it \& the 3 other points do not form a convex 4-gon, contrary to the 4-tuple being red.

\( \Rightarrow \) The set \( A \) forms a convex \( n \)-gon. \( \square \)
How big is $r_2(k, l)$? We have shown

$$r_2(k, l) \leq r_2(k-k, l) + r_2(k, l-1)$$

Corollary:

$$r_2(k, l) \leq \binom{k+l-2}{k-1}$$

Immediately by induction

Corollary:

$$r_2(k, k) \leq 4^k$$

Theorem (Erdős)

$$r_2(k, k) \geq 2^{k/2}$$ for all $k \geq 2$.

Proof. $r_2(2, 2) = 2 \geq 2^{2/2}$. So WMA $k = 3$.

Let $G_n$ be the set of all graphs with vertex-set $\{1, 2, \ldots, n\}$. Let $X \subseteq \{1, 2, \ldots, n\}$ have size $k$. How many graphs in $G_n$ have $X$ as a clique?

$$|G_n| = 2^{\binom{n}{2}}$$

Let $X \subseteq \{1, 2, \ldots, n\}$ have size $k$. How many graphs in $G_n$ have a clique of size $k$?

$$\leq \binom{n}{k} 2^{\binom{k}{2} - \binom{k}{k}}$$
Need to show that for every $n < 2^{k/2}$ there exists a graph on $n$ vertices with no clique or independent set of size $k$. The proportion of graphs in $G_n$ that have a clique of size $k$ is

$$\frac{\binom{n}{k} 2^{\binom{k}{2}} - \binom{k}{2}}{\binom{n}{2}} \leq \frac{n^k}{2} - \frac{1}{k!} \frac{\binom{k}{2} 2^k}{k!} = \frac{k^2/2 - k^{1/2} + (k/2)^{1/2}}{k} \leq \frac{2^{k/2}}{k!} = \frac{2^{k/2}}{k!} \leq \frac{\sqrt{2}}{k!} \leq \frac{\sqrt{2}}{k} \leq 1$$

$\Rightarrow < 1/2$ graphs in $G_n$ have a clique of size $k$

Similarly, $< 1/2$ graphs in $G_n$ have an independent set of size $k$

$\Rightarrow \exists$ graph that has neither \hfill $\square$
So $2^{k/2} \leq r_2(k, k) \leq 2^k$.

Does $\lim_{k \to \infty} (r_2(k, k))^{1/2}$ exist, and if so, what is it? If exist, it is between $\sqrt{2}$ and 4.

For $r_n(k, k)$ the bounds are

$$2^{\frac{2^k}{2} \sum_{n=1}^{k-1}} \leq r_n(k, k) \leq 2^{\frac{2^k}{2} \sum_{n=1}^{k-1}}.$$
Random graphs

Let $0 < p < 1$, it may depend on $n$.

Let $G(n, p)$ be the probability space of all graphs $G$ with $V(G) = \{1, 2, \ldots, n\}$, where $G$ has probability

$$\frac{1}{E(G)} \cdot \frac{|E(G)|}{(1-p)^{\binom{n}{2} - |E(G)|}}$$

Edges exist with probability $p$, independently of each other.

Markov's inequality: Let $X$ be a non-negative random variable on a probability space with $0 < EX < \infty$. Then for all $t > 0$

$$P\left[ X \geq t \cdot EX \right] \leq \frac{1}{t}$$

Ref. We say that a.e. graph in $G(n, p)$ has property $\Pi$ if

$$\lim_{n \to \infty} P\left[ G \text{ has } \Pi \right] \to 1$$
Theorem. Let $0 < p < 1$ be fixed. Then a.e. graph $G$ in $G(n, p)$ has the property that for all distinct vertices $x_1, x_2, ..., x_k, y_1, y_2, ..., y_l$, there exists a vertex $v \in V(G) - \{x_1, x_2, ..., x_k, y_1, y_2, ..., y_l\}$ such that $v$ is adjacent to $x_1, x_2, ..., x_k$ and not adjacent to $y_1, y_2, ..., y_l$.

Proof. For fixed $x_1, x_2, ..., x_k, y_1, y_2, ..., y_l$, let us say that $v$ works if $v \in V(G) - \{x_1, x_2, ..., x_k, y_1, y_2, ..., y_l\}$ and $v$ is adjacent to $x_1, x_2, ..., x_k$ and not adjacent to $y_1, y_2, ..., y_l$. The probability that a given $v$ works is $p^k (1-p)^l$. The probability it does not work is $1 - p^k (1-p)^l$. The probability that no $v$ works is

$$(1 - p^k (1-p)^l)^{n-k-l}$$
The probability that $\exists x_1, \ldots, x_k, y_1, \ldots, y_l$ such that no $v$ works is

\[
\leq n^{k+l} \left(1 - p^k (1-p)^l\right)^{n-k-l} \to 0
\]

as $n \to \infty$.

Consequences: When $p$ is fixed:

1. a.e. graph in $G(n,p)$ has diameter $\leq 2$
2. for fixed $k$ a.e. graph in $G(n,p)$ is $k$-connected

If $G$ is not $k$-connected, then $\exists a, b, y_1, \ldots, y_l$, where $l < k$, such that $\exists a-b$ path in $G \setminus \{y_1, y_2, \ldots, y_l\}$. Apply this to $a, b, y_1, y_2, \ldots, y_l$. For adjacent does not equal to $a, b, y_1, \ldots, y_l$, a contradiction.

3. For every fixed graph $H$, a.e. graph in $G(n,p)$ has an induced subgraph isomorphic to $H$. 